Certain Pairs of Generalized Derivations of Semiprime Rings

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Abstract: *Let R be a semiprime ring with the cancellation property ,U be a set of R, (D,d) and (G,g) be generalized derivations of R ,if R admits to satisfied some conditions ,where d acts as a left centralizer (resp.g acts as a left centralizer).Then there exist idempotents $\epsilon1, \epsilon 2, \epsilon 3 \in C$ and an invertible element $\lambda \in C$ such that $\epsilon i\epsilon j= 0$, for $i{\neq}j$, $\epsilon 1+ \epsilon 2+ \epsilon 3 =1$,and $\epsilon 1f(x)=\lambda\epsilon 1g(x), \epsilon 2g(x)= 0, \epsilon 3f(x)= 0$ hold for all $x \in U$. (*)*

During our paper we will using (*)for denoted to above result.

1.   Introduction

Derivations on rings help us to understand rings better and also derivations on rings can tell us about the structure of the rings. For instance a ring is commutative if and only if the only inner derivation on the ring is zero. Also derivations can be helpful for relating a ring with the set of matrices with entries in the ring. Derivations play a significant role in determining whether a ring is commutative. Derivations can also be useful in other fields. Generalized derivation of operators on various algebraic structures have been an active area of research since the last fifty years due to their usefulness in various fields of mathematics. In [1],Bresar defined the following notation ,an additive mapping $D:R{\rightarrow}R$ is said to be a generalized derivation if there exists a derivation $d:R{\rightarrow}R$ such that $D(xy)=D(x)y+xd(y)$ for all $x,y \in R$. Other properties of generalized derivations were given by B.Hvala [3] ,T.K.Lee [4] and A.Nakajima ([5],[6],[7] and[8]).We note that for a semiprime ring R,if D is a function from R to R and $d:R{\rightarrow}R$ is an additive mapping such that $D(xy)=D(x)y+xd(y)$ for all $x,y \in R$.Then D uniquely determined by d and moreover d must be a derivation by [[1], Remark1] .We denote a generalized derivation $D:R{\rightarrow}R$ determined by a derivation d of R by (D,d). Muhammad A .C. and Mohammed S.S.[11] proved , let R be a semiprime ring and $d: R{\rightarrow} R$ a mapping satisfy $d(x)y = xd(y)$ for all $x,y \in R$. Then d is a centralizer. Molnar [10] has proved , let R be a 2-torsion free prime ring and let $d: R \rightarrow g\ R$ satisfying $d(x)x + xg(x) = o$ for all $x \in R$, then $d(x),g(x) \in Z(R)$ and $d(u)[x,y] = g(u)[x,y] = o$ for all $u , x,y \in R$.J.Vukman

[13] proved, let R be a 2-torsion free semiprime ring and let d:R → R be an additive centralizing mapping on R , in this case, d is commuting on R .B. Zalar [15] has proved, let R be a 2-torsion free semiprime ring and d:R → R an additive mapping which satisfies d(x2) =d(x)x for all x $\in$ R. Then d is a left centralizer. Mohammad A.,AsmaA.andShakir A.[9] proved, let R be a prime ring and U be a non-zero ideal of R . If R admits a generalized derivation D associated with a non-zero derivation d such that D(xy) - xy $\in$ Z(R) for all x,y $\in$ U , then R is commutative.Hvala [4] initiated the algebraic study of generalized derivation and extended some results concerning derivation to generalized derivation . Nadeem[13] proved , let R be a prime ring and U a non-zero ideal of R . If R admits a generalized derivation D with d such that D(xoy) = xoy holds for all x,y $\in$ U , and if D=o or d≠o , then R is commutative, where d is derivation and the symbol xoy stands for the anti-commutatorxy+yx. In this paper we study when R be a prime ring with cancellation property be a set of R, (D,d) and (G,g) be generalized derivations of R satisfied some conditions. Recently, MehsinJabel[19,20 and 21]proved some results concerning generalized derivations on prime and semiprime rings.

2- Preliminaries

Throughout R will represent an associative ring with the center Z(R),R is said to be prime if xRy=o implies x=o or y=o ,and R is semiprime if xRx=o implies x=o.A prime ring is semiprimebut the converse is not true in general. An additive mapping d:R→R is called a derivation if d(xy)=d(x)y+xd(y) holds for all x,y $\in$ R and d is called left centralizer if d(xy)=d(x)y for all x,y $\in$ R. Also, d is called an inner derivation from R to R induced by a ∈R, such that d(x)=[a,x] for all x ∈ R, then d is a derivation. Not every derivation

is an inner derivation, for example see [16].An additive mapping D:R→R is said to be a generalized derivation if there exists a derivation d:R→R such that D(xy)=D(x)y+xd(y)for all x,y ∈ R. However, generalized derivation covers the concept of derivation. Also with d = o, a generalized derivation covers the concept of left multiplier (left centralizer) that is, an additive mapping D satisfying D(xy) = D(x)y for all x,y ∈ R . [2]).As usual , we write [x,y] for xy − yx and make use of the commutator identities [xy,z] = x[y,z]+ [x,z]y and [x,yz] = y[x,z]+ [x,y]z ,y,z ∈ R. Let R be a semiprime ring. The (two-sided) annihilator of an ideal A of R will be denoted by A$^\perp$, and note that A$^\perp$ coincides with the left and right annihilators of A. Note also that A∩ A$^\perp$ = (0) and that Aө A$^\perp$,(where ө denote to inner sum) is an essential ideal of R. For any semiprime ring R one can construct the ring of quotients Q of R [17], For any semiprime ring R one can construct the ring of quotients Q of R[17], As R can be embedded isomorphically in Q, we consider R as a subringof Q . If the element q ∈ Q commutes with every element in R then q belongs to C, the center of Q.C contains the centroid of R, and it is called the extended centroid of R. In general, C is a von Neumann regular ring, and C is a field if and only if R is prime [17, Theorem 5]. The subring CR of Q is called the central closure of R. R is said to be centrally closed if it is equal to its central closure, or, equivalently, its centroid coincides with its extended centroid. The central closure CR is a centrally closed semiprime ring [18, Theorem 3.2] (thus, the extended centroid of CR is C). Let E bean essential ideal of R. It is a basic fact that if φ:E →R is an R-bimodule homomorphism then there exists λ ∈ C such that φ (u) = λ u for any u ∈ E. Another useful fact is that qE = (0) where q ∈ Q implies that q = 0.Amitsur [17]

called an ideal B of R closed if $B = (B^{\perp})^{\perp}$. The annihilator $A^{\perp}$ of any ideal A of R is a closed ideal (namely, we clearly have $A^{\perp} \subseteq ((A^{\perp})^{\perp})^{\perp}$; conversely, since $A \subseteq (A^{\perp})^{\perp}$, it follows that $((A^{\perp})^{\perp})^{\perp} \subset A^{\perp})$. If B is a closed ideal then there exists an idempotent $\varepsilon \in C$ such that $B = R \cap \varepsilon Q$ [17, Corollary 9]. Now suppose that R is centrally closed. We claim that, in this case, we have $B = \varepsilon R$. Indeed, if $x \in B$ then $x \in \varepsilon Q$, so $x = \varepsilon x$; thus, $B \subseteq \varepsilon R$. Conversely, since R is centrally closed, we have $\varepsilon R \subseteq R$, and so $\varepsilon R \subseteq B$. We summarize these observations into the following statement: If R is a centrally closed semiprime ring and A is an ideal of R, then there exists an idempotent $\varepsilon \in C$ such that $A^{\perp} = \varepsilon R$.

1. The Main Results

Theorem 3.1

Let R be a semiprime ring , U be a set of R, (D,d) and (G,g) be generalized derivations of R,if R admits to satisfy [d(x),g(x)]=0 for all $x \in U$ and d acts as a left centralizer(resp.g acts as a left centralizer)**,** then there exist idempotent satisfied (*)**.**

Proof: We have [d(x),g(x)] =o   for all $x \in U$.
Replacing x by xy, we obtain:

[d(x)y,g(xy)]+[xd(y),g(xy)]=o all x**,**$y \in U$
d(x)[y,g(xy)]+[d(x),g(xy)]y+x[d(y),g(xy)]+[x,g(xy)]d(y)=o

for all x,$y \in U$**,** then
d(x)[y,g(x)y]+d(x)[y,xg(y)]+[d(x),g(x)y]y+[d(x),xg(y)]y+x[d(y),g(x)y]+ x[d(y),xg(y)]+[x,g(x)y]d(y)+[x,xg(y)]d(y)=o for all x, $y \in U$.
d(x)[y,g(x)]y+d(x)x[y,g(y)]+d(x)[y,x]g(y)+g(x)[d(x),y]y+
[d(x),g(x)]y^2+x[d(x),g(y)]y+[d(x),x]g(y)y+xg(x)[d(y),y]+
x[d(y),g(x)]y+x^2[d(y),g(y)]+x[d(y),x]g(y)+g(x)[x,y]d(y)+[x,g(x)]y d(y)+x [x,g(y)]d(y)=o for all x, $y \in U$.   (1)

Replacing y by x and according to the relation [d(x),g(x)]=o,
we obtain:

d(x)[x,g(x)]x+d(x)x[x,g(x)]+g(x)[d(x),x]x+[d(x),x]g(x)x+xg(x)]

[d(x),x]+x[d(x),x]g(x)+[x,g(x)]xd(x)+x[x,g(x)]d(x)=o for all x $\in$ U.

$$(2)$$

Then:

$d(x)xg(x)x-d(x)g(x)x^2+d(x)x^2g(x)-d(x)xg(x)x+g(x)d(x)x^2-$
$g(x)xd(x)x+d(x)xg(x)x-xd(x)g(x)x+xg(x)d(x)x-$
$g(x)xd(x)+xd(x)xg(x)-x^2d(x)g(x)+xg(x)xd(x)-$

$g(x)x^2d(x)+x^2g(x)d(x)-xg(x)xd(x)=o$ for all x $\in$ U .   (3)

Then:

$d(x)x^2g(x)-g(x)xd(x)x+d(x)xg(x)x-xd(x)g(x)x+xg(x)d(x)x-$
$xg(x)xd(x)+xd(x)xg(x)-x^2d(x)g(x)-g(x)x^2d(x)+x^2g(x)d(x)=o$

for all x $\in$ U.   (4)

Since d(x)g(x)=g(x)d(x),then above equation become:

$d(x)x^2g(x)-g(x)xd(x)x+d(x)xg(x)x-xg(x)xd(x)+xd(x)xg(x)-$
$g(x)x^2d(x)=o$ for all x $\in$ U.   (5)

Since d is acts as a left centralizer, then
$d(x^3)g(x)-g(x)xd(x^2)+d(x^2)g(x)x-xg(x)xd(x)+xd(x^2)g(x)-$
$g(x)x^2d(x)=o$ for all x $\in$ U. Then :

$d(x)x^2g(x)+xd(x^2)g(x)-g(x)xd(x)x-$
$g(x)x^2d(x)+d(x)xg(x)x+xd(x)g(x)x-$
$xg(x)xd(x)+xd(x)xg(x)+x^2d(x)g(x)-g(x)x^2d(x)=o$ for all x $\in$ U .

According to (5) , we obtain :
$xd(x^2)g(x)-g(x)x^2d(x)+xd(x)g(x)x+x^2d(x)g(x)=o$

for all x $\in$ U.   (6)

Then:

$$xd(x)xg(x)+2x^2d(x)g(x)-g(x)x^2d(x)+xd(x)g(x)x=o$$
$$\text{for all } x \in U. \tag{7}$$

Since $d(x)g(x)=g(x)d(x)$, above equation becomes:

$$xd(x)xg(x)+2x^2g(x)d(x)-g(x)x^2d(x)+xg(x)d(x)x=o$$
$$\text{for all } x \in U. \tag{8}$$

Since d acts as a left centralizer, (8) becomes:

$$xd(x^2)g(x)+2x^2g(x)d(x)-g(x)x^2d(x)+xg(x)d(x^2)=o \text{ for all } x \in U.$$
Then:

$$xd(x)xg(x)+x^2d(x)g(x)+2x^2g(x)d(x)-g(x)x^2d(x)+xg(x)d(x)x+$$
$$xg(x)xd(x)=o \text{ for all } x \in U. \tag{9}$$

According to (8) the equation (9) becomes
$$x^2g(x)d(x)+xg(x)xd(x)= o \text{ for all } x \in U. \tag{10}$$
Since $g(x)d(x)=d(x)g(x)$, we obtain

$$x^2d(x)g(x)+xg(x)xd(x)=o \text{ for all } x \in U. \tag{11}$$
Since d acts as a left centralizer, (11) become :

$$x^2d(xg(x))+xg(x)xd(x)=o \text{ for all } x \in U. \text{ Then}$$
$$x^2d(x)g(x)+x^3d(g(x))+xg(x)xd(x)=o \text{ for all } x \in U. \tag{12}$$
According to (11), the equation (12) reduce to

$$x^3d(g(x))=o \text{ for all } x \in U. \tag{13}$$
Right- multiplying (13) by r and since d acts as a left centralizer, we get:

$$x^3d(g(x))r+x^3g(x)d(r)=o \text{ for all } x \in U, r \in R. \tag{14}$$

According to (13), we obtain:
$$x^3g(x)d(r)=o \text{ for all } x \in U, r \in R. \tag{15}$$
By using the cancellation property on x^3 in (15),we get

$g(x)d(r)=o$ for all $x\in U, r\in R$.   (16)
Replacing r by ruin(16),we get

$g(x)rd(u)=o$ for all $x,u\in U, r\in R$.   (17)
Again using the cancellation property on $g(x)$ in (15),we get

$d(r)=o$ for all $r\in R$.
Right-multiplying above relation by $rg(u)$, we obtain

$d(x)rg(u)=o$ for all $x,u\in U, r\in R$.   (18)
Thus (17) and (18) gives

$d(x)rg(u)=g(x)rd(u)$ for all $x,u\in U, r\in R$.   (19)

Obviously,
The identity $(19) also holds in case x is an element$ from CR. Thus,
there is no loss of generality in assuming that R is centrally closed.
Let $A=Rd(U)R$ and $B= Rg(U)R$. We have $A^\perp=\omega R$ and $B^\perp=\delta R$ for
some idempotent $\omega,\delta \in$ C.
We set $\epsilon_1 = (1 - \omega)(1 - \delta), \epsilon_2 = (1 - \omega)\delta, and \in_3= \omega$.
Clearly, the $\epsilon_i is$ mutually orthogonal idempotents with sum 1.
Since $\delta g(u) \in B^\perp, u \in U, we have \delta g(u)R\delta g(u) = 0 \ for all u \in$
$U,$ which yields $\delta g(u) = 0 \ \ for all u \in U$ .Hence, $\epsilon_2 g(u) =$
$0 \ for all u \epsilon U$.
Similarly, we see that $\in_3 \ d(u) = 0 \ for all u \in U$.
Note that $(\in_1 A)^\perp = (\in_1 B)^\perp = (1 - \in_1)R$.
Hence, $E = \in_1 A \oplus (1 - \in_1)R$ is an essential ideal of R.
Define $\psi: E \to R by$

$$\psi(\in_1 (\sum_{i=1}^{n} x_i d(u_i)y_i) + (1 - \in_1)r)$$

$$= \in_1 (\sum_{i=1}^{n} x_i g(u_i)y_i) + (1 - \in_1)r.$$

In order to show that ψ is well defined, we suppose that
$\epsilon_1(\sum_{i=1}^{n} x_i d(u_i)y_i) = 0 \ for all x, y, u \epsilon U$.
Consequently, $\epsilon_1(\sum_{i=1}^{n} x_i d(u_i)y_i)zg(t) = 0 \ for all x, y, u, t \epsilon U, z \epsilon R$.

Since by (19),we have
$d(u_i)y_i z g(t) = g(u_i)y_i z d(t)$, it follows   that:

$$\epsilon_1(\sum_{i=1}^{n} x_i g(u_i)y_i)z d(t) = 0 \; forall x, y, \qquad t\epsilon U, z\epsilon R.$$

Thus, the element $\epsilon_1(\sum_{i=1}^{n} x_i g(u_i)y_i)$ $liesinA^{\perp}$.

Since $A^{\perp} = \omega R and \epsilon_1 = (1 - \omega)(1 - \delta)$, It follows that:

$\epsilon_1(\sum_{i=1}^{n} x_i g(u_i)y_i) = 0$.This proves that ψ is well defined.

ψ is an R- bimodal homomorphism. Thus, there exists clearly $\lambda\epsilon C$ such that $\psi(u) = \lambda u$ holds for every $u\epsilon E$. Hence, $\epsilon_1 d(u) = \lambda\epsilon_1 g(u)$ for all $u\epsilon U$.
It remains to prove that λ is invertible. Note that $\lambda E = \epsilon_1 B \oplus (1 - \epsilon_1)R. Since \epsilon_1 B \oplus (1 - \epsilon_1)R$ is an essential ideal (namely, $(\epsilon_1 B)^{\perp} = (1 - \epsilon_1)R)$, λ cannot be divisor of zero. Consequently, as C is a von Neumann regular ring, λ is invertible. The proof of the theorem is thus completed.

Theorem 3.2
   Let R be a semiprime ring with cancellation property, U be a set of R, (D,d) and (G,g) be generalized derivations of R, if R admits to satisfy [D(x),G(x)]=o for all x $\in$ U and d acts as a left centralizer(resp.g acts as a left centralizer)**,** then there exist idempotent satisfied (*)**.**

Proof: We have [D(x),G(x)]=o for all x $\in$ U. Replacing x by xy, we obtain [D(x)y,G(xy)]+[xd(y),G(xy)]=o for all x, y $\in$ U. Then D(x)[y,G(xy)]+[D(x),G(xy)]y+x[d(y),G(xy)]+[x,G(xy)]d(y)=o for all x, y $\in$ U.
D(x)[y,G(x)y]+D(x)[y,xg(y)]+[D(x),G(x)y]y+[D(x),xg(y)]y+x[d(y),G(x)y]+x[d(y),xg(y)]+[x,G(x)y]d(y)+[x,xg(y)]d(y)=o for all x, y $\in$ U. Then:
D(x)[y,G(x)]y+D(x)x[y,g(y)]+D(x)[y,x]g(y)+G(x)[D(x),y]y+

x[D(x),g(y)]y+[D(x),x]g(y)y+xG(x)[d(y),y]+x[d(y),G(x)]y+x²[d(y),
g(y)]+x[d(y),x]g(y)+G(x)[x,y]d(y)+[x,G(x)]yd(y)+x[x,g(y)]d(y)=o
for all x, y $\in$ U.
Replacing y by x, we obtain
D(x)[x,G(x)]x+D(x)x[x,g(x)]+G(x)[D(x),x]x+x[D(x),g(x)]x+
[D(x),x]g(x)x+xG(x)[d(x),x]+x[d(x),G(x)]x+x²[d(x),g(x)]+

x[d(x),x] g(x)+[x,G(x)]xd(x)+x[x,g(x)]d(x)=o for all x $\in$ U .

x([D(x),g(x)]x+G(x)[d(x),x]+[d(x),G(x)]x+x[d(x),g(x)]+[d(x),x]
g(x)+[x,g(x)]d(x))+D(x)[x,G(x)]x+D(x)x[x,g(x)]+G(x)[D(x),x]x+

[D(x),x]g(x)x+[x,G(x)]xd(x)=o for all x $\in$ U .Then :

x(D(x)g(x)x-g(x)D(x)x+G(x)d(x)x-G(x)xd(x)+
d(x)G(x)x-G(x)d(x)x+xd(x)g(x)-xg(x)d(x)+d(x)xg(x)-
xd(x)g(x)+xg(x)d(x)-g(x)xd(x))+D(x)xG(x)-
D(x)G(x)x+D(x)x²g(x)-D(x)xg(x)x+G(x)D(x)x²-
G(x)xD(x)x+D(x)xg(x)x-xD(x)g(x)x+xG(x)xd(x)-

G(x)x²d(x)=o for all x $\in$ U .Then:

-xg(x)D(x)x+xd(x)G(x)x+xd(x)xg(x)-
xg(x)xd(x)+D(x)xG(x)x+D(x)x²g(x)-

G(x)xD(x)x-G(x)x²d(x)=o for all x $\in$ U .

Since d acts as a left centralizer, then:

-xg(x)D(x)x+xd(x)G(x)x+xd(x²)g(x)-xg(x)xd(x)+D(x)xG(x)x+

D(x)x²g(x)-G(x)xD(x)x-G(x)x²d(x)=o for all x $\in$ U .   (20)
-xg(x)D(x)x+xd(x)G(x)x+xd(x)xg(x)+x²d(x)g(x)-
xg(x)xd(x)+D(x)xG(x)x+D(x)x²g(x)-

G(x)xD(x)x-G(x)x²d(x)=o for all x $\in$ U.   (21)
Since d acts as a left centralizer (21) reduces to:
-xg(x)D(x)x+xd(x)G(x)x+xd(x²)g(x)+x²d(x)g(x)-
xg(x)xd(x)+D(x)xG(x)x+D(x)x²g(x)-

G(x)xD(x)x-G(x)x^{2}d(x)=o for all x $\in$ U .
According to (20) ,we obtain:

x^{2}d(x)g(x)=o for all x $\in$ U .   (22)
By using the cancellation property on x^{2}d(x) in (22),we get
g(x)=o for all x $\in$ U.   (23)

Left-multiplying (23) by d(u)r gives:

d(u)rg(x)=o for all x ,u $\in$ U,r$\in$R.   (24)
Again right-multiplying (23) by rd(x) gives:

g(u)rd(x)=o for all x ,u $\in$ U,r $\in$ R.
Then by same in Theorem 3.1,we complete the proof of theorem.

Theorem 3.3
   Let R be a semiprime ring with cancellation property, U be a
set of R, (D,d) and (G,g) be generalized derivations of R,if R
admits to satisfy [D(x),G(x)]=[d(x),g(x)] for all x $\in$ U and d acts
as a left centralizer(resp.g acts as a left centralizer), then thereexist
idempotents satisfied (*)**.**
Proof: We have [D(x),G(x)]=[d(x),g(x)] for all x $\in$ U . Then
D(x)G(x)-G(x)D(x)=d(x)g(x)-g(x)d(x) for all x $\in$ U . Since d acts
as a left centralizer, we obtain:

 D(x)G(x)-G(x)D(x)=d(xg(x))-g(x)d(x) for all x $\in$U. Then:
 [D(x),G(x)]=d(x)g(x)-xd(g(x))-g(x)d(x) for all x$\in$U.
According to the relation [D(x),G(x)]=[d(x),g(x)] ,we get:
-xd(g(x))=o for all x $\in$ U.   (25)
Right –multiplying (25)by y and since d acts as a left centralizer,
we obtain –xd(g(x)y)=o for all x $\in$ U . Then:

-xd(g(x))y-xg(x)d(y)=o for all x, y $\in$ U. According to (25), we obtain:

xg(x)d(y)=o for all   x,y∈U.   (26)

By using the cancellation property on xg(x) in (26), gives:

d(y)=o for all y $\in$ U . Then by same in Theorem 3.2, we complete the proof of theorem.

Remark 3. 4

In our theorem we cannot exclude the condition cancellation property, the following example explain that.

Example 3.5

Let R be a ring of matrices 2×2 with cancellation property

R={$\begin{pmatrix} a & 0 \\ o & b \end{pmatrix} / a^2 = a, a, b \in Z$ },where Z is the set of integers ,and the additive map D define as the following

$$D\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} and\, d(x) = [a, x], where$$

$$a = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} then, let$$

$$x = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} and\, y = \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix}. then$$

$$D(xy) = D(x)y + xd(y)$$

$$D\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix}\right) = D\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix} + \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}d\begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix}$$

$$D\begin{pmatrix} ag & 0 \\ 0 & bh \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix} + \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 0 & 0 \\ h-g & 0 \end{pmatrix}$$

$$\begin{pmatrix} ag & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} ag & 0 \\ b(h-g) & 0 \end{pmatrix}. then$$

$$\begin{pmatrix} 0 & 0 \\ b(h-g) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \qquad (**)$$

$$b.\begin{pmatrix} 0 & 0 \\ h-g & 0 \end{pmatrix} = b.\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$then\, by\, using\, the\, cancellation\, property\, on\, b\, , we\, get$

$$\begin{pmatrix} 0 & 0 \\ h-g & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, which\, implies\, the\, th = g$$

With substituting this result in equation (**) , we obtain D is generalized derivation .

Now we choose the $\epsilon_1, \epsilon_2, \epsilon_3 \in$ C (C is the extended centroid of R),let $\begin{pmatrix} w & 0 \\ 0 & 0 \end{pmatrix} \in$ R, where w $\in$ Z the set of integers, when

$\begin{pmatrix} w & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} - \begin{pmatrix} w & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$,which implies that $\begin{pmatrix} w & 0 \\ 0 & 0 \end{pmatrix}$

$\in$ C.Also,it is clear that is idempotnt element of R(where $\begin{pmatrix} w & 0 \\ 0 & 0 \end{pmatrix}^2$

$=\begin{pmatrix} w & 0 \\ 0 & 0 \end{pmatrix}$). For condition (i),

Let $\epsilon_{1=}\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}, \epsilon_2 = \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix}, \epsilon_3 = \begin{pmatrix} w & 0 \\ 0 & 0 \end{pmatrix}$.Then

$\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}+\begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix}+\begin{pmatrix} w & 0 \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} x+y+w & 0 \\ 0 & 0 \end{pmatrix}$,put $x+y+w=q$, then

$\begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}=q\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$,left-multiplying by q, we get

$q\begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}=q^2\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$,where q^2 $=q$, thus,we get

$q\begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}=q\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$,by using the cancellation property from left on

q, we get

$$\begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \text{Then}$$

q=1,which meaning x+y+w=1.Then

$\epsilon_{1+} \epsilon_{2+} \epsilon_{3=1.}$

$\epsilon_2 g(x) = \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & h-s \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$,where $g(x) = \begin{pmatrix} h & 0 \\ 0 & s \end{pmatrix}$,similarly

for $\epsilon_3 d(x) = 0$ hold for all x $\in$ R.Also,we have by using same method which using above with the cancellation property, we get the following

$\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ which implies that $\epsilon_i \epsilon_j = 0$, for i≠j.

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بعض أزواج من الاشتقاقات العامة
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ميساء زكي سلمان

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المستخلص:

لتكن R حلقة شبه أولية مع المركز(Z(R و U مجموعة في الحلقة شبه الأولية R إذا تسمح بتحقيق بعض الشوط عندماd تعمل (D, R ,(d)و(G,g) اشتقاقات عامة وكأنها مركزي أيسر(اعتبر g تعمل وكأنها مركزي أيسر).وD تعمل وكأنها مركزي أيسر(اعتبر G تعمل وكأنها مركزي أيسر) عناصرC ∈ 3 ,2 ϵ ,1 ϵ فانه يوجد متلاشية بحيث تحقق مايلي:

ϵ i ϵ j= 0, for i≠j, ϵ 1+ ϵ 2+ ϵ 3 =1,و ϵ 1f(x)=λ ϵ 1g(x), ϵ 2g(x)= 0,

ϵ 3f(x)= 0 لكل x∈ U.

و إن λ ∈ C عنصر انقلابي.