Differential Transformation Method for Solving Nonlinear Heat Transfer Equations

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Abstract: *In this paper, Differential Transformation Method (DTM) is applied for solving the nonlinear differential equations arising in the field of heat transfer. The solution is considered as an infinite series expansion where converges rapidly to the exact solution. The nonlinear convective–radioactive cooling equation and nonlinear equation of conduction heat transfer with the variable physical properties are chosen as illustrative examples and solutions have been found for each case. Results by DTM with other results calculated by Homotopy Perturbation Method are compatible.*

Keywords:. DTM; Nonlinear equations; Conduction and convection heat transfer

1. Introduction

Since most of the phenomena in our world are essentially nonlinear and hence described by nonlinear equations, there has developed an ever-increasing interest of scientists and engineers in the analytical asymptotic techniques for solving nonlinear problems. Recently, many new numerical techniques have been widely applied to the nonlinear problems. One of these methods the Differential Transformation Method (DTM) attracted great attention due to its versatility and straightforwardness.

The technique that we used is introduced by Zhou [1] in a study about electrical circuits. It gives exact values of the kth derivative of an analytical function at a point in terms of known and unknown boundary conditions in a fast manner and has since been used by many mathematicians and engineers to solve various functional equations. The technique that we used, which is based on Taylor series expansion enables us to obtain a series solution by means of an iterative procedure, which is the main advantage of this technique. The (DTM) was applied to linear and nonlinear ODEs; see Vedat [2, 3], Anwar [4-6].

In this article, the basic idea of the DTM is introduced and its application in two heat transfer equations is studied. This numerical scheme is based upon the Taylor series expansion and, as we shall soon see, is capable of finding the exact solution of many nonlinear differential equations.

This paper is organized as follows: Section 2 describes the differential transform method. In Section 3, the method is implemented to two examples, and finally conclusions are given in Section 4.

2. Differential Transformation Method (DTM)

The differential transformation of the kth derivatives of function $y(x)$ is defined as follows [2]:

$$Y(k)=\frac{1}{k!}\left[\frac{d^k y}{dx^k}\right]_{x=x_0} \qquad (1)$$

and $y(x)$ is the differential inverse transformation of $Y(k)$ defined as follows:

$$y(x)=\sum_{k=0}^{\infty}Y(k).(x-x_0)^k \qquad (2)$$

for finite series of k = N , Eq.(2) can be written as:

$$y(x)=\sum_{k=0}^{N}Y(k).(x-x_0)^k \qquad (3)$$

The following theorems that can be deduced from Eqs.(1) and (3):

Theorem 1. If $y(x)=g(x)\pm h(x)$,then $Y(k)=G(k)\pm H(k)$.

Theorem 2. If $y(x)=\alpha.g(x)$,then $Y(k)=\alpha.G(k)$.

Theorem 3. If $y(x)=\dfrac{dg(x)}{dx}$,then $Y(k)=(k+1).G(k+1)$.

Theorem 4. If $y(x)=\dfrac{d^m g(x)}{dx^m}$,then $Y(k)=((k+m)!/k!).G(k+m)$.

Theorem 5. If $y(x)=g(x).h(x)$,then $Y(k)=\sum_{l=0}^{k} G(l)H(k-l)$.

Theorem 6. If $y(x)=x^m$,then $Y(k)=\delta(k-m)=\begin{cases}1 & if \quad k=m\\ 0 & if \quad k\neq m\end{cases}$.

Theorem 7. If $y(x)=\exp(\alpha.x)$,then $Y(k)=\alpha^k/k!$.

Theorem 8. If $y(x)=\sin(\alpha.x+\lambda)$,then $Y(k)=(\alpha^k/k!)\sin(k\pi/2+\lambda)$.

Theorem 9. If $y(x)=\cos(\alpha.x+\lambda)$,then $Y(k)=(\alpha^k/k!)\cos(k\pi/2+\lambda)$.

3. Numerical applications

Two physical problems are presented and solved by the DTM. The nonlinear convective–radioactive cooling equation and nonlinear equation of conduction heat transfer with the variable physical properties are chosen as illustrative examples.

3.1. Cooling of a lumped system by combined convection and radiation

Consider the following problem of the combined convective–radioactive cooling of a lumped system; see Aziz and Na [7]. Let the system have volume V , surface area A, density ρ , specific heat c , emissivity E , and the initial temperature T_i. At t = 0 , the system is exposed to an environment with convective heat transfer with the coefficient of h and the temperature T_a. The system also loses heat through radiation and the effective sink temperature is T_s. The corresponding governing equation of this cooling problem is as follows

$$c\,V\,\frac{dT}{dt} + h\,A\,(T - T_a) + E\,\sigma A\,(T^4 - T_s^4) = 0 \quad , \quad T(0) = T_i \qquad (4)$$

Under the transformations $\theta = \frac{T}{T_i}$, $\theta_a = \frac{T_a}{T_i}$, $\theta_s = \frac{T_s}{T_i}$, $\tau = \frac{hAt}{\rho c V}$,

and $\varepsilon = \frac{E\sigma T_i^3}{h}$

Equation (4) can be written as

$$\frac{d\theta}{d\tau} + (\theta - \theta_a) + \varepsilon(\theta^4 - \theta_s^4) = 0 \ , \ \theta(0) = 1 \qquad (5)$$

For the sake of simplicity, we take $\theta a = \theta s = 0$. Therefore, we have

$$\frac{d\theta}{d\tau} + \varepsilon\theta^4 + \theta = 0 \qquad , \quad \theta(0) = 1 \qquad (6)$$

The exact solution of above equation was found to be of the form

$$\tau = \frac{1}{3} Ln \frac{1+\varepsilon\theta^3}{(1+\varepsilon)\theta^3} \qquad (7)$$

Expanding $\theta\,(\tau\,)$, using Taylor expansion, about $\tau = 0$ gives the series of the exact solution

$$\theta(\tau) = 1 + (-1-\varepsilon)\tau + \left[\frac{1}{2} + \frac{5}{2}\varepsilon + 2\varepsilon^2\right]\tau^2 + \left[-\frac{1}{6} - \frac{7}{2}\varepsilon - 8\varepsilon^2 - \frac{14}{3}\varepsilon^3\right]\tau^3 + \left[\frac{1}{24} + \frac{85}{24}\varepsilon + \frac{35}{2}\varepsilon^2 + \frac{77}{3}\varepsilon^3 + \frac{35}{3}\varepsilon^4\right]\tau^4 + \left[-\frac{1}{120} - \frac{341}{120}\varepsilon - \frac{82}{3}\varepsilon^2 - \frac{455}{6}\varepsilon^3 - \frac{245}{3}\varepsilon^4 - \frac{91}{3}\varepsilon^5\right]\tau^5 + \cdots \qquad (8)$$

Hossein and Milad [8] solved Eq.(6) using Homotopy Perturbation method and gave the following solution:

$$\theta(\tau) = 1 + (-1-\varepsilon)\tau + \left[\frac{1}{2} + \frac{5}{2}\varepsilon + 2\varepsilon^2\right]\tau^2 + \left[-\frac{1}{6} - \frac{7}{2}\varepsilon - 8\varepsilon^2 - \frac{14}{3}\varepsilon^3\right]\tau^3 + \left[\frac{1}{24} + \frac{85}{24}\varepsilon + \frac{35}{2}\varepsilon^2 + \frac{77}{3}\varepsilon^3 + \frac{35}{3}\varepsilon^4\right]\tau^4 + \cdots \quad (9)$$

To solve Equation (6), by means of DTM, we construct the following recurrence equation

$$\theta(k+1) = \frac{-1}{(k+1)}\left[\theta(k) + \varepsilon \sum_{m_3=0}^{k}\sum_{m_2=0}^{m_3}\sum_{m_1=0}^{m_2}\theta(m_1)\ \theta(m_2-m_1)\ \theta(m_3-m_2)\ \theta(k-m_3)\right] \quad (10)$$

The boundary conditions in Eq. (6) can be transformed at $\tau_0 = 0$ as follows:

$$\theta(0) = 1\ . \quad (11)$$

Utilizing the recurrence relation in (9) and the transformed boundary conditions in Eq. (10), $\theta(k)$ were calculated for k = 1,2,3,4 as in Table (1).

Table (1): $\theta(k)$ for different k

K	recurrence equation $\theta(k+1)$	$\theta(k+1)$
0	$-\theta(0)-\varepsilon$	$-1-\varepsilon$
1	$\frac{-1}{(2)}[\theta(1)+\varepsilon\ \{\sum_{m_3=0}^{1}\sum_{m_2=0}^{m_3}\sum_{m_1=0}^{m_2}\theta(m_1)\ \theta(m_2-m_1)\ \theta(m_3-m_2)\,\theta(1-m_3)\}]$	$\frac{1}{2!}[1+5\varepsilon+4\varepsilon^2]$
2	$\frac{-1}{3}[\theta(2)+\varepsilon\ \{\sum_{m_3=0}^{2}\sum_{m_2=0}^{m_3}\sum_{m_1=0}^{m_2}\theta(m_1)\ \theta(m_2-m_1)\ \theta(m_3-m_2)\ \ \theta(2-\ m_3)\}]$	$-\frac{1}{3!}[1+21\varepsilon+48\,\varepsilon^2+28\,\varepsilon^3]$
3	$\frac{-1}{4}[\theta(3)+\varepsilon\ \{\sum_{m_3=0}^{3}\sum_{m_2=0}^{m_3}\sum_{m_1=0}^{m_2}\theta(m_1)\ \theta(m_2-m_1)\ \theta(m_3-m_2)\ \ \theta(3-m_3)\}]$	$\frac{1}{4!}[1+77\,\varepsilon+252\,\varepsilon^2+232\,\varepsilon^3+56\,\varepsilon^4]$
4	$\frac{-1}{5}[\theta(4)+\varepsilon\ \{\sum_{m_3=0}^{4}\sum_{m_2=0}^{m_3}\sum_{m_1=0}^{m_2}\theta(m_1)\ \theta(m_2-m_1)\ \theta(m_3-m_2)\ \ \theta(4-m_3)\}]$	$-\frac{1}{5!}[1+333\,\varepsilon+\ 2776\,\varepsilon^2+8044\varepsilon^3+3045\,\varepsilon^4+2864\,\varepsilon^5]$

Therefore, the solution of Equation (6) as

$$\theta(\tau)=1-(1+\varepsilon)\tau+\frac{1}{2!}[1+5\varepsilon+4\varepsilon^2]\tau^2-\frac{1}{3!}[1+21\varepsilon+48\,\varepsilon^2+28\,\varepsilon^3]\tau^3+\frac{1}{4!}[1+77\,\varepsilon+252\,\varepsilon^2+232\,\varepsilon^3+56\,\varepsilon^4]\,\tau^4-\frac{1}{5!}[1+333\,\varepsilon+\ 2776\,\varepsilon^2+8044\varepsilon^3+3045\,\varepsilon^4+2864\,\varepsilon^5]\tau^5\ \ldots\ldots \quad (12)$$

The solution (Eq.(12)) of the problem (Eq.(6)) represented in Table (2) and (3), for $(0.1<\tau<0.9)$ and $(\varepsilon=0.1\,,0.2,\ 0.3,\ 0.4)$,

Table (2): Results by DTM, HPM for $\theta(\tau)$ verses τ for $\varepsilon = 0.1, 0.2.$

	$\varepsilon = 0.1$			$\varepsilon = 0.2$		
T	$\theta(\tau)$ DTM	$\theta(\tau)$ HPM	Absolute error I HPM-DTM I	$\theta(\tau)$ DTM	$\theta(\tau)$ HPM	Absolute error I HPM-DTM I
0.1	0.8971519	0.8971406	1.138449E-5	0.8897192	0.8896737	4.553795E-5
0.2	0.8067375	0.8065653	1.721978E-4	0.7953129	0.7946396	6.733537E-4
0.3	0.7263253	0.7255046	8.206964E-4	0.711838	0.7087079	3.130138E-3
0.4	0.6533575	0.6509255	2.432048E-3	0.6325679	0.6235554	9.01258E-3
0.5	0.5843707	0.5788278	5.542934E-3	0.5460925	0.5262384	1.985413E-2
0.6	0.514215	0.5035397	1.067525E-2	0.4334173	0.3967045	3.671286E-2
0.7	0.4352739	0.4170133	1.826066E-2	0.2650634	0.2053045	5.975886E-2
0.8	0.3366847	0.30812	2.856475E-2	-1.833515E-3	-0.0896946	8.786109E-2
0.9	0.2035577	0.1619464	4.161128E-2	-0.426423	-0.5445983	0.1181753

Table (3) Results by DTM, HPM for $\theta(\tau)$ verses τ for $\varepsilon = 0.3, 0.4$.

	$\varepsilon = 0.3$			$\varepsilon = 0.4$		
τ	$\theta(\tau)$ DTM	$\theta(\tau)$ HPM	Absolute error I HPM-DTM I	$\theta(\tau)$ DTM	$\theta(\tau)$ HPM	Absolute error I HPM-DTM I
0.1	0.8825229	0.8824103	1.126528E-4	0.8755469	0.875323	2.23875E-4
0.2	0.7842923	0.7826664	1.625896E-3	0.7734694	0.7703296	3.139853E-3
0.3	0.6962563	0.6889225	7.333875E-3	0.678055	0.6644023	1.365274E-2
0.4	0.6030357	0.5826917	2.034402E-2	0.5578151	0.5217548	3.606033E-2
0.5	0.4755976	0.4328499	4.274762E-2	0.3501579	0.2794273	7.073066E-2
0.6	0.2639324	0.1896412	7.429115E-2	-5.393971E-2	-0.1647186	0.1107789
0.7	-0.1102708	-0.2213187	0.1110478	-0.8252591	-0.9640033	0.1387442
0.8	-0.7609485	-0.9050371	0.1440886	-2.211668	-2.334935	0.1232669
0.9	-1.844987	-2.00314	0.1581537	-4.553447	-4.569213	1.576614E-2

Figures (1) and (2) represent the solution between $\theta(\tau)$ and τ for by HPM and DTM for ($\varepsilon = 0.1, 0.2, 0.3, 0.4$) ,

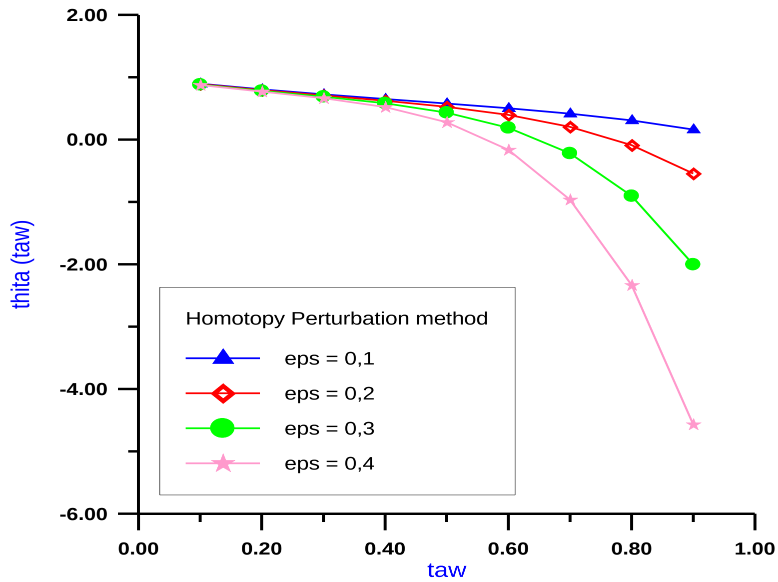

Figure (1) HPM solution between $\theta(\tau)$ and (τ) for $(\varepsilon = 0.1, 0.2, 0.3, 0.4)$

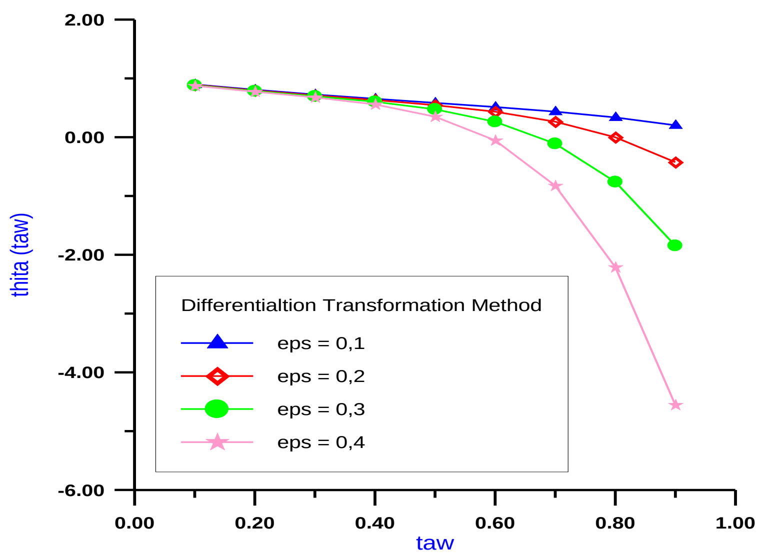

Figure (2) DTM solution between $\theta(\tau)$ and (τ) for $(\varepsilon = 0.1, 0.2, 0.3, 0.4)$

3.2. Cooling of a Lumped System with Variable Specific Heat

Consider the cooling of a lumped system; see Y'aziz et al [9] exposed to a convective environment at temperature Ta with convective heat transfer coefficient h at time t = 0. Let the system have volume V, surface area A, density ρ , specific heat C and initial temperature Ti . Assume that the specific heat C is a linear function temperature of the form

$C = Ca\,[1 + \beta(T - Ta)],$ (13)

where Ca is the specific heat, at temperature Ta and β is a constant. The cooling equation corresponding to this problem is

$$\rho c\, V\, \frac{dT}{dt} + h\, A\, (T - T_a) = 0 \quad , \quad T(0) = T_i \tag{14}$$

Under the transformations $\theta = \frac{T-T_a}{T_i-T_a}$,$\tau = \frac{hAt}{\rho cV}$,and $\varepsilon = \beta(T - T_a)$ Equation (15) can be written as

$$(1 + \varepsilon\theta)\frac{d\theta}{d\tau} + \theta = 0 \qquad \theta(0) = 1 \tag{15}$$

Taylor expansion, about $\tau = 0$ gives the series of the exact solution

$$\theta(\tau) = 1 - \frac{1}{1+\varepsilon}\tau + \frac{1}{2!\,(1+\varepsilon)^3}\tau^2 - \frac{1-2\varepsilon}{3!\,(1+\varepsilon)^5}\tau^3 + \frac{1-8\varepsilon+6\varepsilon^2}{4!\,(1+\varepsilon)^7}\tau^4 - \frac{1-22\varepsilon+58\varepsilon^2-24\varepsilon^3}{5!\,(1+\varepsilon)^9}\tau^5 + \cdots \tag{16}$$

Hossein *et al* [8] solved Eq.(15) using Homotopy Perturbation method and gave the same solution in Eq.(16).

Now to solve Equation (15), by means of DTM, we construct the following equation

$$\frac{d\theta}{d\tau} + \varepsilon\,\theta\,\frac{d\theta}{d\tau} + \theta = 0 \tag{17}$$

The recurrence equation is

$$(k+1)\ \theta(k+1) + \varepsilon \sum_{l=0}^{k} \theta(l)(k-l+1)\,\theta(k-l+1) + \theta(k) = 0 \qquad (18)$$

The boundary conditions in Eq. (14) can be transformed at $\tau_0 = 0$ as follows:

$$\theta(0) = 1\ . \qquad (19)$$

Utilizing the recurrence relation in Eq.(18) and the transformed boundary conditions in Eq. (19), $\theta(k)$ were calculated for k = 1,2,3,4 as in Table (4)

Table (4) $\boldsymbol{\theta(k)}$ for different $\boldsymbol{k}$

K	recurrence equation $\theta(k+1)$	$\theta(k+1)$
0	$[1+\varepsilon]\,\theta(1) + 1 = 0$	$\frac{-1}{[1+\varepsilon]}$
1	$2\,\theta(2) + \varepsilon \sum_{l=0}^{1} \theta(l)\,(2-l)\theta(2-l) + \theta(1) = 0$	$\frac{1}{2!\ [1+\varepsilon]^3}$
2	$3\,\theta(3) + \varepsilon \sum_{l=0}^{2} \theta(l)\,(3-l)\theta(3-l) + \theta(2) = 0$	$\frac{2\varepsilon - 1}{3!\,[1+\varepsilon]^5}$
3	$4\,\theta(4) + \varepsilon \sum_{l=0}^{3} \theta(l)\,(4-l)\theta(4-l) + \theta(3) = 0$	$\frac{6\varepsilon^2 - 8\varepsilon + 1}{4!\,[1+\varepsilon]^7}$
4	$5\,\theta(5) + \varepsilon \sum_{l=0}^{4} (5-l)\theta(l)\,\theta(5-l) + \theta(4) = 0$	$\frac{24\varepsilon^3 - 58\varepsilon^2 + 22\varepsilon - 1}{5!\,[1+\varepsilon]^9}$

Therefore, the solution of Equation (15) as

$$\theta(\tau) = 1 - \frac{1}{[1+\varepsilon]}\tau + \frac{1}{2!}\frac{1}{[1+\varepsilon]^3}\tau^2 - \frac{1-2\varepsilon}{3!\,[1+\varepsilon]^5}\tau^3 + \frac{6\varepsilon^2-8\varepsilon+1}{4!\ [1+\varepsilon]^7}\ \tau^4 - \frac{1}{5!}\left[\frac{-24\varepsilon^3+58\varepsilon^2-22\varepsilon+1}{[1+\varepsilon]^9}\right]\tau^5 + . \qquad (20)$$

The solution in Eq.(20) is compatible with that obtained by Ganji *et al* [10], by applying the HPM and VIM. Figure (3) illustrates the variation of the obtained solution of Equation (20) $\theta(\tau)$ over τ by DTM for ($\varepsilon = 0.1\ , 0.2,\ 0.3,\ 0.4$) .

4. Conclusion

The Differential Transformation Method (DTM) is successful in solving two nonlinear differential equations arising in heat transfer problems. Results were compatible with the Homotopy Perturbation method HPM and variational iteration method VIM Ganji et al [10]. For higher orders of approximation with a greater degree of accuracy more computations must be needed. Thus DTM is an important tool in handling highly nonlinear differential equations with a minimum size of computations and a wide interval of convergence.

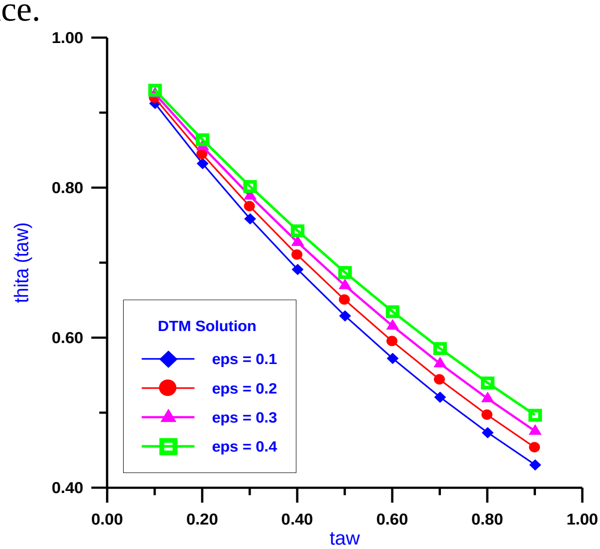

Figure (3) represents the solution between $\theta(\tau)$ and τ for by DTM for ($\varepsilon = 0.1\ , 0.2,\ 0.3,\ 0.4$) .

This emphasizes the fact that this method is applicable to many other systems of nonlinear equations and it is reliable and promising when compared with existing methods.

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استخدام طريقة التحويل التفاضلي في حل معادلات انتقال الحرارة

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المستخلص:

في هذا البحث ، تم تطبيق طريقة التحويل التفاضلي لحل المعادلات التفاضلية غير الخطية والتي تظهر الحاجة اليها في مجال انتقال الحرارة. يمثل الحل متسلسلة متقاربة للحل التام. تم حل معادلة التبريد بالحمل الاشعاعي غير الخطية بالاضافة الى معادلة انتقال الحرارة بالتوصيل ذات المتغيرات الفيزياوية . ان نتائج الحل بطريقة التحويل التفاضلي كانت متطابقة مع نتائج الطرائق الاخرى ومنها طريقة الاضطراب المتماثل.