Journal of Kufa for Mathematics and Computer Vol.2, No.1, may 2014, pp 6-13

Solving Some Kinds of Non-Linear Homogeneous Ordinary Differential Equations of Second Order

By

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Abstract

In this paper ,we will find the solution of some kinds of homogeneous non-linear ordinary differential equations of second order which are defined as the following "The equation that each degree of any term in it is equal in the degree with the other terms in its dependent variable and derivatives ".Specific assumptions is used to find the general solution of these kinds of equations, this assumption transforms the non-linear differential equations of second order to the simple formula of differential equations of first order .

Introduction

Non linear equations are usually more difficult to solve than linear once and the number of classes of these equations are very large ,so we will choose some kinds of homogenous non-linear O.D.Es. of second order which are homogenous in y and its derivatives .

The researcher Kuder [5] .2006 studied the linear O.D.Es. of second order which have the form $y'' + p(x)y' + q(x)y = 0$ and used the assumption $y = e^{\int z(x)dx}$ to find the general solution .

The researcher Abd Al-Sada [1] .2006 studied the linear P.D.Es. of second order with constant coefficients and which have the form

$$AZ_{xx} + BZ_{xy} + CZ_{yy} + DZ_x + EZ_y + FZ = 0 ,$$

where A,B,C,D,E and F are arbitrary constants ,and used the assumption $y = e^{\int u(x)dx+\int v(y)dy}$ to find the complete solution of it .

Finally,the researcher Hani [4] ,2008 , studied the linear P.D.Es. of second order, which have three independent variables, and which have the form

$$AZ_{xx} + BZ_{xy} + CZ_{xt} + DZ_{yy} + EZ_{yt} + FZ_{tt} + GZ_x + HZ_y + IZ_t + JZ = 0 ,$$

where A,B,C,…,J are arbitrary constants, and used the assumption

$$y = e^{\int u(x)dx+\int v(y)dy+\int w(t)dt}$$

to find the complete solution of it .

In this paper ,we will find the solution of some kind of second order nonlinear O.D.Es. which are homogenous of degree two and the general form is

$$f_1(x,y,y',y'')y'' + f_2(x,y,y',y'')y' + f_3(x,y,y',y'')y = 0,$$

where f_1, f_2 *and* f_3 are linear of dependent variable y and derivatives .The assumption $y = e^{\int z(x)\,dx}$ help us to find the general solution of this kind of equations .

Definitions

Def. 1:Bernouli Equation [7]

The form

$$y' = f(x)\,y + g(x)\,yk \qquad \dots (1)$$

is the general form of Bernouli Equation . There are several special cases

a – f(x) and g(x) are constants .The equation is separable .

b – k = 0 .The equation is linear .

c – k = 1 . The equation is separable .

d – k is neither zero nor unity (it is not necessary an integer) .The solution is given by

$$y^{(1-k)} = y_1 - y_2 \qquad \dots (2)$$

s.t. $y_2 = ce^{\Phi}$; $y_2 = (1-k)e^{\Phi}\int e^{-\Phi}g(x)dx$;

$\Phi(x) = (1 - k)\int f(x)dx$

This result can be derived by three different ways

i - let $y^{(k-1)} = u$,therefore

$u'(x) = (1-k)[\,g(x) + f(x)u\,]$this is linear equation and the solution is (2) .

ii – An integrating factor of (1) is

$I(x,y) = y^{-k}e^{-\Phi}$.

The solution follows with two quadratures.

iii –Let$y = ue^{F}$, $F(x) = \int f(x)dx$ and get

$$u'(x) = u^{k}\,g(x)\,e^{(k-1)F}$$

The equation is separable .Restore y after it has been solved and the result is equivalent to (2) .

Def. 2: Riccati Equation [8]

The general form of Riccati equation is written as

$$y' + h(x)y^2 + g(x)y + f(x) = 0 \dots (3)$$

where h(x) , g(x) and f(x) are given functions of x (or constants) .

Def. 3: The Non-Linear Equation of Second Order [2]

The general equation is

$$f(x, y, y', y'') = 0$$

but terms like $y\,y'$, y^2y'' , $x\,y^3$,etc ,will occur when it is given in explicit form .

Def. 4 :Non-Linear Homogeneous O. D. Es. Of Second Order

The equation that each degree of any terms in it is equal in the degree with the other terms of it in its dependent variable and derivatives is called homogeneous non-linear O. D.E. with degree n if any terms of it is of degree n .

The Solutions of Riccati Equation [6]

To find the general solution of the equation

$$Z'(x) + Z^2(x) + P(x)Z(x) + Q(x) = 0 \qquad \dots (4)$$

Which is similar to Riccati equation ,we go back to the forms of the function P(x) and Q(x) ,by the following cases :

1 – If P(x) and Q(x) are constants ,say P(x) = a and Q(x) = b then the equation (4)

becomes

$$Z'(x) + Z^2(x) + aZ(x) + b = 0$$

and the general solution of the last equation is given by

i- $Z(x) = d\tan\big(dc - dx\big) - \frac{a}{2}$;

$d^2 = b - \frac{a^2}{4}$, if $b \neq \frac{a^2}{4}$,

where c is an arbitrary constants.

ii - $z(x) = \frac{1}{x+c} - \frac{a}{2}$, if $b = \frac{a^2}{4}$,

where c is an arbitrary constants

2 – If P(x) and Q(x) are not constants , then there are three cases

i – If Z_1 is a known solution to it , then the general solution of equation (4) is given by

$$Z(x) = \frac{e^{-\int (P+2Z_1)dx}}{\int e^{-\int (P+2Z_1)dx} dx} + Z_1 \qquad \dots(5)$$

ii – If Z_1 and Z_2 are two known solutions to it ,then the general solution of equation(4) is given by

$$Z(x) = \frac{Z_1 - c\, Z_2 e^{-\int (Z_1 - Z_2)dx}}{1 - c e^{\int (Z_1 - Z_2)dx}}$$

where c is an arbitrary constants .

iii – If Z_1 ,Z_2 and Z_3 are three known solutions to it ,then the general solution of equation(4) is given by

$$Z(x) = \frac{Z_1 - c\, J(x) Z_2}{1 - cJ(x)}; J(x) = \frac{Z_3 - Z_1}{Z_3 - Z_2}$$

where c is an arbitrary constants .

To see proofs of the above formulas for the general solution of equation (4) ,see[4]

Special Kinds of Non-Linear Homogenous O.D.Es. of Second Order [3].

There are two kinds

1 – Variable is Missing .

i - The independent variable is missing. The general equation is given by

$$f\,(y, y', y'') = 0,$$

ii - The dependent variable is missing . The general equation is given by

$$f\,(x, y', y'') = 0,$$

2 - The general case .No variable is missing so that the given equation has the form

$$f\,(x, y, y', y'') = 0,$$

and there are two cases .

i -The exact non-linear equation [9] . Suppose that the given equation is

$$\Phi\,(x, y, y', y'') = 0, \qquad \dots(6)$$

using the abbreviations $p = \frac{dy}{dx}$ and $p = \frac{d^2y}{dx^2}$,the necessary and sufficient conditions for exactness are

$$\frac{\partial\Phi}{\partial y} - \frac{d}{dx}\left(\frac{\partial\Phi}{\partial p}\right) + \frac{d^2}{dx^2}\left(\frac{\partial\Phi}{\partial p}\right) = 0,$$

If y″ in equation(6)appears to a degree higher than one ,then the equation can not be exact.

ii - Non-linear homogenous equation in y and its derivatives [6] .In this case

$$F(x, ty, ty', ty'') = t\,k\,F(x, y, y', y'') \qquad \ldots(7)$$

where k is a constant .

Note 1:

We will chose the non-linear O.D.Es of second order which are homogenous of degree two only ,and the general form it is given by

$$f_1(x, y, y', y'')y'' + f_2(x, y, y', y'')y' + f_3(x, y, y', y'')y = 0, \quad \ldots(8)$$

where f_1, f_2 *and* f_3 are linear of dependent variable y and derivatives .

As the number of classes of non-linear homogenous O.D.Es. of second order

are very large, so we will choose some kinds to find the general solution of it , while the other kinds are solved by the same methods .

Solving Special kinds of Non-Linear Homogeneous O.D.Es. of Second Order .

To solve the equation (8) so we will choose some kinds of the non-linear O.D.Es of second order.

1) $A_1 y'' y + A_2 (y')^2 + A_3 y^2 = 0$

2) $A_1 x y'' y + A_2 x (y')^2 + A_3 y y' = 0$

3) $A_1 X^4 (y'')^2 + A_2 X^2 y'' y + A_3 y^2 = 0$

In order to find the general solution of the above equations ,we search a function Z(x) such that the assumption

$$y = e^{\int z(x)dx}, \qquad \ldots(9)$$

represents the general solution of the required equations ,through finding y′ and y″ from the equation (9) ,we get

$$y' = Z(x)e^{\int z(x)dx}$$

$$y'' = (Z'(x) + Z^2(x))e^{\int z(x)dx}$$

by substituting y, y′ and y″ in the above equations , we get :

1) $[A_1 (Z'(x) + Z^2(x)) + A_2 Z^2(x) + A_3]\, e^{2\int z(x)dx} = 0$
2) $[A_1 x(Z'(x) + Z^2(x)) + A_2 x Z^2(x) + A_3 Z(x)]\, e^{2\int z(x)dx} = 0$
3) $[A_1 x^4 (Z'(x) + Z^2(x))^2 + A_2 x^2 (Z'(x) + Z^2(x))^2 + A_3] e^{2\int z(x)dx} = 0$

since

$$y = e^{2\int z(x)dx} \quad , \text{so}$$

1) $[A_1 (Z'(x) + Z^2(x)) + A_2 Z^2(x) + A_3] = 0$
2) $[A_1 x(Z'(x) + Z^2(x)) + A_2 x Z^2(x) + A_3 Z(x)] = 0$
3) $[A_1 x^4 (Z'(x) + Z^2(x))^2 + A_2 x^2 (Z'(x) + Z^2(x))^2 + A_3] = 0$

The last equations are of the first order O.D.Es. and contain independent variable x and dependent variable Z .

The General Solution of Non-Linear Second Order of O.D.Es . of Homogenous Degree .

To find the general solution of non-linear second order of O.D.Es . of homogenous degree ,we follow the following

1) $A_1 y'' y + A_2 (y')^2 + A_3 y^2 = 0$ …(10)

where A_1 ,A_2 and A_3 are arbitrary constants and not identically zero .Then the general solution of equation (10) is given by

$$y(x) = A\cos(K_1 - K_2 x)^B ,$$

where A, B, k_1 and k_2 are arbitrary constants .

2) $A_1 xy''y + A_2 x(y')^2 + A_3 y y' = 0$ …(11)

Where A_1 ,A_2 and A_3 are arbitrary constants and not identically zero .

Then the general solution of equation (11) is given by

$$y(x) = k_1[\frac{B_2 X^{1-B_1}}{1-B_1} + k_2]^{\frac{1}{B_2}}$$

where k_1 and k_2 are arbitrary constants .

3) $A_1 x^4 (y'')^2 + A_2 x^2 y''y + A_3 y^2 = 0$ …(12)

where A_1 ,A_2 and A_3 are arbitrary constants and not identically zero .Then the general solution of equation (12) is given by

$$y(x) = Ae^{\int Z_1 dx} \int e^{-\int 2Z_1 dx} dx \,; A = e^c$$

where A is an arbitrary constant .

Solutions

1) Solution : since

$[A_1 (Z'(x) + Z^2(x)) + A_2 Z^2(x) + A_3] = 0$

This equation is variable separable equation ,we can solve it as follows

$$Z'(x) + B_1^2 + B_2^2 Z^2(x) = 0 :$$

$$B_1 = \sqrt{\frac{A_3}{A_1}} \quad ,B_2 = \sqrt{1 + \frac{A_2}{A_1}}$$

$$\frac{dz}{B_1^2 + B_2^2 Z(x)} = -\,dx$$

$$\tanh^{-1}\left(\frac{B_1}{B_2} Z(x)\right) = k_1 - k_2 x:$$

$$k_1 = c\,B_1 B_2 \ ,k_2 = B_1 B_2$$

$$z(x) = \frac{B_1}{B_2}\tanh(k_1 - k_2 x)$$

Then ,the general solution of equation (10) is given by

$$y(x) = e^{\int \frac{B_1}{B_3} \tanh(k_1 - k_3 x)\, dx}$$

$$= e^{\frac{B_1}{k_2 B_2} \ln\cos(k_1 - k_2 X) + c} \quad ;$$

$$; \quad \cos(k_1 - k_2 x) > 0$$

$= A \cos(k_1 - k_2 x)^B$ …(13)

where $k_1 = e^c$, $B = \frac{B_1}{k_2 B_2}$, k_1 and k_2 are arbitrary constants .

2) Solution : since

$$A_1 x(Z'(x) + Z^2(x)) + A_2 x Z^2(x) + A_3 Z(x) = 0$$

This equation is similar to Bernouli equation (1) and we can solve it as follows

$$z'(x) \frac{B_1}{x} z(x) + B_2 z^2(x) = 0 \ ;$$

$$B_1 = \frac{A_3}{A_1} \quad , B_2 = \frac{A_1 + A_2}{A_1}$$

Then by equation (2) we get

$$z(x) = \frac{e^{-\int \frac{B_1}{x} dx}}{\int B_2 e^{-\int \frac{B_1}{x} dx} dx}$$

So the general solution of equation (11) is given by

$$y(x) = e^{\int [\frac{e^{-\int \frac{B_1}{x} dx}}{\int B_2 e^{-\int \frac{B_1}{x} dx} dx}] \, dx}$$

$$= k_1 \left[e^{\frac{1}{B_2} \ln\left(\frac{B_2 X^{-B_1 + 1}}{-B_1 + 1} + k_2\right)} \right] ;$$

$k_1 = e^c$

$$= k_1 [\frac{B_2 X^{-B_1 + 1}}{-B_1 + 1} + k_2]^{\frac{1}{B_2}}$$

…(14)

where $B_1 = \frac{A_3}{A_1}$, $B_2 = \frac{A_2 + A_1}{A_1}$, $k_1 = e^c$ and k_2 are arbitrary constants .

3) Solution : since

$$A_1 x^4 (Z'(x) + Z^2(x))^2 + A_2 x^2 (Z'(x) + Z^2 (x)) + A_3 = 0$$

Therefore

$$x^2 (Z(x) + Z^2(x)) = \frac{-A_2 \pm \sqrt{A_2^2 - 4A_1 A_3}}{2A_1}$$

$$z'(x) + z^2(x) - \frac{k}{X^2} = 0 \ ;$$

$$k = \frac{-A_2 \mp \sqrt{A_2^2 - 4A_1 A_3}}{2A_1}$$

This equation is similar to Ricati equation and it has a particular solution Z_1 ,then the general solution is given by equation (5) , therefore

$$z(x) = \frac{e^{-\int (2Z_1) dx}}{\int e^{-\int (2Z_1) dx} dx} + Z_1$$

so the general solution of equation (12) is given by

$$y(x) = e^{\int [\frac{e^{-\int (2Z_1) dx}}{\int e^{-\int (2Z_1) dx} dx} + Z_1] \, dx}$$

$$= e^{\ln(\int e^{-\int 2Z_1 dx} dx) + \int Z_1 dx + c}$$

$$= Ae^{\int Z_1 dx} \int e^{-\int 2Z_1 dx} dx \,; A = e^c$$

…(15)

where A is an arbitrary constant .

Examples

Example (1) : To solve the non-linear homogenous O.D.

$$y''y + 3(y')^2 + 4y^2 = 0$$

which the independent variable is missing, we will use the formula as given in (13), then the general solution of it is

$$y(x) = A\cos(4B - 4x)^{\frac{1}{4}}$$

where A and B are arbitrary constants .

Example (2) : To solve the equation

$$xy''y + 2x(y')^2 - 2yy' = 0$$

which is exact non-linear equation since the sufficient and necessary conditions are holds ,we will use the formula as given in (14) ,then the general solution of it is

$$y(x) = k_1(x^3 + k_2)^{\frac{1}{3}}$$

where k_1 and k_2 are arbitrary constants .

Example (3) : To solve the non-linear homogenous equation

$$x^4 \, y''^2 - 4x^2 y''y + 4y^2 = 0$$

we can see the Ricati equation

$$z'(x) + z^2(x) - \frac{k}{X^2} = 0 \;;$$

which is product through solving the original equation and it has a particular solution $Z_1 = \frac{2}{X}$.

We will use the formula as given in (15) , so the general solution of it is given by

$$y(x) = Ae^{\int \frac{2}{X} dx} \int e^{-\int \frac{4}{X} dx} dx$$

$$= \frac{A}{X}\left(Bx^3 - \frac{1}{3}\right)$$

where A and B are arbitrary constants .

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الخلاصة

في هذا البحث سنجد الحل لبعض ألأنواع من المعادلات التفاضلية الاعتيادية اللاخطية ذات الدرجة المتجانسة من الرنبة الثانية والتي نستطيع ان نعرفها كالاتي "المعادلة التفاضلية الاعتيادية التي درجة كل حد فيها تكون مساوية لدرجة حدودها الاخرى في منغيرها العتمد واشتقاقاته " . نعويض محدد يستخدم لأيجاد الحل العام لهذه الانواع من المعادلات وهذا التعويض يحول المعادلات التفاضلية اللاخطية ذات الرتبة الثانية الى معادلات تفاضلية من الرتبة الاولى .