Dominating Sets and Domination Polynomial of Complete Graphs with Missing Edges

Abdul Jalil M. Khalaf and Sahib Shayyal Kahat *
Department of Mathematics
Faculty of Mathematics and Computer Science
University of Kufa, Najaf, IRAQ.

Abstract— Let $G = (V, E)$ be a simple graph. A set $D \subseteq V$ is a dominating set of G, if every vertex in $V - D$ is adjacent to at least one vertex in D. Let K_n be complete graph with order *n*. Let K_n^i be the family of dominating sets of a complete K_n with cardinality i, and let $d(K_n, i) = |k_n^i|$. In this paper, we construct K_n, and obtain a recursive formula for $d(K_n, i)$. Using this recursive formula, we consider the polynomial $D(K_n, x) = \sum_{i=1}^{n} d(K_n, i)x^i$, which we call domination polynomial of complete graphs and obtain some properties of this polynomial.

I. INTRODUCTION

Let $G = (V, E)$ be a simple graph of order $|V| =$ n. A set $D \subseteq V$ is a dominating set of G, if every vertex in $V - D$ is adjacent to at least one vertex in D. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in G. For a detailed treatment of this parameter, the reader is referred to [6]. It is well known and generally accepted that the problem of determining the dominating sets of an arbitrary graph is a difficult one (see [5]). Alikhani and Peng found the dominating set and domination polynomial of cycles and certain graph [1],[2]. Gehet, Khalaf and Hasni found the dominating set and domination polynomial of stars and wheels [3][4]. Let G_n be graph with order n and let G_n^i be the family of dominating sets of a graph G_n with cardinality i and let $d(G_n, i) = |G_n^i|$. We call the polynomial $D(G_n, x) = \sum_{i=\gamma(G)}^{n} d(K_n, i)x^i$, the domination polynomial of graph G [2]. Let K_n^i be the family of dominating sets of a complete graph K_n with cardinality i and let $d(K_n, i) = |k_n^i|$. We call the polynomial $D(K_n, x) = \sum_{i=1}^{n} d(K_n, i)x^i$, the domination polynomial of complete graph. In the next section we construct the families of dominating sets of K_n with cardinality i by the families of dominating sets of K_{n-1} with cardinality i and $i - 1$. We investigate the domination polynomial of complete graphs in section 3. And we study dominating sets and domination polynomial of some cases of complete graphs with missing edges in section 4.

As usual we use $\binom{n}{i}$ for the combination n to i, and we denote the set {1, 2,…, n} simply by [n], and we denote $\rho(v)$ to degree of the vertex v, and let $\Delta(G) = \max\{\rho(v) | \forall v \in V(G)\}$ and $\delta(G) = \min\{\rho(v) | \forall v \in V(G)\}$

.

II. DOMINATING SETS OF COMPLETE GRAPHS

Let K_n, $n \geq 3$, be the complete graph with n vertices $V(K_n) = [n]$ and $E(K_n) = \{(v, u): \forall v, u \in V(K_n)\}$. Let K_n^i be the family of dominating sets of K_n with cardinality i. We shall investigate dominating sets of complete graph. To prove our main results we need the following lemmas:

Lemma 1 [3] The following properties hold 8 graph G.

$(i)|G_n^n| = 1$ $(ii)|G_n^{n-1}| = n$ $(iii)|G_n^i| = 0$ if $i > n$

$(iv)|G_n^0| = 0$

Lemma 2 [3] The following properties are hold by definition of combination$\left(\binom{n}{i} = \frac{n!}{i!(n-1)!}\right) for\ all$ $n \in Z^+$.

$(i)\ \binom{n}{n} = 1$ $(ii)\ \binom{n}{n-1} = n$ $(iii)\ \binom{n}{1} = n$ $(iv)\ \binom{n}{0} = 1$ $(v)\ \binom{n}{i} = 0$ if $i > n$

Theorem 1 Let K_n be complete graph with order n, then $d(K_n, i) = \binom{n}{i}\ \forall\ n \in Z^+$, and $i = 1,2,\dots,n$

Proof.

Let K_n be a complete graph, since every vertex $v \in K_n$ it is adjacent with every other vertex $u \in K_n$ then every subset of K_n with cardinality $i\ \forall\ 1 \le i \le n$ is dominating sets of K_n, therefore $d(K_n, i) = \binom{n}{i}$ ■

Theorem 2 Let K_n be complete graph with order n, then

$d(K_n, i) = d(K_{n-1}, i) + d(K_{n-1}, i-1)\quad \forall\ i > 1,\ n > 1.$

Proof.

(i) We have $\binom{n}{i} = \frac{n(n-1)\dots(n-i+1)(n-i)!}{i!(n-i)!} = \frac{n(n-1)\dots(n-i+1)}{i!} = \frac{(n-1)(n-2)\dots(n-i+1)}{i!} \cdot n,$

$\binom{n-1}{i} = \frac{(n-1)(n-2)\dots(n-i)!}{i!} = \frac{(n-1)(n-2)\dots(n-i+1)!}{i!} \cdot (n-1),$

$\binom{n-1}{i-1} = \frac{(n-1)(n-2)\dots(n-i)!}{(i-1)!} = \frac{(n-1)(n-2)\dots(n-i+1)!}{i!} \cdot i$

Let $m = \frac{(n-1)(n-2)\dots(n-i+1)!}{i!}$, then $d(K_n, i) = \binom{n}{i} = mn,$ and $d(K_{n-1}, i) + d(K_{n-1}, i-1) = m(n-i) + mi = mn - mi + mi = mn = d(K_n, i)$ ■

Using Theorem 1 and Theorem 2, we obtain the coefficients of $D(K_n, x)$ for $1 \le n \le 15$ in Table 1. Let $d(K_n, i) = |k_n^i|$. There are interesting relationships between the numbers $d(K_n, i)$ $(1 \le i \le n)$ in the table.

j / n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1														
2	2	1													
3	3	3	1												
4	4	6	4	1											
5	5	10	10	5	1										
6	6	15	20	15	6	1									
7	7	21	35	35	21	7	1								
8	8	28	56	70	56	28	8	1							
9	9	36	84	126	126	84	36	9	1						
10	10	45	120	210	252	210	120	45	10	1					
11	11	55	165	330	462	462	330	165	55	11	1				
12	12	66	220	495	792	924	792	495	220	66	12	1			
13	13	78	286	715	1287	1716	1716	1287	715	286	78	13	1		
14	14	91	364	1001	2002	3003	3532	3003	2002	1001	364	91	14	1	
15	15	105	455	1365	3003	5005	6535	6535	5005	3003	1365	455	105	15	1

Table 1. $d(K_n, i)$The number of dominating sets of K_n with cardinality i

In the following theorem, we obtain some properties of $d(K_n, i)$

Theorem 3 The following properties hold for coefficients of $D(K_n, x)$, $\forall\ n \in Z^+$:

$(i)\ d(K_n, 1) = n.$

$(ii)\ d(K_n, i) = d(K_n, n-i).$

(iii) If n is even number, then d(K_n, $\frac{n+1}{2}$) = d(K_n, $\frac{n-1}{2}$)

$(iv)\ \gamma(K_n) = 1$

Proof.

Let K_n be a complete graph, then

(i)By Theorem 1 $d(K_n, 1) = \binom{n}{1} = n$, then $d(K_n, 1) = n.$

(ii) We have $\binom{n}{1} = \binom{n}{n-1}$, then $d(K_n, i) = d(K_n, n-i)$ (by Lemma 2 and Theorem 1)

(iii) It is hold from (ii)

(iv) since $\{v\}\ \forall v \in V(K_n)$ is dominating set of (K_n), then $\gamma(K_n) = 1.$ ■

III. DOMINATION POLYNOMIAL OF A COMPLETE GRAPHS

In this section we introduce and investigate the domination polynomial of complete graphs.

Definition

Let k_n^i be the family of dominating sets of a complete K_n with cardinality i, and let $d(K_n, i) = |k_n^i|$, and since $\gamma(K_n) = 1$. Then the domination polynomial $D(K_n, x)$ of K_n is defined as

$$D(K_n, x) = \sum_{i=1}^{n} d(K_n, i)x^i$$

Theorem 4 The following properties hold for all $D(K_n, x)\quad \forall\ n \ge 3$

$(i)\ D(K_n, x) = D(K_{n-1}, x) + xD(K_{n-1}, x) + x$

$(ii)\ D(K_n, x) = \sum_{i=1}^{n} \binom{n}{i} x^i$

Proof.

(i) from definition of the domination polynomial and Theorem 2, we have

$D(K_n, x) = \sum_{i=1}^{n} d(K_n, i)x^i = \sum_{i=1}^{n} d(K_{n-1}, i)x^i$

$+ \sum_{i=1}^{n} d(K_{n-1}, i-1)x^i$, we have $d(K_n, i) = 0$ if $i > n$ or $i = 0$ (Lemma1), then $\sum_{i=1}^{n} d(K_{n-1}, i)x^i = \sum_{i=1}^{n-1} d(K_{n-1}, i)x^i = D(K_{n-1}, x)$ and (since $d(K_{n-1}, i-1)x^{i-1} = \binom{n-1}{i-1} = \binom{n-1}{0} = 1$ if $i = 1$ and $\sum_{i=2}^{n} d(K_{n-1}, i-1)x^{i-1} = \sum_{i=1}^{n-1} d(K_{n-1}, i)x^i$), then

$\sum_{i=1}^{n} d(K_{n-1}, i-1)x^i = x\sum_{i=1}^{n} d(K_{n-1}, i-1)x^{i-1}$
$= x[\sum_{i=1}^{n-1} d(K_{n-1}, i)x^i - 1] = xD(K_{n-1}, x) + x$

therefore $D(K_n, x) = D(K_{n-1}, x) + xD(K_{n-1}, x) + x$

(ii) $D(K_n, x) = \sum_{i=1}^{n} d(K_n, i)x^i = \sum_{i=1}^{n}\binom{n}{i}x^i$ (by Theorem 1) •

Example 1

Let K_6 be complete graph with order 6,
then , $\gamma(K_6) = 1$ *and* $D(K_6, x) = \sum_{i=1}^{6}\binom{6}{i}x^i =$
$6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$. *(see Fig-l)*

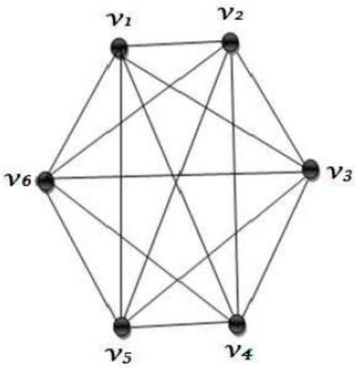

Fig-1: G=K6 has $\binom{6}{i}$ dominating set with cardinality i

IV. DOMINATING SETS AND DOMINATION POLYNOMIAL OF COMPLETE GRAPH WITH MISSING EDGES

Let K_n be complete graph with order n, and let ε be number of missing edges from K_n, we denoted by $K_n(-\varepsilon)$ to complete graph with number of missing edges.

We mine trail if $el, e2, e3, \dots, ek$ missing edges, then el adjacent $e2, e2$ adjacent $e3, \dots,$ *and* $ek - l$ adjacent ek
To prove our main results in following theorems we need the following proposition

Proposition 1

Let $G_n = K_n(-\varepsilon)$ the following properties hold for all graph G_n. The following properties are clear see figure (2,3,4,5).

(i) $\delta(G) = n - 2$ *if all missing edges are independent*
(ii) $\delta(G) = n - 3$ *if all missing edges are trail*
(iii) $\delta(G) = n - 1 - \varepsilon$ *if all missing edges are adjacent.*
(iv) $n - 1 - \delta(G) \le \varepsilon$
(v) *Number of vertices with degree* $< n-1$ *is* $\varepsilon + 1$ *if all missing edges are adjacent*
(vi) *Number of vertices with degree* $< n-1$ *is* $\varepsilon + 1$ *if all missing edges are trail*
(vii) *Number of vertices with degree* $< n-1$ *is* 2ε *if all missing edges are independent* $(viii)$ *Number of vertices with degree* $n-2$ *is* 2ε *if all missing edges are independent* (ix) *Number of vertices with degree* $n-3$ *is* $\varepsilon - 1$*and number of vertices with degree* $n-2$ *is* 2 *if all missing edges are trail*
(x) *Number of vertices with degree* $n-2$ *is* ε *and number of vertices with degree* $n-1-\varepsilon$ *is* 1 *if all missing edges are adjacent.*

Theorem 5 *Let* $G_n = K_n(-\varepsilon)$ *, then the following properties hold for G:*

(i) $d(G_n, i) = \binom{n}{i}$ $\forall$ $i > n - 1 - \delta(G)$
(ii) $d(G_n, i) = \binom{n}{i} - \binom{\varepsilon}{i}$ $\forall$ $i \le \varepsilon$ *if all missing edges are adjacent*
(iii) $d(G_n, i) = \binom{n}{i}$ $\forall$ $i > \varepsilon$
(iv) $d(G_n, i) = \binom{n}{i}$ $\forall$ $i > 1$ *if all missing edges are independent*
(v) $d(G_n, 1) = n - 2\varepsilon$ *if all missing edges are independent*
(vi) $d(G_n, 1) = n - (\varepsilon + 1)$ *if all missing edges are adjacent*
(vii) $d(G_n, 1) = n - (\varepsilon + 1)$ *if all missing edges are trail*
$(viii)$ $d(G_n, 2) = \binom{n}{2} - (\varepsilon - 1)$ *if all missing edges are trail*
(ix) $d(G_n, i) = \binom{n}{i}$ $\forall$ $i > 2$ *if all missing edges are trail*
(x) $d(G_n, 1) = n - \varepsilon$ *if all missing edges are trail such that*

formation C_ε

(xi) $d(G_n,2)=\binom{n}{2}-\varepsilon$ *if all missing edges are trail such that formation* C_ε

Proof.

By the properties in Proposition 1, we prove the following proposition:

(i) Let v be vertex with degree $\delta(G)$, then it has $n-1-\delta(G)$ of missing edges, then we have $n-1-\delta(G)$ vertices are nonadjacent with v, hence every subset of G_n with cardinality $i > n-1-\delta(G)$ is dominating set of G_n, then $d(G_n,i)=\binom{n}{i}\ \forall\ i > n\text{-}1\text{-}\ \delta(G)$ (see Fig 2,3,4,5)

(ii) since all missing edges are adjacent, then we have ε vertices are nonadjacent with one vertex, then we have only $\binom{\varepsilon}{i}\ \forall\ i\le\varepsilon$ are not dominating set of G_n, therefore $d(G_n,i)=\binom{n}{i}-\binom{\varepsilon}{i}\ \forall\ i\le\varepsilon$ (see Fig-2).

(iii) We have $\varepsilon\ge n-1-\delta(G)$ and since $i>\varepsilon$ then $i\ge n-1-\delta(G)$ and $d(G_n,i)=\binom{n}{i}\ \forall\ i>\varepsilon$ by (i) (see Fig-2,3,4,5)

(iv) since all missing edges are independent, then $\delta(G)=n-2$, therefore $d(G_n,i)=\binom{n}{i}\ \ \forall\,\text{i} >1$ by (i) (see Fig-3)

(v) since all missing edges are independent, and we have ε number of missing edges, then we have 2ε vertices with degree *n* - 2, and since $\{v\}\forall\ v$ vertices with degree *n* -2 is not dominating set of G_n, therefore $d(G_n,1)=n-2\varepsilon$. (see Fig-3)

(vi) since all missing edges are trail, and we have ε of missing edges, then we have $\varepsilon+1$ vertices with degree < *n* - 1, and since $\{v\}\forall\ v$ vertices with degree < *n* - 1 is not dominating set of G_n, therefore $d(G_n,1)=n-(\varepsilon+1)$ (see Fig-4)

(vii) since all missing edges are adjacent, and we have ε of missing edges, then we have ε+1 vertices with degree < *n* - 1, and since $\{v\}\forall\ v$ vertices with degree < *n* - 1 is not dominating set of of G_n, therefore $d(G_n,1)=n-(\varepsilon+1)$ (see Fig-2)

$(viii)$ $\forall v\ \in V(G)$ *if* $\rho(v)=n-3$, then there exist set with cardinality 2 is not dominating set of G, since all missing edges are trail, then we have $(\varepsilon$ - 1) vertices with degree *n* - 3, hence we have $(\varepsilon$ - 1) sets are not dominating set of G, therefore $d(G_n,2)=\binom{n}{2}-(\varepsilon-1)$ (see Fig-4)

(ix) since all missing edges are trail, then $\delta(G)=n-3$, therefore $d(G_n,i)=\binom{n}{i}\ \forall\ i>2$ by (i) (see Fig-4)

(x) Since all cycle *V(C)=E(C)*, then we have ε vertices with degree < *n* -1, therefore $d(G_n,1)=n-\varepsilon$ by (i) (see Fig-5)

(xi) $\rho(v)=n-3\ \forall v\in C$, then $d(G_n,2)=\binom{n}{2}-\varepsilon$ by $(viii)$ (see Fig-5) • ∎

Theorem 6 *Let* $G_n=K_n$ (- ε),*then the following properties hold for G:*

(i) *D(*K_n *(- ε), x) = (n -l- ε)x+* $\sum_{i=2}^{n}(\binom{n}{i}-\binom{\varepsilon}{i})x^i$ *if all missing edges are adjacent*

(ii) *D(*K_n *(- ε), x) = (n -2 ε)x+* $\sum_{i=2}^{n}\binom{n}{i}x^i$ *if all missing edges are independent*

(iii) *D(*K_n *(- ε), x) = (n -1-ε)x+(*$\binom{n}{2}$*-(ε-1))*$x^2+\sum_{i=3}^{n}\binom{n}{i}x^i$ *if all missing edges are trail*

(iv) *D(*K_n *(- ε), x) = (n -ε)x+(*$\binom{n}{2}$*-ε)*$x^2+\sum_{i=3}^{n}\binom{n}{i}x^i$ *if all missing edges are trail such that formation* C_ε.

Proof.

(i) Since all missing edges are adjacent by Theorem 5 in (ii) and (vi) $D(K_n(-\varepsilon),x)=$ $d(K_n(-\varepsilon),1)x+\sum_{i=2}^{n}d(K_n(-\varepsilon),i)x^i$ *=(n-l- ε)x* $+\sum_{i=2}^{n}(\binom{n}{i}-\binom{\varepsilon}{i})x^i$

(ii) Since all missing edges are independent by Theorem 5 in (iv) and (v) $D(K_n(-\varepsilon),\ x)$ = $d(K_n(-\varepsilon),\ 1)x+\sum_{i=2}^{n}d(K_n(-\varepsilon),i)x^i=$ *(n -2 ε)x* $+\sum_{i=2}^{n}\binom{n}{i}x^i$

(iii) Since all missing edges are trail by Theorem 5 in (vii), (viii) and (ix) $D(K_n(-\varepsilon),x)=d(K_n(-\varepsilon),\ 1)x+d(K_n(-\varepsilon),\ 2)x^2+\sum_{i=3}^{n}d(K_n(-\varepsilon),\ i)x^i=$

$(n-1-\varepsilon)x+(\binom{n}{2}-(\varepsilon-1))x^2+\sum_{i=3}^{n}\binom{n}{i}x^i$

(iv) Since all missing edges are trail such that formation C_ by theorem 5 in (ix),(x)

and (vi) $D(K_n(-\varepsilon),x) = d(K_n(-\varepsilon),\ 1)x + d(K_n(-\varepsilon),\ 2)x^2 + \sum_{i=3}^{n} d(K_n(-\varepsilon),\ i)x^i$

$=(n-\varepsilon)x+(\binom{n}{2}-\varepsilon)x^2+\sum_{i=3}^{n}\binom{n}{i}x^i$. ∎

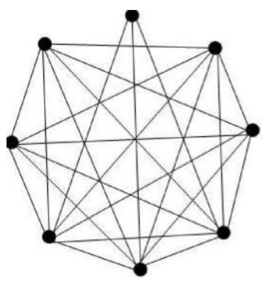

Fig-2: $G=K_8(-4)$**all missing edges are adjacent**

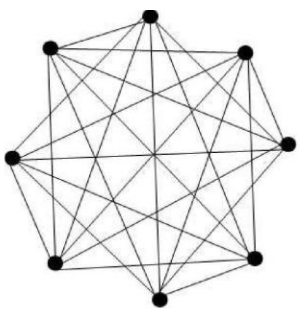

Fig-3: $G=K_8(-4)$**all missing edges are independent**

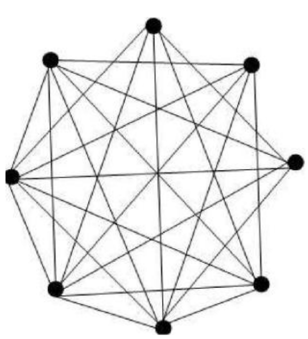

Fig-4: $G=K_8(-4)$**all missing edges are trail**

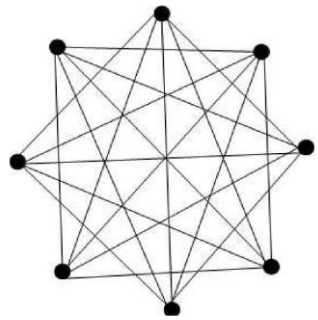

Fig-5: $G=K_8(-4)$ **all missing edges are trail formation** $\mathbf{C_8}$

Example 2

Let $G_n=K_8\ (-4)$, *be complete graph with four missing edges are adjacent, then Domination Polynomial of (G) is* $D(K_8\ (-4),x)=(3)x+\sum_{i=2}^{8}(\binom{8}{i}-\binom{4}{i})x^i= 3x + 22x^2 + 52x^3 + 69x^4 + 56x^5 + 28x^6 + 8x^7 + x^8$ *by Theorem* 6 *(see Fig-2)*

V. REFERENCES

[1] S. Alikhani, Y. H. Peng, Dominating Sets and Domination Polynomial of Cycles, Global Journal of Pure and Applied Mathematics,4(2008) 2, 151-162.

[2] S. Alikhani, Y. H. Peng, Dominating Sets and Domination Polynomial of Certain Graphs, II, Opuscula Mathematica 30, no. 1 (2010), 37-51.

[3] S. Sh. Gehet, A. M. Khalaf, R. Hasni, Dominating Sets and Domination Polynomial of Stars, submitted.

[4] S. Sh. Gehet, A. M. Khalaf, R. Hasni, Dominating Sets and Domination Polynomial of Wheels, submitted.

[5] M. R. Garey, D. S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completness. Freeman, New York, 1979.

[6] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, New York, 1998.