ISSN (1681- 6870)

Cascade System Reliability for Weibull - Fréchet Distribution

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Received 19/10/2020 **Accepted 6/12/2020**

Abstract: *In this paper, the properties of cascade reliability model considering Weibull strength and Fréchet stress distribution have been studied, concentrating on calculating the reliability function for systems of one, two and three components. For these systems, three estimation methods were suggested and the values of reliability function for each method were calculated. The results of simulation showed positive relationship between the parameters of reliability and Weibull parameters and negative relationship between reliability function and Fréchet parameters*

Keywords: Weibull distribution, Fréchet distribution, cascade models, Reliability function.

معولية النظام التتابعي لتوزيع ويبل- فرجت

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المستخلص
تم تقدير دالة المعولية لنظام Cascade لنماذج الاجهاد - المتانة لتوزيع ويبل فرجت ولنظام معولية مؤلف من مكون واحد ومكونين وثلاثة مكونات .ولإيجاد تقدير دالة المعولية تم استخدام ثلاثة طرائق تقدير واظهرت نتائج المحاكاة وجود علاقة طردية بين قيم معلمات توزيع ويبل و قيمة دالة المعولية كما اظهرت نتائج المحاكاة وجود علاقة عكسية بين قيمة معلمات توزيع فرجت وقيمة دالة المعولية.

الكلمات المفتاحية: توزيع ويبل، توزيع فرجت، النموذج التتابعي، دالة المعولية

1. Introduction

Reliability is defined as the measure of ability of a system or any part of the system to achieve the function for which it was designed to without failure. Mathematically the reliability is defined as follows:

If R(t) represents the reliability of a system or a component in the system at time (t) thenc[10]

$$R(t) = pr(T > t)\,;for\ t \geq 0$$

Where (T) is a continuous random variable representing cumulative time for system or component work time.

One of the reasons for failure of a system or a component occurs when the exerted pressure is greater than system strength. In this case the reliability is define mathematically as

$$R(t) = pr(x > y)$$

Where x is strength and y is stress.

A system is consisted of a group of associated components or items. The system differs according to type of connection of components.

ISSN (1681- 6870)

2. Weibull Distribution

Weibull one of the most important models of reliability and survival functions. Density and cumulative distribution functions for Weibull distribution are[c12]

$$F(x,\alpha,\beta) = 1 - e^{-\left(\frac{x}{\beta}\right)^{\alpha}} \ ,x,\alpha,\beta \geq 0 \tag{1}$$

$$f(x/\alpha\,;\beta) = \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^{\alpha}} \tag{2}$$

Where $(\boldsymbol{\alpha})$ is represent shape parameter; $(\boldsymbol{\beta})$ is represent scale parameter.

3. Fréchet Distribution

Fréchet distribution is considered as one of the distributions for extreme values. It is Suggested by Maurice Fréchet. It is known by extreme value type -II distribution and also known as inverse Weibull distribution. The cumulative distribution function of Fréchet distribution is[c13]

$$F(y;\lambda;\gamma) = exp\left(-\left(\frac{y}{\gamma}\right)^{-\lambda}\right) \ y,\lambda,\gamma > 0 \tag{3}$$

$$f(y;\lambda;\gamma) = \frac{\lambda}{\gamma}\left(\frac{y}{\gamma}\right)^{-\lambda-1} e^{-\left(\frac{y}{\gamma}\right)^{-\lambda}} \tag{4}$$

Where $(\boldsymbol{\lambda})$ is represent shape parameter; $(\boldsymbol{\gamma})$ is represent scale parameter.

4. Reliability

The reliability is defined as probability of the system or component to continue achieving what is required to do after time (t).

The reliability function is a positive Mentone decreasing function, falls in the range (0-1] System Reliability Theory Models and Statistical Methods

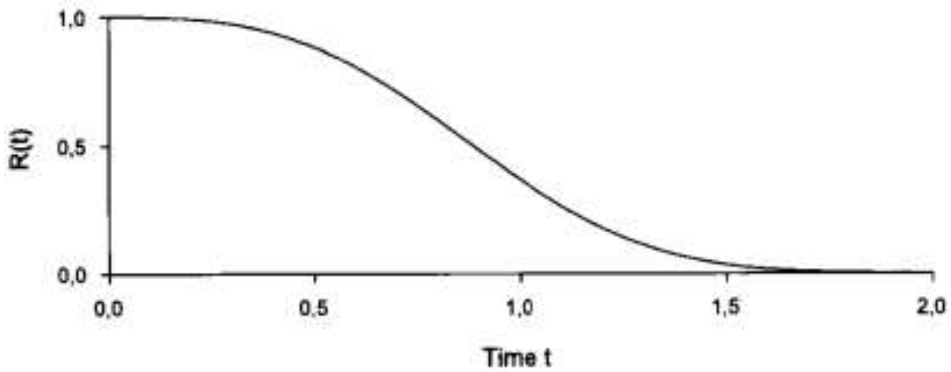

Figure (1): Curve of reliability function

A system is defined as a group of different parts, components or items connected with each other. There are different types of connections, parallel, series or k-out of n.

The reliability of the system depends on marginal reliability of all components. Sometimes the component $(i; i = 1,2,\dots,n)$ fails if it is exposed to stress or external force which is greater than its strength. That means the system keeps working as long as the exerted stress (temperature, high pressure…etc.) on this component is less than its strength. In this type of systems, reliability is expressed as the relationship between strength variable (x) and stress variable(y). the failure of the system, in this case happens when stress variable is greater than strength variable for all components in the systems.

5. Cascade model for stress-strength

Redundancy system is a type of parallel systems in which part of the system works while the other part doesn't, until the first part fails. The redundancy system is divided into two types: (Redundancy Low Level) and Redundancy (High Level)

Cascade models are spatial case of redundancy model. In cascade models, after each filer the exerted stress on components $(\boldsymbol{i}; \boldsymbol{i} = \mathbf{1}, \mathbf{2}, \dots, \boldsymbol{n})$ is rested by a factor called (attenuation factor). For this resin cascade models are regarded as a spatial case of redundancy system[c9].

$$Y_2 = KY_1\,, Y_3 = KY_2 = K^2Y_1 \dots Y_i = K^{i-1}Y_1$$

$$y_i = ky_{i-1} = k^*y_1$$

The system continuous working even though the failure (n-1) of the components.

$$x_i \le y_i\ \forall\ i = 1,2,3 \dots n-1\ ;\ x_n > y_n$$

The total reliability of the system is

$$R_n = \sum_{i=1}^{n} R_i\,; i = 1,2,\dots,n \tag{5}$$

Cascade systems are multi-component systems. For example, in (Cascade 2+1) system two components work the third is standby Estimating the Reliability Function of (2+1) Cascade Model. Whereas in cascade (1+1) system for each working component, there is another component which is in standby case. This is the genital situation.

6. Reliability of Cascade for Weibull -Fréchet

The joint probability density function for Weibull- Fréchet distribution will be as follows:

$$f(x,y,\alpha,\beta,\theta;\lambda) = \frac{\alpha}{\beta}\frac{\lambda}{\gamma}\left(\frac{x}{\beta}\right)^{\alpha-1}\left(\frac{y}{\theta}\right)^{-(\lambda+1)} e^{-\left(\frac{x}{\beta}\right)^{\alpha}-\left(\frac{y}{\gamma}\right)^{-\lambda}} \tag{6}$$

To find reliability of cascade system ($\boldsymbol{R_n}$) is found by the sum of marginal reliabilities of its components as follows:

- **Case of n=1**

Assuming ($\boldsymbol{x_i}$) represent strength variable for the (ith) components and (yi) represent stress variable for the same component, cascade system reliability for Weibull Fréchet models in a single component system is

$$R(1) = pr(x_1 > k_1^*y_1) = \int_0^\infty \int_{x_1 = k_1^* y_1}^\infty f(x_i)g(y_i)dx_1dy_1$$

$$R(1) = \int_0^\infty \bar{F}(k_1^*y_1)g(y_1)dy \tag{7}$$

$$R(1) = \int_0^\infty \left(e^{-\left(\frac{k_1^*y_1}{\beta}\right)^{\alpha}}\right)\frac{\lambda}{\gamma}\left(\frac{y_1}{\gamma}\right)^{-\lambda-1} e^{-\left(\frac{y_1}{\gamma}\right)^{-\lambda}} dy_1 \tag{8}$$

ISSN (1681- 6870)

$$R(1) = \int_0^\infty \frac{\lambda}{\gamma}\left(\frac{y_1}{\gamma}\right)^{-\lambda-1} e^{-\left(\frac{k_1^* y_1}{\beta}\right)^\alpha - \left(\frac{y_1}{\gamma}\right)^{-\lambda}} dy_1 \quad (9)$$

- **Case of n=2:**

Assuming $(\boldsymbol{x_1;x_2})$ represent strength variables for both components while (y1, y2) represent stress variables of them, cascade system reliability for Weibull Fréchet models in a double component system is

$$R(2) = pr(x_1 \le y_1; x_2 > y_2)$$

$$R(2) = pr(x_1 \le k_1^* y_1; x_2 > k_2^* y_1)$$

$$R(2) = \int_0^\infty \left(\int_0^{k_1^* y_1} f_1(x_1)d(x_1)\right)\left(\int_{k_2^* y_1}^\infty f_1(x_1)dx_2\right) g(y_1)dy_1$$

$$R(2) = \int_0^\infty F_1(k_1^* y_1)\bar{F}(k_2^* y_1)\, g(y_1)dy_1$$

$$R(2) = \frac{\lambda}{\gamma}\int_0^\infty \left(1 - e^{-\left(\frac{k_1^* y_1}{\beta}\right)^\alpha}\right) e^{-\left(\frac{k_2^* y_1}{\beta}\right)^\alpha} \left(\frac{y_1}{\gamma}\right)^{-\lambda-1} e^{-\left(\frac{y_1}{\gamma}\right)^{-\lambda}} dy_1$$

$$\boldsymbol{R(2)} = \frac{\boldsymbol{\lambda}}{\boldsymbol{\gamma}}\left(\int_0^\infty \left(\frac{\boldsymbol{y_1}}{\boldsymbol{\gamma}}\right)^{-\lambda-1}\left(\boldsymbol{e}^{\left(\frac{k_2^* y_1}{\beta}\right)^\alpha - \left(\frac{y_1}{\gamma}\right)^{-\lambda}} - \boldsymbol{e}^{-\left(\frac{k_1^* y_1}{\beta}\right)^\alpha - \left(\frac{k_2^* y_1}{\beta}\right)^\alpha - \left(\frac{y_1}{\gamma}\right)^{-\lambda}}\right) \boldsymbol{dy_1}\right) \quad (10)$$

- **Case n=3**

In the same way, assuming (x_i) represent strength variables of the three components and (yi) represent stress variables for the same components, cascade system reliability for Weibull Fréchet models in the system is

$$R(3) = pr(x_1 \le k_1^* y_1; x_2 \le k_2^* y_2; x_3 > k_3^* y_3)$$

$$R(3) = pr(x_1 \le k_1^* y_1; x_2 \le k_2^* y_1; x_3 > k_3^* y_1)$$

$$R(3) = \int_0^\infty \left(\int_0^{k_1^* y_1} f_1(x_1)d(x_1)\right)\left(\int_0^{k_2^* y_2} f_2(x_2)d(x_2)\right)\left(\int_{k_3^* y_1}^\infty f_3(x_3)d(x_3)\right) d(y_1)$$

$$R(3) = \int_0^\infty F_1(k_1^* y_1)\, F_2(k_2^* y_1)\bar{F}_3(k_3^* y_3) g(y_1)dy_1$$

$$R(3) = \frac{\lambda}{\gamma}\int_0^\infty \left(1 - e^{-\left(\frac{k_1^* y_1}{\beta}\right)^\alpha}\right)\left(1 - e^{-\left(\frac{k_2^* y_1}{\beta}\right)^\alpha}\right)\left(\frac{y}{\gamma}\right)^{-\lambda-1} e^{-\left(\frac{y}{\gamma}\right)^{-\lambda} - \left(\frac{k_3^* y_1}{\beta}\right)^\alpha} dy_1 \quad (11)$$

In general (n)

ISSN (1681- 6870)

$$R(n) = \int_{0}^{\infty} \left(\int_{y_1}^{\infty} f_1(x_1) d(x_1) \right) \left(\int_{ky_1}^{\infty} f_2(x_2) d(x_2) \right) \dots \left(\int_{0}^{k^{n-1}y_1} f_n(x_n) d(x_n) \right) d(y_1) \quad (12)$$

$$R(n) = \int_{0}^{\infty} F_1(y_1)\, F_2(ky_1) \dots F_{n-1}(ky_1) \bar{F}_n(k^2 y_n) g(y_1) dy_1$$

This study will concentrate on finding reliability of three component cascade system, so

$$R_3 = R(1) + R(2) + R(3)$$

7. Maximum Likelihood Estimation Methods

It is one of the most important and greatest used estimation methods. estimators in this method have many characteristics that a good estimator has. Its estimators have invariant characteristic. The joint probability density likelihood function for Weibull Fréchet destination is

$$L(x; y/\alpha\,; \beta; \lambda; \theta) = \left(\frac{\alpha}{\beta}\right)^n \left(\frac{\lambda}{\gamma}\right)^n \prod_{i=1}^{n} \left(\frac{x_i}{\beta}\right)^{\alpha-1} \left(\frac{y_i}{\theta}\right)^{-(\lambda+1)} e^{-\sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\alpha} - \sum_{i=1}^{n}\left(\frac{y_i}{\gamma}\right)^{-\lambda}} \quad (13)$$

By taking the log. Of equation (13) results:

$$\mathcal{L} = \mathrm{n}\ln\left(\frac{\alpha}{\beta}\right) + n ln\left(\frac{\lambda}{\gamma}\right) + (\alpha - 1)\sum ln\left(\frac{x_i}{\beta}\right) - (\lambda + 1)\sum ln\left(\frac{y_i}{\gamma}\right) - \sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\alpha} - \sum_{i=1}^{n}\left(\frac{y_i}{\theta}\right)^{-\lambda} \quad (14)$$

partial differentiation for parameters $(\boldsymbol{\alpha}; \boldsymbol{\beta}; \boldsymbol{\lambda}; \boldsymbol{\gamma})$ are shown below

$$\frac{\partial \mathcal{L}}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} ln\left(\frac{x_i}{\beta}\right) - \sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\alpha} ln\left(\frac{x_i}{\beta}\right) \quad (15)$$

$$\frac{\partial \mathcal{L}}{\partial \beta} = -\frac{n}{\beta} - \frac{n(\alpha - 1)}{\beta} + \frac{\alpha}{\beta}\sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\alpha} \quad (16)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} ln\left(\frac{y_i}{\gamma}\right) + \sum_{i=1}^{n}\left(\frac{y_i}{\gamma}\right)^{-\lambda} ln\left(\frac{y_i}{\gamma}\right) \quad (17)$$

$$\frac{\partial \mathcal{L}}{\partial \gamma} = -\frac{n}{\gamma} + \frac{n(\lambda - 1)}{\gamma} - \frac{\lambda}{\gamma}\sum_{i=1}^{n}\left(\frac{y_i}{\gamma}\right)^{-\lambda} \quad (18)$$

Equalizing all the above equations (15-18) by zero results system of nonlinear equations, the system of equations cannot be solved by analytic mathematical methods. Numerical methods are used to find the solutions

ISSN (1681- 6870)

8. The least square method:

This method is one of the important methods of estimation. It has several characteristics that make it the best methods and widest used. This method is based on minimizing the sum of squares of errors.

$$LSE=\sum_{i=1}^{n}\left(\frac{\alpha}{\beta}\frac{\lambda}{\gamma}\left(\frac{x_i}{\beta}\right)^{\alpha-1}\left(\frac{y_i}{\gamma}\right)^{-\lambda-1}e^{-\left(\frac{x_i}{\beta}\right)^{\alpha}-\left(\frac{y_i}{\gamma}\right)^{-\lambda}}-\frac{i}{n-1}\right)^2 \tag{19}$$

Partial differentiation of (23) according to Least squares method is found to be

$$\frac{\partial LSE}{\partial\alpha}=2\sum_{i=1}^{n}\left(\frac{\alpha}{\beta}\frac{\lambda}{\gamma}\left(\frac{x_i}{\beta}\right)^{\alpha-1}\left(\frac{y_i}{\gamma}\right)^{-\lambda-1}e^{-\left(\frac{x_i}{\beta}\right)^{\alpha}-\left(\frac{y_i}{\gamma}\right)^{-\lambda}}-\frac{i}{n-1}\right)\left(\frac{\lambda}{\beta\gamma}\left(\frac{x_i}{\beta}\right)^{\alpha-1}\left(\frac{y_i}{\gamma}\right)^{-\lambda-1}e^{-\left(\frac{x_i}{\beta}\right)^{\alpha}-\left(\frac{y_i}{\gamma}\right)^{-\lambda}}\left(1+\alpha ln\left(\frac{x_i}{\beta}\right)-\alpha\left(\frac{x_i}{\beta}\right)^{\alpha}ln\left(\frac{x_i}{\beta}\right)\right)\right) \tag{20}$$

$$\frac{\partial LSE}{\partial\beta}=2\sum_{i=1}^{n}\left(\frac{\alpha}{\beta}\frac{\lambda}{\gamma}\left(\frac{x_i}{\beta}\right)^{\alpha-1}\left(\frac{y_i}{\gamma}\right)^{-\lambda-1}e^{-\left(\frac{x_i}{\beta}\right)^{\alpha}-\left(\frac{y_i}{\gamma}\right)^{-\lambda}}-\frac{i}{n-1}\right)\left(\frac{\alpha\lambda}{\beta^2\gamma}\left(\frac{x_i}{\beta}\right)^{\alpha-1}\left(\frac{y_i}{\gamma}\right)^{-\lambda-1}e^{-\left(\frac{x_i}{\beta}\right)^{\alpha}-\left(\frac{y_i}{\gamma}\right)^{-\lambda}}\left(\alpha\left(\frac{x_i}{\beta}\right)^{\alpha}-\alpha\right)\right) \tag{21}$$

$$\frac{\partial LSE}{\partial\lambda}=2\sum_{i=1}^{n}\left(\frac{\alpha}{\beta}\frac{\lambda}{\gamma}\left(\frac{x_i}{\beta}\right)^{\alpha-1}\left(\frac{y_i}{\gamma}\right)^{-\lambda-1}e^{-\left(\frac{x_i}{\beta}\right)^{\alpha}-\left(\frac{y_i}{\gamma}\right)^{-\lambda}}-\frac{i}{n-1}\right)\left(\frac{\alpha}{\beta\gamma}\left(\frac{x_i}{\beta}\right)^{\alpha-1}\left(\frac{y_i}{\gamma}\right)^{-\lambda-1}e^{-\left(\frac{x_i}{\beta}\right)^{\alpha}-\left(\frac{y_i}{\gamma}\right)^{-\lambda}}\left(1-cln\left(\frac{y_i}{\gamma}\right)ln\left(\frac{y_i}{\gamma}\right)\right)\right) \tag{22}$$

$$\frac{\partial LSE}{\partial\gamma}=2\sum_{i=1}^{n}\left(\frac{\alpha}{\beta}\frac{\lambda}{\gamma}\left(\frac{x_i}{\beta}\right)^{\alpha-1}\left(\frac{y_i}{\gamma}\right)^{-\lambda-1}e^{-\left(\frac{x_i}{\beta}\right)^{\alpha}-\left(\frac{y_i}{\gamma}\right)^{-\lambda}}-\frac{i}{n-1}\right)\left(\frac{\alpha\gamma}{\beta\gamma}\left(\frac{x_i}{\beta}\right)^{\alpha-1}\left(\frac{y_i}{\gamma}\right)^{-\lambda-1}e^{-\left(\frac{x_i}{\beta}\right)^{\alpha}-\left(\frac{y_i}{\gamma}\right)^{-\lambda}}\left(\lambda-\lambda ln\left(\frac{y_i}{\gamma}\right)^{-\lambda}\right)\right) \tag{23}$$

Equalization system equations (20-23) by zero gives system of nonlinear equations, the system of equations cannot be solved by analytic mathematical methods. Numerical methods are used to find the solutions

9. Bayes Estimation Method

In This method parameters are assumed as random variables that follows probable distribution. To estimate these parameters, prior function must be found. In this paper the marginal prior function is assumed to be distributed as Gama distribution as follows:

ISSN (1681- 6870)

$$g_1(\alpha) = \frac{\psi_1^{\eta_1}}{\Gamma(\eta_1)} \alpha^{\eta_1} e^{-\psi_1 \alpha}$$

$$g_2(\beta) = \frac{\psi_2^{\eta_2}}{\Gamma(\eta_2)} \beta^{\eta_2} e^{-\psi_2 \beta}$$

$$g_3(\lambda) = \frac{\omega_1^{\delta_1}}{\Gamma(\delta_1)} \lambda^{\delta_1} e^{-\omega_1 \lambda}$$

$$g_4(\gamma) = \frac{\omega_2^{\delta_2}}{\Gamma(\delta_2)} \gamma^{\delta_2} e^{-\omega_2 \gamma}$$

The joint prior function for $(\alpha; \beta; \lambda; \gamma)$ is

$$g(\alpha;\beta;\lambda;\gamma) = \frac{\psi_1^{\eta_1}\psi_2^{\eta_2}\omega_1^{\delta_1}\omega_2^{\delta_2}}{\Gamma(\eta_1)\Gamma(\eta_2)\Gamma(\delta_1)\Gamma(\delta_2)} \alpha^{\eta_1}\beta^{\eta_2}\lambda^{\delta_1}\gamma^{\delta_2} e^{-\psi_1\alpha-\psi_2\beta-\omega_1\lambda-\omega_2\gamma} \quad (24)$$

Where $(\boldsymbol{\psi_1}; \boldsymbol{\psi_2}; \boldsymbol{\omega_1}; \boldsymbol{\omega_2})$ are hyper parameters.

The Likelihoods for Weibull – Fréchet is

$$L(x; y/\alpha\,; \beta; \lambda; \theta) = \prod_{i=1}^{n} \left(\frac{x_i}{\beta}\right)^{\alpha-1} \left(\frac{y_i}{\theta}\right)^{-(\lambda+1)} e^{-\left(\frac{x_i}{\beta}\right)^{\alpha} - \left(\frac{y_i}{\theta}\right)^{-\lambda}}$$

$$L(x; y/\alpha\,; \beta; \lambda; \theta) = \left(\frac{\alpha}{\beta}\frac{\lambda}{\theta}\right)^n \prod_{i=1}^{n} \left(\frac{x_i}{\beta}\right)^{\alpha-1} \left(\frac{y_i}{\theta}\right)^{-(\lambda+1)} e^{-\left(\frac{x_i}{\beta}\right)^{\alpha} - \left(\frac{y_i}{\theta}\right)^{-\lambda}} \quad (25)$$

According to Bayes theory, the posterior function will be as follows

$$\pi(\alpha;\beta;\lambda;\theta/x\,;y) = \frac{g(\alpha;\beta;\lambda;\gamma)\left(\frac{\alpha}{\beta}\frac{\lambda}{\theta}\right)^n \prod_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\alpha-1}\left(\frac{y_i}{\theta}\right)^{-(\lambda+1)} e^{-\left(\frac{x_i}{\beta}\right)^{\alpha}-\left(\frac{y_i}{\theta}\right)^{-\lambda}}}{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty g(\alpha;\beta;\lambda;\gamma)\left(\frac{\alpha}{\beta}\frac{\lambda}{\theta}\right)^n \prod_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\alpha-1}\left(\frac{y_i}{\theta}\right)^{-(\lambda+1)} e^{-\left(\frac{x_i}{\beta}\right)^{\alpha}-\left(\frac{y_i}{\theta}\right)^{-\lambda}}} \quad (26)$$

Use square error loss function,

$$L = \left(\hat{\theta} - \theta\right)^2 \quad (27)$$

the Bayes estimator of unknown parameters $(\alpha; \beta; \lambda; \gamma)$ under square error loss function is posterior mean

$$E(u(\theta\,/\,t)) = \int\int\int\int_0^\infty u(\theta)\,\pi^*(\theta)d\theta$$

$$E(u(\theta\,/\,t)) = \frac{\int\int\int\int_0^\infty u(\theta)\,\pi(\theta)d\theta}{\int\int\int\int_0^\infty \pi(\theta)d\theta} \quad (28)$$

ISSN (1681- 6870)

Where $\theta = (\alpha\ ;\beta\ ;\lambda\ \ ;\gamma)$

Equation (28) is contains a ratio of integrals which cannot be solved analytically, the Lindley's approximation procedure will be employed to estimate the unknown parameters. For evaluating the posterior expectation of an arbitrary function, Lindley consider an approximation as follows

$$E(u(\theta\ /\ t)) = \frac{\int u(\theta)v(\theta)e^{\mathcal{L}(\theta)}dx}{\int v(\theta)e^{\mathcal{L}(\theta)}dx} \tag{29}$$

In applying the Lindley's approximation procedure, it Where u(θ); v(θ) are arbitrary function of (θ); L(θ) is the log is assumed that v(θ) is a prior distribution for unknown parameters; u(θ) being some of interest. Ratio of integration can be approximated asymptotically as follows

$$\widehat{\Theta} = u(\hat{\theta}) + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}(u_{ij} + 2u_i\rho_j)\hat{\sigma}_{ij} + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{l=1}^{m}L_{ijk}u_i\sigma_{ij}\sigma_{kl} \tag{30}$$

$$\hat{\alpha}_{Bayes} = \alpha + \frac{1}{2}(\mathcal{L}_{111}\sigma_{11}^2 + \mathcal{L}_{221}(\sigma_{21}^4 + \sigma_{22}\sigma_{11}) + 3\mathcal{L}_{112}\sigma_{11}\sigma_{21} + \mathcal{L}_{222}\sigma_{22}\sigma_{21}) + \sigma_{11}\rho_1 + \sigma_{12}\rho_2 \tag{31}$$

$$\hat{\alpha}_{Bayes} = \alpha + \frac{1}{2}(\mathcal{L}_{111}\sigma_{11}^2 + \mathcal{L}_{221}(\sigma_{21}^4 + \sigma_{22}\sigma_{11}) + 3\mathcal{L}_{112}\sigma_{11}\sigma_{21} + \mathcal{L}_{222}\sigma_{22}\sigma_{21}) + \sigma_{11}\rho_1 + \sigma_{12}\rho_2 \tag{31}$$

$$\hat{\beta}_{Bayes} = \beta + \frac{1}{2}(\mathcal{L}_{111}\sigma_{11}\sigma_{12} + \mathcal{L}_{112}(\sigma_{12}^4 + \sigma_{11}\sigma_{22}) + 3\mathcal{L}_{122}\sigma_{12}\sigma_{22} + \mathcal{L}_{222}\sigma_{22}^2) + \sigma_{22}\rho_2 + \sigma_{12}\rho_1 \tag{32}$$

$$\mathcal{L}_{111} = \frac{\partial\mathcal{L}^3}{\partial\alpha^3} = \frac{2n}{\alpha^3} - \left(\sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\alpha} ln\left(\frac{x_i}{\beta}\right)^3\right) \tag{33}$$

$$\mathcal{L}_{112} = \frac{\partial\mathcal{L}^3}{\partial\alpha^2\partial\beta} = -\frac{1}{\beta}\sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\alpha} ln\left(\frac{x_i}{\beta}\right)\left(\alpha ln\left(\frac{x_i}{\beta}\right) - 2\right) \tag{34}$$

$$\mathcal{L}_{122} = \frac{\partial\mathcal{L}^3}{\partial\alpha\partial\beta^2} = \frac{n}{\beta^2} - \frac{1}{\beta^2}\sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\alpha}\left(\alpha^2 ln\left(\frac{x_i}{\beta}\right) + \alpha ln\left(\frac{x_i}{\beta}\right) + 2\alpha + 1\right) \tag{35}$$

$$\mathcal{L}_{222} = \frac{\partial\mathcal{L}^3}{\partial\beta^3} = -\frac{2n\alpha}{\beta^3} + \frac{(\alpha^3 + 3\alpha^2 - 2\alpha)}{\beta^3}\sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\alpha} \tag{36}$$

$$\sigma_{11} = \left(-\frac{\partial\mathcal{L}^2}{\partial\alpha^2}\right)^{-1} = \frac{\alpha^2}{n + \alpha^2\beta^{-\alpha}\left(\sum x_i^{\alpha}\left(ln(\beta) - ln(x_i)\right)^2\right)} \tag{37}$$

$$\sigma_{12} = \left(-\frac{\partial\mathcal{L}^2}{\partial\alpha\partial\beta}\right)^{-1} = \frac{\beta}{n + \beta^{-\alpha}(\sum x_i^{\alpha}(\alpha ln(\beta) - \alpha ln(x_i) - 1))} \tag{38}$$

$$\sigma_{22} = \left(-\frac{\partial\mathcal{L}^2}{\partial\beta^2}\right)^{-1} = \frac{\beta^2}{\alpha(-n + \beta^{-\alpha}\alpha\sum x_i^{\alpha} + \beta^{-\alpha}\sum x_i^{\alpha})} \tag{39}$$

ISSN (1681- 6870)

$$\rho = ln\left(\frac{\psi_1^{\eta_1}\psi_2^{\eta_2}\omega_1^{\delta_1}\omega_2^{\delta_2}}{\Gamma(\eta_1)\Gamma(\eta_2)\Gamma(\delta_1)\Gamma(\delta_2)}\alpha^{\eta_1}\beta^{\eta_2}\lambda^{\delta_1}\gamma^{\delta_2}e^{-\psi_1\alpha-\psi_2\beta-\omega_1\lambda-\omega_2\gamma}\right) \tag{40}$$

$$\rho_1 = \frac{\partial v}{\partial \alpha} = \frac{\eta_1}{\alpha} - \psi_1 \tag{41}$$

$$\rho_2 = \frac{\partial v}{\partial \beta} = \frac{\eta_2}{\beta} - \psi_2 \tag{42}$$

$$\hat{\lambda}_{Bayes} = \lambda + \frac{1}{2}(\mathcal{L}_{333}\sigma_{33}^2 + \mathcal{L}_{433}(\sigma_{34}^4 + \sigma_{33}\sigma_{44}) + 3\mathcal{L}_{334}\sigma_{33}\sigma_{34} + \mathcal{L}_{444}\sigma_{34}\sigma_{44}) + \sigma_{33}\rho_3 + \sigma_{34}\rho_4 \tag{43}$$

$$\hat{\gamma}_{Bayes} = \gamma + \frac{1}{2}(\mathcal{L}_{333}\sigma_{33}\sigma_{34} + \mathcal{L}_{334}(\sigma_{34}^4 + \sigma_{33}\sigma_{44}) + 3\mathcal{L}_{344}\sigma_{34}\sigma_{44} + \mathcal{L}_{444}\sigma_{44}^2) + \sigma_{44}\rho_4 + \sigma_{34}\rho_3 \tag{44}$$

$$\mathcal{L}_{333} = \frac{\partial^3\mathcal{L}}{\partial\lambda^3} = \frac{2n}{\lambda^3} + \sum_{i=1}^{n}\left(\frac{y_i}{\gamma}\right)^{-\lambda} ln\left(\frac{y_i}{\gamma}\right)^3 \tag{45}$$

$$\mathcal{L}_{334} = \frac{\partial^3\mathcal{L}}{\partial\lambda^2\partial\gamma} = -\frac{1}{\gamma}\sum_{i=1}^{n}\left(\frac{y_i}{\gamma}\right)^{-\lambda} ln\left(\frac{y_i}{\gamma}\right)\left(\lambda ln\left(\frac{y_i}{\gamma}\right) - 2\right) \tag{46}$$

$$\mathcal{L}_{444} = \frac{\partial^3\mathcal{L}}{\partial\gamma^3} = -\frac{2n(\lambda-2)}{\gamma^3} - \frac{(\lambda^3 - 3\lambda^2 + 2\lambda)}{\gamma^3}\sum_{i=1}^{n}\left(\frac{y_i}{\gamma}\right)^{-\lambda} \tag{47}$$

$$\mathcal{L}_{443} = \frac{\partial^3\mathcal{L}}{\partial\lambda\partial\gamma^2} = -\frac{n}{\gamma^2} - \frac{1}{\gamma^2}\sum_{i=1}^{n}\left(\frac{y_i}{\gamma}\right)^{-\lambda}\left(\lambda^2 ln\left(\frac{y_i}{\gamma}\right) - 2\lambda - \lambda ln\left(\frac{y_i}{\gamma}\right) - 1\right) \tag{48}$$

$$\sigma_{33} = \left(-\frac{\partial\mathcal{L}^2}{\partial\lambda^2}\right)^{-1} = \frac{\lambda^2}{n + \lambda^2\gamma^{-\lambda}\left(\sum y_i^{-\lambda}\left(ln(\gamma) - ln(y_i)\right)^2\right)} \tag{49}$$

$$\sigma_{34} = \left(-\frac{\partial^2\mathcal{L}}{\partial\lambda\partial\gamma}\right)^{-1} = \frac{\gamma}{-n + \gamma^{\lambda}\left(\sum y_i^{-\lambda}(\lambda ln(\gamma) - \lambda ln(y_i) + 1)\right)} \tag{50}$$

$$\sigma_{44} = \left(-\frac{\partial^2\mathcal{L}}{\partial\gamma^2}\right)^{-1} = \frac{\gamma^2}{-2n + n\lambda + \gamma^{\lambda}\lambda^2\left(\sum y_i^{-\lambda}\right) - \gamma^{\lambda}\lambda\left(\sum y_i^{-\lambda}\right)} \tag{51}$$

$$\rho_3 = \frac{\delta_1}{\lambda} - \omega_1 \tag{52}$$

$$\rho_4 = \frac{\delta_2}{\gamma} - \omega_2 \tag{53}$$

ISSN (1681- 6870)

10. Simulation

Monte-Carlo Method has been used to compare different methods of estimation of reliability function for cascade systems of models. The assumed sizes of samples are as tabled below:

Table (1): Value simple sizes

Small samples	10
Medium samples	35 , 50
large samples	75 ; 100

Assumed values of parameters are as in table (2)

Table (2): Value of assumed parameters

Model	α	β	λ	γ
Model 1	2	2	2	2
Model 2	0.5	1	5	12
Model 3	12	3	6	1.5
Model 4	10	8	1.5	2.5
Model 5	14	12	0.5	0.2

The values selected are based on the probable of relationships among the parameters. mean square error criteria is used to compare different methods of estimation of reliability function.

$$\mathrm{MSE}(\hat{R}) = \frac{1}{\mathrm{L}} \sum_{\mathrm{i}=1}^{\mathrm{L}} \left(\hat{\mathrm{R}}_{\mathrm{i}} - \mathrm{R}\right)^2 \tag{54}$$

Where (L) represents number of replications of each experiment which was equal to 100

11. Simulation results

Simulation is performed for cascade models, for strength-stress Weibull- Fréchet distributions of one, two or three components. Marginal reliability and reliability of the system with defaults parameters were calculated as shown in table (3). Table (4) displays the average of mean squares errors for MLM , LSM and Bayes method. Whereas the number of Preferences of each method based on the criteria of mean squares errors are tabled on table (5)

Table (3): values of marginal reliabilities and system reliability $(k^* = \frac{1}{i})$

Models	α	β	γ	λ	$R(1)$	$R(2)$	$R(3)$	R3
Model1	2.0	2.0	2.0	2.0	0.279732	0.371522	0.202561	0.853815
Model2	0.5	1.0	5.0	12.0	0.101645	0.177914	0.205105	0.484664
Model3	12.0	3.0	6.0	1.5	0.051401	0.290646	0.263033	0.60508
Model4	10.0	8.0	1.5	2.5	0.981549	0.015165	0.002373	0.999086
Model5	14.0	12.0	0.5	0.2	0.586233	0.041957	0.029015	0.657205

12. Conclusion

From simulation results conclusion

1- The reliability of marginal with on component for the first, third and fourth models is greater than that with two or three components. Whereas, for the second and third models the reliability of marginal with two or three components is greater than that of one component.
2- It's clear from simulation results that weibull distribution parameters and reliability of the system are directly proportional with each other. While Fréchet distribution parameters are inversely proportional to system's reliability.

ISSN (1681- 6870)

3- The average mean squares errors decrease as the sizes of samples increase.
4- For large samples the estimators of MLM are more accurate than that estimated in other methods.
5- For small samples the estimators of Bayes method are more accurate than that estimated in other methods.
6- For medium samples the estimators of LSM are more accurate than that estimated in other methods.

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ISSN (1681- 6870)

Table (4): Average values of MSE for Maximum Likelihood Method , Least Square estimated and Bayes method for cascaded Weibull-Fréchet distribution.

Models	n	MLM				LSM				Bayes			
Model1	10	1.54507	1.68897	1.56687	1.56687	0.97021	0.98912	0.96239	0.82311	0.86177	0.88933	0.92386	0.91077
	35	0.98421	0.98089	0.95546	0.84911	0.44689	0.60283	0.76867	0.76867	0.55069	0.73548	0.40257	0.40257
	50	0.01625	0.05060	0.06997	0.06997	0.03271	0.04060	0.08997	0.04997	0.07058	0.07658	0.06387	0.09387
	75	0.01396	0.01759	0.07047	0.07047	0.02799	0.03449	0.06340	0.07666	0.06644	0.06305	0.06176	0.08176
	100	0.01045	0.03439	0.05666	0.05666	0.01952	0.04822	0.07041	0.07041	0.01324	0.05722	0.07070	0.06541
Model2	10	1.91170	1.94441	1.35828	1.35828	0.91225	0.97871	0.96287	1.16287	0.97040	0.94077	0.95819	0.92867
	35	0.82348	0.63695	0.92985	0.92298	0.70071	0.69560	0.97778	0.97778	0.70715	0.59816	0.77781	0.77781
	50	0.19911	0.16176	0.20465	0.10465	0.19524	0.18221	0.10704	0.14070	0.11396	0.11759	0.60465	0.10407
	75	0.06139	0.00947	0.09865	0.09865	0.01779	0.08816	0.09937	0.09372	0.07870	0.07228	0.03715	0.09954
	100	0.04318	0.03882	0.10994	0.10994	0.04393	0.07338	0.04996	0.04996	0.05204	0.08030	0.09643	0.06430
Model3	10	1.79866	1.99691	1.87897	1.87897	1.66921	1.78929	1.24340	1.14340	1.59669	1.13079	1.14339	1.10740
	35	0.68201	0.77903	0.58608	0.58608	0.53287	0.59093	0.43989	0.78440	0.53287	0.59093	0.43989	0.43989
	50	0.10728	0.10229	0.11972	0.11972	0.12886	0.12870	0.11905	0.12130	0.10444	0.12739	0.13031	0.12130
	75	0.07989	0.02221	0.08984	0.08984	0.09678	0.07023	0.08041	0.08041	0.09845	0.08035	0.08986	0.09041
	100	0.00781	0.00130	0.01011	0.00101	0.01781	0.01968	0.03643	0.00643	0.01748	0.01210	0.01216	0.01216
Model4	10	1.70640	1.23568	1.62542	1.62542	0.90351	0.71714	0.81668	1.91668	0.92334	0.67717	0.81122	0.18112
	35	0.72284	0.71048	0.46105	0.46105	0.42810	0.48387	0.33604	0.31336	0.23474	0.24593	0.33604	0.25614
	50	0.17275	0.52345	0.10974	0.10974	0.13543	0.52345	0.12325	0.13255	0.17474	0.54462	0.12392	0.12433
	75	0.07301	0.03689	0.06887	0.06887	0.09361	0.09382	0.09565	0.09565	0.09093	0.08919	0.06880	0.08197
	100	0.00729	0.00185	0.00755	0.00755	0.00910	0.01615	0.58650	0.08650	0.01002	0.01469	0.01865	0.01586
Model5	10	1.83651	1.28844	1.29298	1.29298	0.83651	0.88435	0.92978	0.92978	0.77755	0.96568	0.84715	0.71523
	35	0.42819	0.44001	0.22548	2.25481	0.42887	0.43358	0.28683	0.48683	0.41947	0.52268	0.27787	0.25779
	50	0.24886	0.30358	0.25568	0.25568	0.26060	0.30221	0.56781	0.25770	0.38604	0.25137	0.06776	0.77588
	75	0.01527	0.01410	0.02524	0.02524	0.08527	0.03364	0.07135	0.03352	0.06595	0.04203	0.07136	0.07135
	100	0.00285	0.00426	0.00226	0.00226	0.00293	0.00945	0.00173	0.00917	0.00889	0.00865	0.00892	0.00949

ISSN (1681- 6870)

Table (5): Number of preferences of MLM, LSM and Bayes method according to MSE.

Sample	Methods				Mlm	Lsm	Bayes
10	BAYES	BAYES	BAYES	LSM	0	1	3
35	LSM	LSM	BAYES	BAYES	0	2	2
50	MLM	LSM	BAYES	LSM	1	2	1
75	MLM	MLM	BAYES	MLM	3	0	1
100	MLM	MLM	MLM	MLM	4	0	0
10	LSM	BAYES	BAYES	BAYES	0	1	3
35	LSM	BAYES	BAYES	BAYES	0	1	3
50	BAYES	BAYES	LSM	BAYES	0	1	3
75	LSM	MLM	BAYES	LSM	1	2	1
100	MLM	MLM	LSM	LSM	2	2	0
10	BAYES	BAYES	BAYES	BAYES	0	0	4
35	LSM	LSM	LSM	BAYES	0	3	1
50	BAYES	MLM	LSM	MLM	2	1	1
75	MLM	MLM	LSM	LSM	2	2	0
100	MLM	MLM	MLM	MLM	4	0	0
10	LSM	BAYES	BAYES	BAYES	0	1	3
35	BAYES	BAYES	LSM	BAYES	0	1	3
50	LSM	MLM	MLM	MLM	3	1	0
75	MLM	MLM	BAYES	MLM	3	0	1
100	MLM	MLM	MLM	MLM	4	0	0
10	BAYES	LSM	BAYES	BAYES	0	1	3
35	BAYES	LSM	MLM	BAYES	1	1	2
50	MLM	BAYES	BAYES	MLM	2	0	2
75	MLM	MLM	MLM	MLM	4	0	0
100	MLM	MLM	LSM	MLM	3	1	0
total					39	24	37