2024, 21(3): 1080-1085
https://dx.doi.org/10.21123/bsj.2023.8230
P-ISSN: 2078-8665 - E-ISSN: 2411-7986

Baghdad Science Journal

Best approximation in b-modular spaces

Hiba Adel Jabbar [1], ***Samira Naji Kadhim*** [2], ***Salwa Salman Abed**** [3]

[1] Al-Mahaj School, Baghdad Directorate of Education Al-Rusafa First, Iraqi Ministry of Education, Baghdad, Iraq.
[2]Department of Mathematics, College of Science for Women, University of Baghdad, Baghdad, Iraq.
[3]Department of Mathematics, College of Education for Pure Sciences-Ibn Al-Haitham, University of Baghdad, Baghdad, Iraq.
*Corresponding Author.

Received 14/12/2022, Revised 21/07/2023, Accepted 23/07/2023, Published Online First 20/08/2023, Published 01/03/2024

Abstract

In this paper, some basic notions and facts in the b-modular space similar to those in the modular spaces as a type of generalization are given. For example, concepts of convergence, best approximate, uniformly convexity etc. And then, two results about relation between semi compactness and approximation are proved which are used to prove a theorem on the existence of best approximation for a semi-compact subset of b-modular space.

Keywords: b-Modular vector spaces, Best approximation, Fixed point, Semi-compactness, Set valued mapping.

Introduction

The primary problems of best approximation theory from Ky Fan's point of view a convexity advantage which require introducing a mapping with some hypotheses. In this article, the focus was on Ky Fan's type best approximation: Let Ω be convex compact subset of a normed linear space Γ, f is a continuous function on Ω and u is an element of Γ; that approximately to u from the elements in Ω would be a vector $v \in \Omega$ such that $\|v - f(v)\| = d\,(f(v), \Omega)$[1], where d is a metric distance induced by norm .

The value of Ky Fan's Theorem[1] is due to its use in elicitation many fixed-point theorems as a corollary depending on weaker assumptions in many fields of nonlinear analysis[1]. This theorem has been of great importance in nonlinear analysis, approximation theory, fixed point theory and variational inequalities. The result is equivalent to the well-known topological fixed point theorem due to Brouwer introduced by Kanster, Knratowski, and Mazurkiewiez. They produced a very important result depending on Sperner's lemma, presently, it is known as the (KKM-map) principle. This principle was employed to give a simplified proof of Brouwer's theorem. An extension of KKM- map) theorem was presented by Ky Fan in topological vector spaces and gave several interesting applications, specially, in fixed point theory and best approximation theory. Fixed point theorems have been used at many places in approximation theory. Later on, many results were developed using fixed point theorem to prove the existences of best approximation. By H. Kaneko sufficient conditions for the existence of a coincidence point of continuous multivalued mappings are derived in p-normed spaces (0 < p < 1). As applications, some results on the set of best approximation for this

2024, 21(3): 1080-1085
https://dx.doi.org/10.21123/bsj.2023.8230
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
Baghdad Science Journal

class of mappings are obtained. A. Latif obtained the coincidence point theorem for Banach spaces or thesis sources (for more details see [1-4]).

The aim of this article is to establish some relations among approximate compact set, compact set and proximinal set in modular spaces. A first attempt was made by Birnbaum and Orlicz[5]. Their approach considers spaces of functions with some properties different from those provided by the L_p – norm ($p \geq 1$). This work found various applications in differential and integral equations. Another generalization was given by Luxemburg. The main idea is to consider, in a measure space, a functional that has the properties of a norm plus a monotony condition. For this historical narrative see[6, 7]. For known current notion of modular spaces see [8], who introduced it as a generalization of metric spaces. This idea has been based on replacing the particular form of the functional (integral formula) by an abstractly one which has some features was called a modular in (named by Nakano in 1950)[6, 7]. It was then redefined with some modification by Musielak and Ortiz in 1959 [6].

Example 1 Let θ be a non-negative convex Orlicz function such that $\theta(0) = 0$. The Orlicz space is a modular function generated by

$$\rho(f) = \int_R^{\cdot} \theta(|f(t)|)dm(t)$$

Example 2 Let μ be a ϑ- finite measure. $Ƒ = \{\tau: R \to R$: τ is measure preserving transformations, $\mu_\tau(E) = \mu(\tau^{-1}(E))\}$. The group $Ƒ$ is a modular function space generated by $\rho(f) = sup_{\tau\in f} \int_R^{\cdot} |f(t)|^p d\mu_\tau(t)$ is a Musielak-Orlicz. Note, throughout the work, R is a symbol of real numbers

Example 3: Let μ be a σ – finite measure. $Ƒ = \{\tau : R \to R$ measure preserving transformation, $\mu_\tau(E) = \mu(\tau^{-1}(E))\}$. The group $Ƒ$ is modular function space with

$$\rho(f) = sup_{\tau\in f} \int_R^{\cdot} |f(t)|^p d\mu_\tau(t)$$

is a Lorentz ρ-space.

Example 4 Let φ be an Orlicz function. $Ƒ = \{\tau: R \to R$ is measure preserving transformations and $\mu_\tau(E) = \mu(\tau^{-1}(E))\}, \mu$ is a φ – finite measuer . The group $Ƒ$ is a modular function space generated by

$$\rho(f) = sup_{\tau\in f} \int_R^{\cdot} \varphi(|f(t)|)d\mu_\tau(t)$$

is a Orlicz-Lorentz space.

In 2017, the best approximation modular spaces have been defined by Abed [8] and results about proximinal set, Chebysev set are proven also, see[9]. Various results in modular spaces and other related spaces about fixed point problem can be seen in Turkoglu and Nesrin[9], Albundi[10], Abdul Jabbar and Abed[11, 12], Ahmed[13], Mohammed and Abed[14], Pathak and Beg[15], Abed and Abdul Jabbar[16, 17], Ege and, Alac[18] also Abed and Salman[19].

Preliminaries

A generalization of modular space should be mentioned here as defined as modular b-metric space[17].

Definition 1 Let Γ be a vector space over $F(= R \text{ or } ₵)$, a function $\zeta:\Gamma \to [0,\infty]$ is called b-modular if

(i) $\zeta(v^\Lambda) = 0$ if and only if $v^\Lambda = 0$.
(ii) $\zeta(\alpha v^\Lambda) = \alpha\zeta(v^\Lambda)$, $\alpha \in F$ and $|\alpha| = 1, \forall v^\Lambda \in \Gamma$.
(iii) $\zeta(\alpha v^\Lambda + \beta u^\Lambda) \leq b[\zeta(v^\Lambda) + \zeta(u^\Lambda)]$ iff $\alpha, \beta \geq 0$, for all $u^\Lambda, v^\Lambda \in \Gamma, b \geq 1$.

If (iii) replaced by

(iii) $\zeta(\alpha v^\Lambda + \beta u^\Lambda) \leq b[\alpha\zeta(v^\Lambda) + \beta\zeta(u^\Lambda)]$, for $\alpha, \beta \geq 0$, $\alpha + \beta = 1$, for all $v^\Lambda, u^\Lambda \in \Gamma$ $b \geq 1$. Then Γ modular ζ is called convex b-modular.

When b=1, it will be convex modular. Similar to [8] set the following

Definition 2 A function ζ defines an identical b-modular space Γ_ζ, as follows

$$\Gamma_\zeta = \{v^\Lambda \in \Gamma: \zeta(\alpha v^\Lambda) \to 0 \text{ whenever } \alpha \to 0\}.$$

Example 5 The space $l_p = \{u = \{\hat{u}_n\} \subset R, \sum_1^\infty |\hat{u}_n|^p < \infty\}$, $0 < p < 1$, with modular $\zeta(u) = (\sum_1^\infty |\hat{u}_n|^p)^{1/p}$ is b-modular space with b=2 by similar details in[4].

Remark 1

i) If $u^\Lambda = 0$ then $\zeta(\alpha v^\Lambda) = \zeta\left(\frac{\alpha}{\beta}\beta v^\Lambda\right) \leq \zeta(\beta v^\Lambda)$, for all $\alpha, \beta \text{ in } F$, $0 < \alpha < \beta$, by condition (iii) above and by similar reasoning in [6] and this shows that ζ is increasing function.
ii) The family of all ζ-balls in the space Γ_ζ gives a topology.

In the sense in [13], the following definition is stated

Definition 3 The distance between $v^{\Lambda} \in \Gamma_\zeta$ and $B \subset \Gamma_\zeta$ is
$D_\zeta(v^{\Lambda}, B) = \inf\{\zeta(v^{\Lambda} - u^{\Lambda});\ u^{\Lambda} \in B\}$.

Definition 4 Let Γ_ζ be a b-modular space

(a) A sequence $\{v_n^{\Lambda}\} \subset \Gamma_\zeta$ converges to $v^{\Lambda} \in \Gamma_\zeta$ if $\zeta(v_n^{\Lambda} - v^{\Lambda}) \to 0$ as n→ ∞. It is called ζ-convergent to v^{Λ}write $v_n^{\Lambda} \xrightarrow{\zeta} v^{\Lambda}$

(b) A sequence $\{v_n^{\Lambda}\} \subset \Gamma_\zeta$ is called $\zeta-$ Cauchy if $\zeta(v_n^{\Lambda}\text{-}v_m^{\Lambda}) \to 0$ as $n \to \infty$.

(c) If every $\zeta-$ Cauchy sequence in Γ_ζ is $\zeta-$ convergent to a point in Γ_ζ then Γ_ζ is called $\zeta-$ complete.

(d) If every sequence $\{v_n^{\Lambda}\} \subset B \subset \Gamma_\zeta$ is $\zeta-$ convergent to $v^{\Lambda} \in B$ then $B \subset \Gamma_\zeta$ is called $\zeta-$ closed. And B is bounded if $daim_\zeta(B) < \infty$ (daimeter w.r.t. ζ)

(e) A subset $B \subset \Gamma_\zeta$ is called $\zeta-$ compact if every sequence $\{v_n^{\Lambda}\} \subset B$ contains a $\zeta-$ convergent subsequence.

Here, Γ_ζ be a b- modular space its b-modular function is ζ and $\emptyset \neq \Omega \subset \Gamma_\zeta$. Let Γ_ζ and N_ρ be two b-modular spaces and $S: \Gamma_\zeta \to N_\rho$ each $u^{\Lambda} \in \Gamma_\zeta, \emptyset \neq S(u^{\Lambda}) \subseteq N_\rho$ be a set-valued mapping.

Definition 5 [9] Let $v^{\Lambda} \in \Gamma_\zeta$, v^{Λ} is a fixed point of S if $v^{\Lambda} \in Sv^{\Lambda}$ (or $v^{\Lambda} = Sv^{\Lambda}$, when S is single valued)

Reformed the concept of upper semi-continuous mapping (shortly, $u.s.c.$).

Definition 6 [16] A set-valued mapping F is $u.s.c.$, if the set $\{u^{\Lambda} \in \Gamma_\zeta: F(x) \cap B \neq \emptyset\}$ is closed whenever B is closed subset of N_ρ. In natural way, the following are defined, see [14].

Definition 7 If Ω is a subset of Γ_ζ then:

(i) Ω is called proximinal if for all $v^{\Lambda} \in \Gamma_\zeta$, there exists a $u^{\Lambda} \in \Omega$ such that $\zeta|(v^{\Lambda} - u^{\Lambda})| = D_\zeta(v^{\Lambda}, \Omega)$.

(ii) Ω is called Chebysev if for each $v^{\Lambda} \in \Gamma_\zeta$, there is a unique element $u^{\Lambda} \in \Gamma_\zeta$ such that $\zeta|(v^{\Lambda} - u^{\Lambda})| = D_\zeta(v^{\Lambda}, \Omega)$.

Definition 8 A collection of all best approximation of $v^{\Lambda} \in \Gamma_\zeta$ by Ω is

$$P_\Omega(v^{\Lambda}) = \{u^{\Lambda} \in \Omega: \zeta(v^{\Lambda} - u^{\Lambda}) = D_\zeta(v^{\Lambda}, \Omega)\}$$

and $P_\Omega: M \to 2^\Omega$ is said to the metric projection on Γ_ζ, where 2^Ω is the class of all nonempty subset of Ω.

Results and Discussion

The definition of a semi compact set in a b-modular vector space is introduced in this section.

Definition 9 A subset Ω of Γ_ζ is called a semi-compact if for every $v^{\Lambda} \in \Gamma$ and every sequence $<v_n^{\Lambda}>$ in Ω with $\lim_{n\to\infty} \zeta(v^{\Lambda} - v_n^{\Lambda}) = D_\zeta(v^{\Lambda}, \Omega)$, there exists a subsequence $\langle v_{n_i}^{\Lambda} \rangle$ converges to $w \in \Omega$.

Example 6 Consider the closed unit ball in the 1-modular space$l_p = \{u = \{\hat{u_n}\}: \subset R, \sum_1^\infty |\hat{u_n}|^p < \infty\}$, $0 < p < 1$, with modular$\zeta(u) = (\sum_1^\infty |\hat{u_n}|^p)^{1/p}$, it is semi-compact but not compact.

Remark 2 If Ω is compact subset ofΓ_ζ, then Ω is a semi-compact. To show, let $v^{\Lambda} \in \Gamma_\zeta$ and $<v_n^{\Lambda}>$ be a sequence in Ω with $\lim_{n\to\infty} \zeta(v^{\Lambda} - v_n^{\Lambda}) = D_\zeta(v^{\Lambda}, \Omega)$. Since Ω is compact set, then by Definition 4 (e) there is a convergent subsequence $\langle v_{n_i}^{\Lambda} \rangle$ of $<v_n^{\Lambda}>$ in Ω. Below, note that the converse is not true

Definition 10 A b-modular space Γ_ζ is uniformly convex if ∀ε > 0, ∃δ(ε) > 0, such that if $\zeta(v^{\Lambda}) = \zeta(u^{\Lambda}) = 1$ and $(v^{\Lambda} - u^{\Lambda}) \geq \varepsilon$, then $\zeta\left(\frac{1}{2}(v^{\Lambda} + u^{\Lambda})\right) \leq 1 - \delta$.

When b=1, Abdul Jabbar and Abed [13] gave many facts about uniformly convex modular space.

Proposition 1 If Ω is closed convex subset of a uniformly convex spaceΓ_ζ then it is semi-compact.

Proof: SupposeΓ_ζ uniformly convex, $\Omega \subset \Gamma_\zeta$, Ω is convex and closed and $u^{\Lambda} \in \Gamma_\zeta$ and $< u_n^{\Lambda} > \subseteq \Gamma_\zeta$ such that $\zeta(u_n^{\Lambda} - u^{\Lambda}) \to D_\zeta(u^{\Lambda}, \Omega)$.Then sup $\zeta(u_n^{\Lambda}) < \infty$. The closeness and convexity of Ω implies there is $u_0^{\Lambda} \in \Omega$ and a sequence $<u_n^{\Lambda}> \subseteq \Omega$ such that $u_n^{\Lambda} \to u_0^{\Lambda}$. As $\lim_{n\to\infty} \zeta(u_n^{\Lambda} - u^{\Lambda}) = u_0^{\Lambda} - u^{\Lambda}$. So,

$$\zeta(u_0^{\Lambda} - u^{\Lambda}) \leq \lim_{n\to\infty} \inf \zeta(u_n^{\Lambda} - u^{\Lambda}) = D_\zeta(u^{\Lambda}, \Omega) \leq \zeta(u_0^{\Lambda} - u^{\Lambda})$$

that is $\zeta(u_0^{\Lambda} - u^{\Lambda}) = D_\zeta(u^{\Lambda}, \Omega)$. By definition of $<u_n^{\Lambda}>$, getting$(u_n^{\Lambda} - u^{\Lambda}) \to D_\zeta(u^{\Lambda}, \Omega) = \zeta(u_0^{\Lambda} - u^{\Lambda})$.

2024, 21(3): 1080-1085
https://dx.doi.org/10.21123/bsj.2023.8230
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
Baghdad Science Journal

Since Γ_ζ is a uniformly convex, then getting$(u_n^\Lambda - u^\Lambda) \to (u_0^\Lambda - u^\Lambda)$, that is then $u_n^\Lambda \to u^\Lambda \in A$, then A is a semi compact.

Theorem 1 If Ω is semi compact subset of Γ_ζ, then Ω is a proximinal and closed

Proof: Let $v^\Lambda \in \Gamma_\zeta$. by definition of $D_\zeta(v^\Lambda,\Omega)$, from the set of the numbers$\{\zeta(v^\Lambda - u^\Lambda) : u^\Lambda \epsilon\, \Omega\}$. Now, construct a sequence $\langle \zeta(v^\Lambda - u_n^\Lambda)\rangle$ such that

$\lim_{n\to\infty} \zeta(v^\Lambda - u_n^\Lambda) = D_\zeta(v^\Lambda, \Omega)$

since Ω is a semi-compact. Then from $\langle u_n^\Lambda\rangle$, there is a subsequence converging to a point $u_0^\Lambda \in \Omega$. Hence, by the continuity of ζ getting

$\zeta(v^\Lambda - u_0^\Lambda) = \zeta\left(v^\Lambda - \lim_{i\to\infty} u_{n_i}^\Lambda\right) = \lim_{i\to\infty} \zeta(v^\Lambda - u_{n_i}^\Lambda) = D_\zeta(v^\Lambda,\Omega)$

when $u_0^\Lambda \in P_\Omega(v^\Lambda)$, the proof of proximinal is complete. Finally, for an accumulation point v^Λ of Ω, then $\exists u^\Lambda \in \Omega$ such that $(v^\Lambda - u^\Lambda) = D_\zeta(v^\Lambda,\Omega) = 0$, so $v^\Lambda \in \Omega$, and Ω is closed set.

Returning to Remark 2, to show the opposite fails, consider $\Gamma_\zeta = l^2(R)$ uniformly convex complete space with convex modular $\gamma(x) = \sqrt{\sum_1^\infty |x_i|^2}$, $\Omega = \{v \in \Gamma_\zeta ; \zeta(v) \le r\}$, $r > 0$, defined by ${u^\Lambda}_1 = 0$ and ${u^\Lambda}_n = \left(1, \frac{1}{n}, \underbrace{0,\dots,0}_{n-1}, 1,0,\dots\right)$, $n \ge 2$ is proximal but not semi compact.

Theorem 2 Let $\emptyset \ne \Omega \subseteq \Gamma_\zeta$ and Ω be a semi-compact. If $\zeta(u)^\Lambda < \infty$, for each u. Then P_Ω maps Γ_ζ into $CB(\Omega)$ is $u.s.c.$, where $CB(\Omega) = \{\Sigma : \emptyset \ne \Sigma \subset \Omega, \Sigma \text{ is closed and bounded}\}$.

Proof: By Theorem 1, Ω is proximinal set. Then $P_\Omega(v)^\Lambda$ is non– empty for each v^Λ in Γ_ζ. So, $P_\Omega(v)$ is closed and bounded (in the sense of [9]).

Thus $P_\Omega(v^\Lambda)$ maps Γ_ζ into $CB(\Omega)$. Now, let $\sum \in CB(\Omega)$ and define the set

$B = \{v^\Lambda \in \Gamma_\zeta : P_\Omega(v^\Lambda) \cap \sum \ne \emptyset\}$

To complete the proof, it is enough to prove B is closed. Let $\langle v_n^\Lambda\rangle$ be a sequence in B, converging to $v^\Lambda \in \Gamma_\zeta$. Assume $\langle v_n^\Lambda\rangle \subseteq B$, then there is a sequence $\langle u_n^\Lambda\rangle \subseteq \Omega$ such that $\langle u_n^\Lambda\rangle \in P_\Omega(v_n^\Lambda) \cap \sum$, $(n = 1,2,\dots)$. By $\langle u_n^\Lambda\rangle \in P_\Omega(v_n^\Lambda)$, $(n = 1,2,\dots)$. You have

$\lim_{n\to\infty} D_\zeta(v_n^\Lambda,\Omega) = \lim_{n\to\infty} \zeta(v_n^\Lambda - u_n^\Lambda)$

$$\begin{aligned}\Rightarrow D_\zeta(v^\Lambda,\Omega) &= \lim_{n\to\infty} \zeta(v^\Lambda - u_n^\Lambda)\\ &\le b[\lim_{n\to\infty} \zeta(v^\Lambda - v_n^\Lambda) + \lim_{n\to\infty} \zeta(v_n^\Lambda - u_n^\Lambda)]\\ &= \lim_{n\to\infty} \zeta(v^\Lambda - u_n^\Lambda)\\ &= D_\zeta(v^\Lambda,\Omega)\end{aligned}$$

Thus $\lim_{n\to\infty} \zeta(v^\Lambda - u_n^\Lambda) = D_\zeta(v^\Lambda,\Omega)$. Consequently, the semi compactness provides a subsequence $\langle u_{n_k}^\Lambda\rangle$ of $<u_n^\Lambda>$ converging to $u_0^\Lambda \in \Omega$, so, there is a subsequence $\langle v_{n_k}^\Lambda\rangle$ of $\langle v_n^\Lambda\rangle$. Now, since $u_0^\Lambda \in \Omega$, then

$$\begin{aligned}D_\zeta(v^\Lambda, \Omega) &\le \zeta(v^\Lambda - u_0^\Lambda)\\ &\le b[\zeta(v^\Lambda - u_{n_k}^\Lambda) + \zeta(u_{n_k}^\Lambda - u_0^\Lambda)]\\ &\le b^2 \zeta(v^\Lambda - v_{n_k}^\Lambda) + b^2 D_\zeta(v_{n_k}^\Lambda, \Omega) + b\zeta(u_{n_k}^\Lambda - u_0^\Lambda)\end{aligned}$$

for $k \to \infty$, $\zeta(v^\Lambda - u_0^\Lambda) = D_\zeta(v^\Lambda, \Omega)$, that is $u_0^\Lambda \in P_\Omega(v^\Lambda)$. Also, since $\sum$ is a closed and $\langle u_{n_k}^\Lambda\rangle \subseteq$ M, $\lim_{k\to\infty} u_{n_k}^\Lambda = u_0^\Lambda$ have $u_0^\Lambda \in P_\Omega(v^\Lambda) \cap \sum$. Thus, the proof is complete.

Theorem 3 Suppose that Ω Theorem 2, and $P_\Omega: \Gamma_\zeta \to 2^\Omega$ is the metric projection of Γ_ζ onto Ω. If $\sum$ compact subset of Γ then $P_\Omega(\sum) = \cup\{P_\Omega(v^\Lambda) : v^\Lambda \in \sum\}$ is compact.

Proof: Assume $\langle u_n^\Lambda\rangle$ be a sequence in $P_\Omega(\sum)$. So, there is a sequence $\langle v_n^\Lambda\rangle \subseteq \sum$ such that for each n $u_n^\Lambda \in P_\Omega(v_n^\Lambda)$, that is $\zeta(v_n^\Lambda - u_n^\Lambda) = D_\zeta(v_n^\Lambda, \Omega)$. Since $\sum$ is compact, then it may assume that there is a $v^\Lambda \in \sum$ with $v_n^\Lambda \to v^\Lambda$ and

$D_\zeta(v^\Lambda, \Omega) \le \zeta(v^\Lambda - u_n^\Lambda) \le b[\zeta(v^\Lambda - v_n^\Lambda) + D_\zeta(v_n^\Lambda, \Omega)]$

Therefore,

$$\begin{aligned}\lim_{n\to\infty} D_\zeta(v^\Lambda,\Omega) &\le \lim_{n\to\infty} \zeta(v^\Lambda - u_n^\Lambda)\\ &\le \lim_{n\to\infty} \zeta(v^\Lambda - v_n^\Lambda)\\ &\le \lim_{n\to\infty} D_\zeta(v_n^\Lambda,\Omega)\end{aligned}$$

$D_\zeta(v^\Lambda,\Omega) \le \lim_{n\to\infty} \zeta(v^\Lambda - u_n^\Lambda) \le D_\zeta(v^\Lambda,\Omega) \Rightarrow$

$D_\zeta(v^\Lambda,\Omega) = \lim_{n\to\infty} \zeta(v^\Lambda - u_n^\Lambda)$

By semi compactness of Ω and $\langle u_n^\Lambda\rangle \subseteq P_\Omega(C) \subseteq \Omega$, the above steps imply to the existences of$u^\Lambda \in \Omega$ and subsequence $\langle u_{n_i}^\Lambda\rangle$ of $\langle u_n^\Lambda\rangle$ with $u_{n_i}^\Lambda \to u^\Lambda$. This prove that $P_\Omega(\sum)$ is compact.

Conclusion

This paper includes many basic concepts and facts in the convex modular vector space, which were employed to obtain some results, such as, a semi-compact subset of modular space is closed proximinal which means that, $p_{\Omega}v^{\Lambda} \neq \emptyset$ for each $v^{\Lambda} \in \Omega$ and images of p_{Ω} is compact. In the future, you can use the results to obtain some applications in other fields, such as control.

Open Problem

It possible to combine the set $P_{\Omega}(\sum)$ into work in [11, 12] and [16] to posing the following question: Could the limit of convergence iterative sequences in [11, 12] and [16] be an invariant best approximation**?**

Acknowledgment

The authors are deeply indebted to the referees of this paper who helped us improve it in several places.

Authors' Declaration

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are ours. Furthermore, any Figures and images, that are not ours, have been included with the necessary permission for re-publication, which is attached to the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in University of Baghdad.

Authors' Contribution Statement

This work was carried out in collaboration between all authors. S S, the owner of the research idea and she reviewed and processed the work. S N, and H A, written and proved the results. All authors read and approved the final manuscript.

References

1. Singh S, Watson B, Srivastava P. Fixed point theory and best approximation: The KKM-map Principle. Dordrecht, Germany: Springer; 1997. p. 232. https://www.amazon.com/Fixed-Point-Theory-Best-Approximation/dp/0792347587
2. Abed AN, Abed SS. Convergence and stability of iterative scheme for a monotone total asymptotically non-expansive mapping. Iraqi J Sci. 2022; 63(1): 241-250. https://ijs.uobaghdad.edu.iq/index.php/eijs/article/view/4331/2130
3. Radhakrishnan M, Rajesh S, Sushama A. Some fixed point theorems on non-convex sets. Appl Gen Topol. 2017; 18(2): 377-390. https://www.researchgate.net/publication/320345812_Some_fixed_point_theorems_on_non-convex_sets
4. Jaworowski J, Kirk WA, Park S. Antipodal points and fixed points. Lect. Notes Ser. 1995; 28: 55-97. https://www.researchgate.net/publication/266752851_Antipodal_points_and_fixed_points
5. Kozlowski WM. Modular Function Spaces. Series of Monographs and Textbooks in Pure and Applied Mathematics. Vol. 122. New York: Marce Dekker Inc.; 1988. p. 252. https://www.scirp.org/(S(351jmbntvnsjt1aadkposzje))/reference/referencespapers.aspx?referenceid=2125146
6. Harjulehto P, Hästö P. Extension in generalized Orlicz spaces. Nonlinear Stud. 2019; 26(4): 861–868. https://arxiv.org/pdf/1910.03893.pdf
7. Harjulehto P, Hästö P. Orlicz spaces and generalized Orlicz spaces. Lecture Notes in Mathematics. Vol. 2236. New York: Springer Cham; 2019. p. 180. http://harjulehto.fi/pdf/Book-gOrlicz.pdf
8. Abed SS. On invariant best approximation in modular spaces. Glob. J Pure Appl Math. 2017; 13(9): 5227-5233. https://www.ripublication.com/gjpam17/gjpamv13n9_102.pdf
9. Turkoglu S, Nesrin M. Fixed point theorems in a new type of modular metric spaces. J Fixed Point Theory Appl. 2018; 25: 1-9. https://fixedpointtheoryandalgorithms.springeropen.com/articles/10.1186/s13663-018-0650-3

10. Albundi ShS, Iterated function system in Ø-metric spaces. Bol da Soc Parana de Mat, 2022; 40: 1-10. https://periodicos.uem.br/ojs/index.php/BSocParanMat/article/view/52556/751375153614
11. Abdul Jabbar MF, Abed SS. Some results on normalized duality mappings and approximating fixed points in convex real moduler spaces. Baghdad Sci J. 2021; 18(4): 1218-1225. https://bsj.uobaghdad.edu.iq/index.php/BSJ/article/view/4322
12. Abdul Jabbar MF, Abed SS. The convergence of iteration scheme to fixed points in modular spaces. Iraqi J Sci. 2020; 60(10): 2197-2202. https://ijs.uobaghdad.edu.iq/index.php/eijs/article/view/1037
13. Kadhim AJ. New common fixed Points for total asymptotically nonexpansive mapping in cat0) Space. Baghdad Sci J. 2021; 18(4): 1286-1293. https://bsj.uobaghdad.edu.iq/index.php/BSJ/article/view/4639
14. Mohammed NJ, Abed SS. On invariant approximations in modular spaces. Ira. J Sci. 2021; 62(9): 3097-3101. https://ijs.uobaghdad.edu.iq/index.php/eijs/article/view/3525
15. Pathak HK, Beg I. Fixed point of multivalued contractions by altering distances with application to nonconvex Hammerstein type integral inclusions. Fixed Point Theory. 2021; 22(1): 327-342. https://www.math.ubbcluj.ro/~nodeacj/volumes/2021-No1/211-pat-beg-3051-final.php
16. Abed SS, Abduljabbar MF. Approximating fixed points in modular spaces. Karbala Int J Mod Sci. 2020; 6(2): 121-128. https://kijoms.uokerbala.edu.iq/cgi/viewcontent.cgi?article=1353&context=home
17. Abed S S, Abdul Jabber M.F, Equivalence between iterative schemes in modular spaces. J Interdiscip Math.2019; 22(.8): 1529–1535. https://www.tandfonline.com/doi/abs/10.1080/09720502.2019.1706850
18. Ege ME, Alaca C. Some results for modular b-Metric spaces and an application to system of linear equations. Azerb J Math. 2018; 8(1): 1-13. https://www.azjm.org/volumes/0801/0801-1.pdf
19. Salman BB, Abed SS, A new iterative sequence of (λ,ρ)-firmly nonexpansive multivalued mappings in modular function Spaces. Math. Mod Eng . Prob . 2022; 10 (1): 212-219. file:///C:/Users/DELL/Downloads/mmep_10.01_24%20(3).pdf

افضل تقريب في فضاءات b - المعيارية

هبة عادل جبار[1]، سميرة ناجي كاظم[2]، سلوى سلمان عبد[3]

[1] مدرسة المهج، مديرية تربية بغداد الرصافة الاولى، وزارة التربية العراقية، بغداد، العراق.
[2]قسم الرياضيات، كلية العلوم للبنات، جامعة بغداد، بغداد، العراق.
[3]قسم الرياضيات، كلية التربية للعلوم الصرفة- ابن الهيثم، جامعة بغداد، بغداد، العراق.

الخلاصة

في البداية، نعطي بعض المفاهيم والحقائق الأساسية في الفضاء ب- المعياري عل غرارتلك الموجودة في المساحات المعيارية كنوع من التعميم ، على سبيل المثال ، مفاهيم التقارب ، أفضل التقريب ، التحدب المنتظم وما إلى ذلك ، ثم تم البرهنة على نتيجتين حول العلاقة بين شبه التراص والتقريب والتي استخدمت لإثبات مبرهنة حول وجود أفضل تقريب لمجموعة جزئية شبه متراصة من الفضاء ب- المعياري.

الكلمات المفتاحية: b- فضاء المعياري، أفضل تقريب، النقطة الصامدة، شبه التراص، تطبيق متعدد القيم.