THE COMPACT OF THE COMPOSITION OPERATOR C_λ

Ammar Ali Neamah [†] ,
Mathematical Department,
Faculty of Mathematics and Computer Science
Kufa University

Aqeel Mohammed Hussein [‡]
Mathematical Department,
College of Education
Qadisyia University

Abstract

Let U denote the unit ball in the complex plane, the Hardy space H^2 is the set of functions $f(z)=\sum_{n=0}^{\infty} f^{\wedge}(n)\, z^n$ holomorphic on U such that $\sum_{n=0}^{\infty} \left|f^{\wedge}(n)\right|^2 < \infty$ with $f^{\wedge}(n)$ denotes the Taylor coefficient of f .

Let ψ be a holomorphic self-map of U, the composition operator C_ψ induced by ψ is defined on H^2 by the equation
$C_\psi f = f \circ \psi \quad (f \in H^2)$

We have studied the composition operator induced by the bijective map λ and discussed the adjoint of the composition operator .We have look also at some known properties of composition operator and tried to get the analogue properties in order to show how the results are changed by changing the map ψ in U.

In order to make the work accessible to the reader , we have included some known results with the details of the proofs for some cases and proved some results.

Introduction :

This search consists of two sections. In section one ,we are going study to the map λ and properties of λ , and also discuss λ as inner map.

In section two, we are going study to the composition operator C_λ induced by adjoint of the operator C_λ induced by the map λ and also discuss the compactness of the operator C_λ .

1. Section One

We are going study to the map λ and properties of λ , and also discuss λ is an inner map.

Definition (1.1) : [4]

The set $U = \{z \in C : |z| < 1\}$ is called unit ball in complex C and $\partial U = \{z \in C : |z| = 1\}$ is called boundary of U.

Definition (1.2) :

For $\beta \in U$, define $\lambda(z) = \dfrac{\bar{\beta}z - 1}{z - \beta} \quad (z \in U)$.

Since the denominator equal zero only at $z = \beta$, the function λ is holomorphic on the ball$\{\ |z| < |\beta|\ \}$. Since $\beta \in U$. Then this ball contain U .Hence λ take U into U and holomorphic on U .

Proposition (1.3) :

For $\beta \in U$, $|\lambda(z)|^2 - 1 = \dfrac{\left(1-|z|^2\right)\left(1-|\beta|^2\right)}{|z-\beta|^2}$

Proof :

$$|\lambda(z)|^2 - 1 = \left|\frac{\bar{\beta}z - 1}{z-\beta}\right|^2 - 1 = \frac{|\bar{\beta}z - 1|^2}{|z-\beta|^2} - 1 =$$

$$\frac{|\bar{\beta}z - 1|^2 - |z-\beta|^2}{|z-\beta|^2} = \frac{(\bar{\beta}z-1)(\beta\bar{z}-1) - (z-\beta)(\bar{z}-\bar{\beta})}{|z-\beta|^2}$$

$$= \frac{|\beta|^2|z|^2 - \bar{\beta}z - \beta\bar{z} + 1 - |z|^2 + \bar{\beta}z + \beta\bar{z} - |\beta|^2}{|z-\beta|^2}$$

the map λ and properties of C_λ, discuss the

$$= \frac{\left(1-|z|^2\right)\left(1-|\beta|^2\right)}{|z-\beta|^2}$$

Proposition (1.4) :

If $\beta \in U$, then λ take ∂U into ∂U.

Proof :

Let $z \in \partial U$, then $|z| = 1$, hence $|z|^2 = 1$. By (1.3) $|\lambda(z)|^2 - 1 = 0$, therefore $|\lambda(z)|^2 = 1$, hence $|\lambda(z)| = 1$, hence $\lambda(z) \in \partial U$, hence λ take ∂U into ∂U.

Definition (1.5) : [7]

Let $\psi : U \to U$ be holomorphic map on U, ψ is called an inner map if $|\psi(z)| = 1$ almost everywhere on ∂U.

Proposition (1.6) :

λ is an inner map .

Proof :

From (1.4) λ take ∂U into ∂U and $|\lambda(z)| = 1$. By (1.5) λ is an inner.

2. Section Two

We are going study to the composition operator C_λ induced by the map λ , and it's properties, also discuss the adjoint of the operator C_λ , and the compactness of C_λ .

Definition (2.1) : [4]

Let U denote the unit ball in the complex plane, the Hardy space H^2 is the set of functions $f(z) = \sum_{n=0}^{\infty} f\,^\wedge(n)\, z^n$ holomorphic on U, where z ∈ C such that $\sum_{n=0}^{\infty} |f\,^\wedge(n)|^2 < \infty$ with $f\,^\wedge(n)$ denotes the Taylor coefficient of f .

Remark (2.2) :[1]

We can define an inner product of the Hardy space functions as follows:

Let f(z) = $\sum_{n=0}^{\infty} f\,^\wedge(n)\, z^n$ and $g(z) = \sum_{n=0}^{\infty} g\,^\wedge(n)\, z^n$, then the inner product of f and g is defined by : $\langle f, g\rangle = \sum_{n=0}^{\infty} f\,^\wedge(n)\, \overline{g\,^\wedge(z)}$.

Definition (2.3) :[9]

Let $\alpha \in U$,define $K_\alpha(z) = \dfrac{1}{1-\overline{\alpha} z}$ $(z \in U)$.

Since α ∈ U then $|\alpha| < 1$ hence the geometric series $\sum_{n=0}^{\infty} |\alpha|^{2n}$ is convergent and thus $K_\alpha \in H^2$ and $K_\alpha(z) = \sum_{n=0}^{\infty} \left(\overline{\alpha}\right)^n z^n$.

Definition (2.4) : [4]

Let $\psi : U \to U$ be holomorphic map on U, the composition operator C_ψ induced by ψ is defined on H^2 by the equation $C_\psi f = f \circ \psi$ $\left(f \in H^2\right)$.

Definition (2.5) : [2]

Let T be a bounded operator on a Hilbert space H, then the norm of an operator T is defined by $\|T\| = \sup\{\|Tf\| : f \in H, \|f\| = 1\}$.

Theorem (2.6) : [10]

If $\psi : U \to U$ is holomorphic map on U, then $f \circ \psi \in H^2$ and $\|f \circ \psi\| \le \sqrt{\dfrac{1+|\psi(0)|}{1-|\psi(0)|}}\ \|f\|$ for every $f \in H^2$.

The goal of this theorem $C_\psi : H^2 \to H^2$.

Definition (2.7) :

The composition operator C_λ induced by λ is defined on H^2 as follows $C_\lambda f = f \circ \lambda$.

Proposition (2.8) :

For each $f \in H^2$ we have $f \circ \lambda \in H^2$ and and $\|f \circ \lambda\| \le \sqrt{\frac{1+|\lambda(0)|}{1-|\lambda(0)|}}\ \|f\|\cdot$

Proof :

Since $\lambda : U \to U$ is holomorphic map on U, then by (2.6) $f \circ \lambda \in H^2$ and $\|f \circ \lambda\| \le \sqrt{\frac{1+|\lambda(0)|}{1-|\lambda(0)|}}\ \|f\|$, hence C_λ: $H^2 \to H^2$

Remark (2.9) : [4]

1) One can easily show that $C_\kappa C_\psi = C_{\psi \circ \kappa}$ and hence $C_\psi^n = C_\psi C_\psi \cdots C_\psi = C_{\psi \circ \psi \circ \cdots \circ \psi} = C_{\psi_n}$
2) C_ψ is the identity operator on H^2 if and only if ψ is identity map from U into U and holomorphic on U.
3) It is simple to prove that $C_\kappa = C_\psi$ if and only if $\kappa = \psi$.

Definition (2.10) : [3]

Let T be an operator on a Hilbert space H, The operator T^* is the adjoint of T if $\langle Tx, y\rangle = \langle x, T^* y\rangle$ for each $x, y \in H$.

Theorem (2.11) : [5]

$\{K_\alpha\}_{\alpha \in U}$ forms a dense subset of H^2.

Theorem (2.12) : [9]

If $\psi : U \to U$ is holomorphic map on U, then for all $\alpha \in U$

$C_\psi^* K_\alpha = K_{\psi(\alpha)}$

Definition (2.13) : [10]

Let H^∞ be the set of all bounded holomorphic maps on U .

Definition (2.14) : [6]

Let $g \in H^\infty$, the Toeplits operator T_g is the operator on H^2 given by :

$$(T_g f)(z) = g(z)\ f(z) \quad (f \in H^2,\ z \in U).$$

Theorem (2.15) : [6]

If $\psi : U \to U$ is holomorphic map on U, then $C_\psi T_g = T_{g \circ \psi} C_\psi \quad (g \in H^\infty)$

Remark (2.16) : [8]

For each $f \in H^2$, it is well- know that $T_h^*\ f = T_{\bar{h}}\ f$, such that $h \in H^\infty$.

Proposition (2.17) :

If $\beta \in U$, then $C_\lambda^* = T_g\ C_\gamma\ T_h^*$, where $h(z) = (z - \beta)$, $g(z) = \frac{1}{z - \bar{\beta}}$, $\gamma(z) = \frac{\beta z - 1}{z - \bar{\beta}}$.

Proof :

By (2.16), $T_h^*\ f = T_{\bar{h}}\ f$ for each $f \in H^2$. Hence for all $\alpha \in U$,

$$\left\langle T_{\bar{h}}^*\ f, K_\alpha \right\rangle = \left\langle T_{\bar{h}} f, K_\alpha \right\rangle = \left\langle f, T_{\bar{h}}^*\ K_\alpha \right\rangle \ \cdots\cdots \ (1)$$

On the other hand ,

$$\left\langle T_{\bar{h}}^*\ f, K_\alpha \right\rangle = \langle f, T_h K_\alpha \rangle = \langle f, h(\alpha) K_\alpha \rangle \ \cdots\cdots \ (2)$$

From (1) and (2) one can see that $T_{\bar{h}}^*\ k_\alpha = h(\alpha)\ k_\alpha$. Hence $T_h^*\ k_\alpha = \overline{h(\alpha)}\ k_\alpha$.

Calculation give

$$C_\lambda^* k_\alpha(z) = k_{\lambda(\alpha)}(z)$$

$$= \frac{1}{1 - \overline{\gamma(\alpha)}\ z} = \frac{1}{1 - \frac{(\beta\bar{\alpha} - 1)\ z}{\bar{\alpha} - \bar{\beta}}}$$

$$= \frac{1}{\frac{\bar{\alpha} - \bar{\beta} - \beta\ \bar{\alpha}\ z + z}{\bar{\alpha} - \bar{\beta}}} = \frac{\bar{\alpha} - \bar{\beta}}{(z - \bar{\beta}) - \bar{\alpha}\ (\beta\ z - 1)}$$

$$= \overline{(\alpha - \beta)} \cdot \frac{1}{z - \bar{\beta}} \cdot \frac{1}{1 - \bar{\alpha}\left(\frac{\beta\ z - 1}{z - \bar{\beta}}\right)}$$

$$= \overline{h(\alpha)} \cdot T_g\ k_\alpha(\gamma(z)) = T_g\ \overline{h(\alpha)}\ k_\alpha(\gamma(z))$$

$$= T_g\ \overline{h(\alpha)}\ C_\gamma\ k_\alpha(z) = T_g\ C_\gamma\ \overline{h(\alpha)}\ k_\alpha(z)$$

$= T_g\ C_\gamma\ T_h^*\ k_\alpha(z)$, therefore

$C_\lambda^* k_\alpha(z) = T_g\ C_\gamma\ T_h^*\ k_\alpha(z)\ \ (z \in U)$.

But $\overline{\{K_\alpha\}_{\alpha \in U}} = H^2$, then $C_\lambda^* = T_g\ C_\gamma\ T_h^*$

Definition (2.18) : [11]

Let T be an operator on a Hilbert space H , T is called compact, if every sequence $\langle x_n \rangle$ in H is weakly converges to x in H ((i.e. $x_n \xrightarrow{W} x$ if $\langle x_n, u\rangle \to \langle x, u\rangle, \forall u \in H$)) then Tx_n is strongly converges to Tx ((i.e. $x_n \xrightarrow{S} x$ if $\|x_n - x\| \to 0$))

Theorem (2.19) : [9]

If $\psi : U \to U$ is holomorphic map on U, then C_ψ is not compact if and only if ψ take ∂U into ∂U.

Proposition (2.20) :

If $\beta \in U$, then C_λ is not compact composition operator.

Proof :

From (1.4), λ take ∂U into ∂U. By (2.19) C_λ is not compact composition operator.

Theorem (2.21) :

If $\psi : U \to U$ is holomorphic map on U, then $C_\psi C_\lambda^*$ is compact if and only if $C_\psi C_\gamma$ is compact , where $C_\lambda^* = T_g\ C_\gamma\ T_h^*$, $\gamma(z) = \dfrac{\beta z - 1}{z - \bar{\beta}}$.

Proof :

Suppose that $C_\psi C_\gamma$ is compact . Note that $C_\psi C_\lambda^* = C_\psi\ T_g\ C_\gamma\ T_h^*$ (since $C_\lambda^* = T_g\ C_\gamma\ T_h^*$ by (2.17)) = $T_{g \circ \psi}\ C_\psi\ C_\gamma\ T_h^*$ (since $C_\psi\ T_g = T_{g \circ \psi} C_\psi$ by (2.15)). $T_{g \circ \psi}$ and T_h^* are bounded operators then $C_\psi C_\lambda^*$ is compact by (2.18)

Conversely, Suppose that $C_\psi C_\lambda^*$ is compact. Note that $C_\psi C_\gamma = C_\psi (C_\gamma^*)^* = C_\psi (T_g\ C_\lambda\ T_h^*)^* = C_\psi\ T_h\ C_\lambda^*\ T_g^* = T_{h \circ \psi}\ C_\psi\ C_\lambda^*\ T_g^*$ (since $C_\psi\ T_h = T_{h \circ \psi} C_\psi$ by (2.15)).

Since $C_\psi C_\lambda^*$ is compact operator , $T_{h \circ \psi}$ and T_g^* are bounded operators by (2.13) and (2.14) then $C_\psi C_\gamma$ is compact by (2.18).

Corollary (2.22) :

If $\psi : U \to U$ is holomorphic map on U, then $C_\psi C_\lambda^*$ is not compact if and only if there exist points $z_1, z_2 \in \partial U$ such that $(\gamma \circ \psi)(z_1) = z_2$ for each $z_2 \in \partial U$.

Proof :

By (2.21) $C_\psi C_\lambda^*$ is not compact if and only if $C_\psi C_\gamma = C_{\gamma \circ \psi}$ is not compact. Since $\gamma : U \to U$ and $\psi : U \to U$ are holomorphics on U, then also $\gamma \circ \psi$. Thus by (2.19) $C_{\gamma \circ \psi}$ is not compact if and only if $\gamma \circ \psi$ take ∂U into ∂U. So, there exist points $z_1, z_2 \in \partial U$ such that $(\gamma \circ \psi)(z_1) = z_2$ for each $z_2 \in \partial U$.

Theorem (2.23) :

If $\psi : U \to U$ is holomorphic map on U, then $C_\lambda^* C_\psi$ is compact if and only if $C_\gamma C_\psi$ is compact , where $C_\lambda^* = T_g\ C_\gamma\ T_h^*$, $\gamma(z) = \dfrac{\beta z - 1}{z - \bar{\beta}}$.

Proof :

Suppose that $C_\gamma C_\psi$ is compact . Note that

$C_\lambda^* C_\psi = T_g\ C_\gamma\ T_h^*\ C_\psi$ (since $C_\lambda^* = T_g\ C_\gamma\ T_h^*$ by (2.17))

$= T_g\ C_\gamma\ T_{\bar{h}}\ C_\psi$ (by (2.16))

$= T_g\ T_{\bar{h} \circ \gamma}\ C_\gamma\ C_\psi$ (since $C_\gamma\ T_{\bar{h}} = T_{\bar{h} \circ \gamma} C_\gamma$ by(2.15)).

Since $C_\gamma C_\psi$ is compact operator , T_g and $T_{\bar{h} \circ \gamma}$

Since $C_\psi C_\gamma$ is compact operator ,
are bounded operators then $C_\lambda^* C_\psi$ is compact by (2.18)

Conversely, Suppose that $C_\lambda^* C_\psi$ is compact . Note that

$$C_\gamma C_\psi = \left(C_\gamma^*\right)^* C_\psi = \left(T_g\ C_\lambda\ T_h^*\right)^* C_\psi$$

$$\text{(since } C_\gamma^* = T_g\ C_\lambda\ T_h^*\text{)}$$

$$= T_h\ C_\lambda^*\ T_g^*\ C_\psi$$

Note that , by (2.11) it is enough to prove the compactness on the family $\{K_\alpha\}_{\alpha \in U}$. Hence for each $z \in U$ we have

$$C_\gamma C_\psi K_\alpha(z) = T_h\ C_\lambda^*\ T_g^*\ C_\psi K_\alpha(z)$$

$$= T_h\ C_\lambda^*\ T_g^*\ K_\alpha(\psi(z)) = T_h\ C_\lambda^*\ \overline{g(\alpha)}\ K_\alpha(\psi(z))$$

$$\text{(since } T_g^*\ K_\alpha = \overline{g(\alpha)}\ K_\alpha\text{)}$$

$$= \overline{g(\alpha)}\ T_h\ C_\lambda^*\ K_\alpha(\psi(z))$$

$= \overline{g(\alpha)}\ T_h\ C_\lambda^*\ C_\psi\ K_\alpha(z)$ Since $C_\lambda^* C_\psi$ is compact , T_h is bounded and $g \in H^\infty$, then $C_\gamma C_\psi$ is compact by (2.18).

Corollary (2.24) :

If $\psi : U \to U$ is holomorphic map on U, then $C_\lambda^* C_\psi$ is not compact if and only if there exist points $z_1, z_2 \in \partial U$ such that $(\psi \circ \gamma)(z_1) = z_2$ for each $z_2 \in \partial U$.

Proof :

By (2.23) $C_\lambda^* C_\psi$ is not compact if and only if $C_\gamma C_\psi = C_{\psi\circ\gamma}$ is not compact . Since $\gamma : U \to U$ and $\psi : U \to U$ are holomorphic on U, then also $\psi \circ \gamma$. Thus by (2.19) $C_{\psi\circ\gamma}$ is not compact if and only if $\psi \circ \gamma$ take ∂U into ∂U . So, there exist points $z_1, z_2 \in \partial U$ such that $(\psi \circ \gamma)(z_1) = z_2$ for each $z_2 \in \partial U$.

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المستخلص:

ليكن U يرمز إلى كرة الوحدة في المستوى العقدي، إن فضاء هاردي H^2 هو مجموعة كل الدوال $f(z) = \sum_{n=0}^{\infty} f^{\wedge}(n)\ z^n$ التحليلية على U بحيث أن $\sum_{n=0}^{\infty} \left|f^{\wedge}(n)\right|^2 < \infty$ و $f^{\wedge}(n)$ يرمز إلى تيلر للدالة f.

لتكن $\psi : U \to U$ دالة تحليلية على U ، المؤثر التركيبي المعرف بـ ψ يعرف على فضاء هاردي H^2 بالشكل التالي: $C_{\psi} f = f \circ \psi \quad (f \in H^2)$

درسنا في هذا البحث المؤثر التركيبي المعرف من الدالة المتقابلة λ حيث ناقشنا المؤثر المرافق للمؤثر التركيبي المعرف بالدالة λ. بالإضافة إلى ذلك نظرنا إلى بعض النتائج المعروفة وحاولنا الحصول على نتائج مناظرة لنتمكن من ملاحظة كيفية تغير النتائج عندما تتغير الدالة التحليلية ψ . ومن أجل جعل مهمة القارئ أكثر سهولة، عرضنا بعض النتائج المعروفة عن المؤثرات التركيبية وعرضنا براهين مفصلة وكذلك برهنا بعض النتائج .