Stability of Linear Large Scale System

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Abstract

In this paper the stability of linear large scale system is studied, by using definite matrices. The concept of family of linear system is introduced, by using permutation matrices. A proposition is introduced to study the linear system in the large.

1. Introduction

The study of behavior and stability of large scale dynamic systems is difficult when it is taken in one piece. However, this system may be transformed into lower order interconnected subsystems. The stability of the large scale dynamic system is studied via the stability of the individual subsystem and the properties of the interconnected functions.

For updating studies, Martynuk [1] introduced a new direction in non linear large dynamic systems. This direction is closely connected with a new class of matrix valued in particular of Lyapunov function. Ohta et al [2] study Lyapunov stability of nonlinear hereditary composite systems using vector Lyapunov functions method with nonlinear comparison systems. Griggs et al [3] studied feedback interconnection consisting of two nonlinear systems is shown to be input-output stable when a “mixed” small gain and passivity assumption is placed on each of the systems.

In this paper, the important proposition is introduced to study the new properties of matrices. Furthermore, test examples are constructed and used to illustrate each of the studied propositions in order to compare the effectiveness of propositions.

2. Some Properties of Definite Matrices

In this section the basic definitions and theorems of matrices was studied. New properties of definite matrices was introduced.

Definition 1

The $n \times n$ two matrices A and B are called congruent if there exist non singular matrix P such that $P^T AP = B$ [4].

Lemma 1

$P^T AP$ is symmetric if A is symmetric [4].

Lemma 2

Let Q and Q' be the quadratic forms of A congruent B respectively, then Q(x)=Q'(y) [4].

Theorem 1

Let A be $n \times n$ real symmetric matrix, then the following properties are equivalent:

(1) $\boldsymbol{det}(\boldsymbol{A_j}) > 0$, $j = 1,2, \dots, n$
(2) $\boldsymbol{Q}(\boldsymbol{x}) = \boldsymbol{x^T A x} > 0$ **for all n-vector x**
(3) Eigenvalues of A are positive [4].

Theorem 2

Let A be $n \times n$ real symmetric matrix, then the following properties are equivalent:

(1) $(-\boldsymbol{1})^j \boldsymbol{det}(\boldsymbol{A_j}) < 0$, $j = 1,2, \dots, n$
(2) $\boldsymbol{Q}(\boldsymbol{x}) = \boldsymbol{x^T A x} < 0$ **for all n-vector x**
(3) Eigenvalues of A are negative [4].

Theorem 3

Let P be $n \times n$ real symmetric matrix has a form:

$$\boldsymbol{P} = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ \vdots & & & \vdots \\ P_{n1} & P_{n2} & & P_{nn} \end{bmatrix}$$

And E represents the elementary row operation which adds multiples of row to another, and let $\overline{P}$ represents the upper triangular matrix which obtained from P by using steps of E, where:

$$\overline{\boldsymbol{P}} = \begin{bmatrix} \overline{P_{11}} & \overline{P_{12}} & \cdots & \overline{P_{1n}} \\ 0 & \overline{P_{22}} & \cdots & \overline{P_{2n}} \\ \vdots & 0 & \ddots & \\ 0 & 0 & \cdots & \overline{P_{nn}} \end{bmatrix}$$

Then the matrix P is positive definite [negative definite] if $\overline{P_{ii}} > 0$ [$\overline{P_{ii}} < 0$] $\boldsymbol{for\ i} = \boldsymbol{1, 2, 3, \dots, n}$.[5]

Proposition 1

For positive definite p we have $\boldsymbol{P^T A P}$ is definite if A if A is definite.

Proof: without loss of generality by take A is positive definite.

1) **By lemma (1) $P^T AP$ is symmetric.**
2) **Let $B = P^T AP$, since A is positive definite, then by theorem (1) $Q(x) > 0$ and by applying lemma (1-2), we have $Q'(y) > 0$.**

Then by using theorem (1-1) B is positive definite.

In the same manner, by take theorem (2), to prove $P^T AP$ be negative definite.

3. Stability of Linear Large Scale System

In this section, the basic theorem of linear large scale system was studied. New proposition of definite matrix was introduced to make a family of linear system, finely introduced simple method of Martynuk approach for linear large scale system for introduced system.

Theorem 4 :For the linear system $x' = Ax$ the zero solution is asymptotically stable iff there exists a positive definite matrix R which is unique solution of the Lyapunov matrix equation for any given positive definite Q [6,7].

Proposition 2

For the linear system $x' = Ax$, if A is stable then the family of matrices $P^T(A^T + A)P$ is stable for all non singular permutation matrices of the identity matrix I.

Proof:

Since A is stable, then by using theorem 4 there exists two positive definite matrices, R and Q such that

$$Q = -(A^T R + RA)$$

Let R be the identity, then

$$Q = -(A^T + A)$$

By using non singular matrix P, we have

$$P^T QP = -P^T(A^T R + RA)P$$

$$P^T QP = P^T SP\ , S = -(A^T + A)$$

Since Q is positive definite, then by using proposition 1, we have $P^T QP$ is positive definite. Then $P^T SP$ is positive definite. Then by using note, we have $P^T(A^T + A)P$ is negative definite.

Now consider the following linear system

$$x' = Ax \qquad (1)$$

Where A is $n \times n$ constant matrix. Since the matrix A has a negative real part then the system (1) is stable.

It is natural to assume that a large scale of (1) is composed of interconnected subsystem given by

$$x_i' = A_{ii}x_i + \sum_{\substack{j=1 \\ i\neq j}}^{m}\{A_{ij}\}x_{ij} \qquad i = 1, 2, \ldots\ldots, m \qquad (2)$$

Where $x_i \in R^{n_i}$, $x_{ij} \in R^{n_i} \times R^{n_j}$ and $x_{ij} = (x_i, x_j)^T$ the state vectors of (2), $\{A_{ij}\}$ are 2×2 block matrices given from the matrix A.

The m isolated subsystems of (2) are given by

$$x_i' = A_{ii}x_i \qquad (3)$$

Where A_{ii} are the main diagonal elements of the matrix A.

To study the stability of interconnected system

Choose the scalar Lyapunov function

$$v_{ii}(x_i) = x_i^T P_{ii} x_i \; for \quad i = 1, 2, \ldots\ldots, m \qquad (4)$$

The matrices P_{ii} are given by solving the Lyapunov equations

$$A_{ii}^T P_{ii} + P_{ii}A_{ii} = -G_{ii} \quad (i = 1, 2, \ldots\ldots, m) \qquad (5)$$

The interconnection structure of system (2-2) given by

$$g_{ij}(x_i, x_j) = \sum_{\substack{j=1 \\ i\neq j}}^{m}\{A_{ij}\}x_{ij} \qquad i \neq j, \; i = (1, 2, \ldots\ldots, m) \qquad (6)$$

The matrices $\{A_{ij}\}$ are given by

$$\{A_{ij}\} = \begin{bmatrix} A_{ii} & A_{ij} \\ A_{ji} & A_{jj} \end{bmatrix} \qquad (7)$$

where $A_{ij} \in A$ $\; i, j = 1, 2, \ldots m$, $i \neq j$.

Choose the scalar Lyapunov function

$$v_{ij}(x_i, x_j) = v_{ji}(x_j, x_i) = x_i^T P_{ij} x_j \qquad (8)$$

Since $P_{ij} = P_{ji} \rightarrow A_{ij} = A_{ji}$.

From above relation we can get the matrices P_{ij} we get the following equations

By using the matrices $\{A_{ij}\}$ we can solve the following Lyapunov equations

$$A_{ij}^T P_{ij} + P_{ij}A_{ij} = -G_{ij} \qquad (9)$$

with $\eta_i > 0$ and $\eta_j > 0$

$i \neq j \quad i, j = 1,2,3, \ldots\ldots\ldots m$

Form (4) and (8) we get

$$v_{ii}(x_i) \leq -\lambda_M(P_{ii})\|x_i\|^2 \qquad x_i \in R^{n_i} \qquad (10)$$

$$v_{ij}(x_i, x_j) \leq \lambda_M^{\frac{1}{2}}(P_{ij}P_{ij}^T)\|x_i\|\|x_j\|, \quad (x_i, x_j) \in R^{n_i} \times R^{n_j} \qquad (11)$$

For function $v(x,\eta)=\eta^T U(x)\eta$ is positive definite and the test matrix S will be in the form

$$S=\begin{bmatrix} -\lambda_M(P_{11}) & \cdots\cdots & \lambda_M^{1/2}(P_{ij}P_{ij}^T) \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ \lambda_M^{1/2}(P_{ij}P_{ij}^T) & \cdots\cdots & -\lambda_M(P_{ii}) \end{bmatrix}$$
(12)

Here $\lambda_M(P_{ii})$ are the maximum eigenvalues of the matrices P_{ii} and $\lambda_M^{1/2}(P_{ij}P_{ij}^T)$ is the norm of the matrices P_{ij} .

The following proposition gives the result of stability of linear large scale system in the general for all m interconnected subsystem.

Proposition 3

Suppose that for the system (1) with interconnected system (2) the following conditions are satisfied.

1) Each isolated subsystem is asymptotically stable.
2) The matrix valued function is positive definite.
3) The test matrix is negative definite.

Then the equilibrium point x=0 of system (1) is asymptotically stable.

Proof: - since (8) is positive definite. It is enough to show that the total derivative is negative definite.

Now from (10) and (11) and by letting vector $u=(\|x_1\|,\|x_2\|,\ldots,\|x_m\|)^T$

we get

$$Dv(x,\eta)=\sum_{i=1}^{m} x_i^T P_{ii}x_i + 2\sum_{i=1}^{m}\sum_{j=i+1}^{m} x_i^T P_{ij}x_j \le \sum_{i=1}^{m} -\lambda_M(P_{ii})\|x_i\|^2 + 2\sum_{i=1}^{m}\sum_{j=i+1}^{m} \lambda_M^{1/2}(P_{ij}P_{ij}^T)\|x_i\|\|x_j\| = u^T S u$$

Since S is negative definite, hence, according to Lyapunov direct method the proof of proposition is complete.

4. Examples

For explaining our proposed propositions, and without of loss of generality, following examples of small order are implemented.

Example1

By Considering the following linear system

$$x'=\begin{bmatrix} -1 & 0 & 1 \\ 1 & -2 & 0 \\ 0 & 1 & -1 \end{bmatrix} x$$

Cleary all permutation matrices are

$\boldsymbol{P_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\boldsymbol{P_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $\boldsymbol{P_3} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\boldsymbol{P_4} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, $\boldsymbol{P_5} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $\boldsymbol{P_6} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

And $\boldsymbol{D = A^T + A} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -4 & 1 \\ 1 & 1 & -2 \end{bmatrix}$

All family of matrices of D are given in the following by using proposition (2)

$\boldsymbol{D_1 = D = P_1 D P_1^T}$

$\boldsymbol{D_2 = P_2 D P_2^T} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -4 \end{bmatrix}$,

$\boldsymbol{D_3 = P_3 D P_3^T} = \begin{bmatrix} -4 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$

$\boldsymbol{D_4 = P_4 D P_4^T} = \begin{bmatrix} -4 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$,

$\boldsymbol{D_5 = P_5 D P_5^T} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -4 \end{bmatrix}$

$\boldsymbol{D_6 = P_6 D P_6^T} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -4 & 1 \\ 1 & 1 & -2 \end{bmatrix}$

Each of above symmetric matrix is negative definite. This matrices make a 6-families of stable system deduced from one stable system, by using proposition (2).

Example 2

By Considering the following composite linear system

$$\boldsymbol{x'} = \begin{bmatrix} -3 & -2 & 2 & -2 \\ 3 & -4 & 1 & -1 \\ 3 & 3 & -4 & -3 \\ 4 & -4 & 0 & -6 \end{bmatrix} \boldsymbol{x}$$

The matrices $\{\boldsymbol{A_{ij}}\}$ **are given by the follows**

$\{\boldsymbol{A_{ij}}\} = \begin{bmatrix} A_{ii} & A_{ij} \\ A_{ji} & A_{jj} \end{bmatrix}$ $\boldsymbol{i, j} = \boldsymbol{1, 2, 3, 4}$, $\boldsymbol{i \neq j \; and \; i} < j$

$\{\boldsymbol{A_{12}}\} = \begin{bmatrix} -3 & -2 \\ 3 & -4 \end{bmatrix}$, $\{\boldsymbol{A_{13}}\} = \begin{bmatrix} -3 & 2 \\ -4 & -3 \end{bmatrix}$, $\{\boldsymbol{A_{14}}\} = \begin{bmatrix} -3 & -2 \\ 4 & -6 \end{bmatrix}$

$\{\boldsymbol{A_{23}}\} = \begin{bmatrix} -4 & 1 \\ 3 & -4 \end{bmatrix}$, $\{\boldsymbol{A_{34}}\} = \begin{bmatrix} -4 & -3 \\ 6 & -6 \end{bmatrix}$ **and** $\{\boldsymbol{A_{24}}\} = \begin{bmatrix} -4 & -1 \\ -4 & -6 \end{bmatrix}$

The interconnected system of (6) is given by

$$\boldsymbol{x_1'} = -3\boldsymbol{x_1} + \begin{bmatrix} -3 & -2 \\ 3 & -4 \end{bmatrix} \boldsymbol{x}_{12} + \begin{bmatrix} -3 & 2 \\ -4 & -3 \end{bmatrix} \boldsymbol{x}_{13} + \begin{bmatrix} -3 & -2 \\ 4 & -6 \end{bmatrix} \boldsymbol{x_{14}}$$

$$x_2' = -4x_2 + \begin{bmatrix} -3 & -2 \\ 3 & -4 \end{bmatrix} x_{21} + \begin{bmatrix} -4 & 1 \\ 3 & -4 \end{bmatrix} x_{23} + \begin{bmatrix} -4 & -1 \\ -4 & -6 \end{bmatrix} x_{24}$$

$$x_3' = -4x_3 + \begin{bmatrix} -3 & 2 \\ -4 & -3 \end{bmatrix} x_{31} + \begin{bmatrix} -4 & 1 \\ 3 & -4 \end{bmatrix} x_{32} + \begin{bmatrix} -4 & -3 \\ 6 & -6 \end{bmatrix} x_{34}$$

$$x_4' = -6x_4 + \begin{bmatrix} -3 & -2 \\ 4 & -6 \end{bmatrix} x_{41} + \begin{bmatrix} -4 & -1 \\ -4 & -6 \end{bmatrix} x_{42} + \begin{bmatrix} -4 & -3 \\ 6 & -6 \end{bmatrix} x_{43}$$

Where $x_{12} = (x_1, x_2)^T$, $x_{13} = (x_1, x_3)^T$, $x_{21} = (x_2, x_1)^T$, $x_{23} = (x_2, x_3)^T$, $x_{24} = (x_2, x_4)^T$, $x_{31} = (x_3, x_1)^T$, $x_{34} = (x_3, x_4)^T$, $x_{41} = (x_4, x_1)^T$, $x_{42} = (x_4, x_2)^T$ and $x_{43} = (x_4, x_3)^T$

For the main diagonal elements of matrix valued function choose the following Lyapunov functions

$$v_{ii} = x^2 \;\; for\; i = 1,2,3,4$$

Each isolated system is asymptotically stable by Lyapunov direct method as in [1].

The other elements of matrix valued function chosen the following Lyapunov functions

$$v_{ij}(x_i, x_j) = v_{ji}(x_i, x_j) = x_i^T P_{ij} x_j$$

Where P_{ij} **given by**

$$A_{ij}^T P_{ij} + A_{ij} P_{ij} = -G_{ij} \;\; i,j = 1,2,3,4 \;\; i \neq j \qquad)$$

The matrices P_{ij} **are given by solving the Lyapunov equations in last equation**

By assume $G_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $for\; i,j = 1,2,3,4\; i \neq j$

$$P_{12} = \begin{bmatrix} 0.1508 & 0.0238 \\ 0.0238 & 0.1429 \end{bmatrix}, \quad P_{13} = \begin{bmatrix} 0.1471 & -0.0294 \\ -0.0294 & 0.2059 \end{bmatrix}$$

$$P_{23} = \begin{bmatrix} 0.1442 & 0.0769 \\ 0.0769 & 0.1827 \end{bmatrix}, \quad P_{34} = \begin{bmatrix} 0.1036 & 0.0286 \\ 0.0286 & 0.1119 \end{bmatrix}$$

$$P_{14} = \begin{bmatrix} 0.1410 & 0.0385 \\ 0.0385 & 0.1090 \end{bmatrix}, \quad P_{24} = \begin{bmatrix} 0.0214 & -0.0700 \\ -0.0700 & 0.1300 \end{bmatrix}$$

By using eq. (4) and eq. (5) we can get the following matrix

$$A = \begin{bmatrix} -1 & 0.0292 & 0.0476 & 0.0278 \\ 0.0292 & -1 & 0.0589 & 0.0427 \\ 0.0476 & 0.0589 & -1 & 0.0187 \\ 0.0278 & 0.0427 & 0.0187 & -1 \end{bmatrix}$$

Since A is negative definite then the equilibrium x=0 of given system is asymptotically stable by applying proposition (3).

5. Conclusion

By mean of stability theory and from the important of the matrices in the study the stability theory, we have given a new result on the matrices whose using in this work, proposition (1) given a special case of permutation matrices in stability of linear system.

A proposition (2) gives the simple method to study the stability of linear large scale system.

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