Solution of Max – Min Composite Fuzzy Relational; Equation

By

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Abstract

Let A be the set of solutions of a max – min product fuzzy relation equation on finite spaces. In this paper we use some algorithms to solve max – min product relation equation.

Introduction

The concept of fuzzy relation equations introduced by Sanchez [5], is a generalization of well known Boolean equations.

Let A and B be two fuzzy sets of two finite spaces X, Y, respectively and R a fuzzy relation of the set $X \times Y$. Considered the following fuzzy relation equation:

A o R = B ………(1)

Where "o" is the max –min product composition. Speaking with terminology of systems theory, A and B represent a class of fuzzy inputs and a class of fuzzy equation (1).

In this paper, we illustrate other algorithms, to solve equation (1).

Preliminaries

Let I = [0, 1] be the real unite interval and we set for every $a, b \in I$, $\bar{a} = 1 - a$, $a \wedge b = \min\{a, b\}$, $a \vee b = \max\{a, b\}$, [1], of course we have:

1- $\overline{a \vee b} = \bar{a} \wedge \bar{b}$, $\overline{a \wedge b} = \bar{a} \vee \bar{b}$ (De Morgan's Laws).

2- $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$, $(a \wedge b) \vee c = (a \vee b) \wedge (b \vee c)$ [distributtivity laws].

Let X={x_1, x_2,…, x_n}, Y={y_1, y_2, …,y_m) be finite sets, F(X)= $\{A : \mu_A : X \to I\}$ the set of all fuzzy sets of X and I_r={1,2,…,r} the set of the first r natural numbers.

Following Zedeh's [1, 2], we remember that F(x) is a complete distributive lattice with the pointwise operations defined for every $x_i \in X$, $i \in I_n$ as :

1- $\overline{A(x_i)} = 1 - A(x)$.

2- $(A \wedge B)(x_i) = A(x_i) \wedge B(x_i)$, $(A \vee B)(x_i) = A(x_i) \vee B(x_i)$.

3- $(A \circ B)(x_i) = A(x_i) \circ B(x_i)$.

and the natural ordering

$A \leq B$ if $A(x_i) \leq B(x_i)$ where A,B $\in$ F(X)

Let $x_i \in X$, $y_j \in Y$, $i \in I_n$, $j \in J_m$, we recall the following definitions:-

<u>Definition 1 [1]</u>

Afuzzy relation R between two finite sets X and Y is an element of F(X$\times$Y).

<u>Definition 2 [1]</u>

Let $\mathcal{A}$ denote the set of all possible vectors A=[$a_i / i \in I_n$] such that,

$a_i \in [0,1]$ for all $i \in I_n$ and let the partial ordering $\mathcal{A}$ be defined as follows:

For any pair ${}^1A$, ${}^2A \in \mathcal{A}$, ${}^1A \leq {}^2A$ and only if ${}^1a_i \leq {}^2a_i$ for all $i \in I_n$.

<u>Definition 3 [3,4]</u>

An element $\hat{A}$ of S(R,B) is called a maximal solution of Eq.(1), if for all $A \in S(R,B)$, $A \geq \hat{A}$ implies $A = \hat{A}$.

Is it well established that whenever the solution set S(R,B) $\neq \phi$, it is always contains a unique maximal solution, $\hat{A}$.

<u>Definition 4 [4,5]</u>

An element Ă of S(R,B) is called minimal solution of Eq.(1) if, for all $A \in S(R,B)$, $A \leq \breve{A}$ implies $A = \breve{A}$ and when S(R,B) $\neq \phi$ it may contain several minimal solution.

A General Solutions to Fuzzy Relation Equation

Let R $\in$ F(X $\times$ Y)and A $\in$ F(X) we define $\boldsymbol{A} \circ \boldsymbol{R} = \boldsymbol{B}$ ……(1), B $\in$ F(Y), the max-min product composition of R and A as:

$$\boldsymbol{B}(\boldsymbol{y}_j) = (\boldsymbol{A} \circ \boldsymbol{R})(\boldsymbol{y}_j) = \bigvee_{j=1}^{m} \bigwedge_{i=1}^{n} [\boldsymbol{A}(\boldsymbol{x}_i, \boldsymbol{R}(\boldsymbol{x}_i, \boldsymbol{y}_j)] \ldots\ldots\ldots\ldots(2)$$

Let the membership matrices of A,R and B be denoted by A[a_i] , R= [$r_{i,j}$], B=[b_j], respectively, where $\boldsymbol{a}_i = \mu_{\boldsymbol{A}}(\boldsymbol{x}_i),\ \boldsymbol{r}_{i,j} = \mu_{\boldsymbol{R}}(\boldsymbol{x}_i, \boldsymbol{y}_j), \boldsymbol{b}_j = \mu_{\boldsymbol{B}}(\boldsymbol{y}_j)$ for all $\boldsymbol{i} \in \boldsymbol{I}_n$ and $\boldsymbol{j} \in \boldsymbol{J}_m$.

This mean that all the entries in this matrices A,R and B are real numbers in unit interval I=[0,1].

When matrices A and B are given and Matrix B is to be determined from Eq.(1) the problem is trivial. It is solved simply by performing the max-min multiplication – like operation on A and R as defined by Eq.(20. Clearly the solution in this case exists and is unique. The problem becomes far from trivial when one of the two matrices on the left –hand side of Eq.(1) is unknown. In this case, the solution is neither guaranteed to exit nor to be unique.

Since B in Eq.(1) is obtained by composing A and R, it is suggestive to view the problem of determining A from B and R as a decomposition of B with respect to R. Let us assume that a pair of specific matrices B and R from Eq.(1) is given and that we wish to determine the set of all particular matrices of the from A that satisfy Eq.(1).

Let each particular matrix A that satisfies Eq.(1) be called its solution and let S(R,B)={ $\boldsymbol{A}/\boldsymbol{A} \circ \boldsymbol{R} = \boldsymbol{B}$ } denote the set of all solutions, (the solution set). It follows immediately from Eq.(2) that if

$$\max_{i \in \boldsymbol{I}_n} \boldsymbol{r}_{i,j} \prec \boldsymbol{b}_j$$

then no value $\boldsymbol{a}_i \in [0,1]$ exist that satisfy Eq.(1)and , no matrix A exists that satisfies the matrix equation thus S(R,B)= ϕ .

This proposition allows us, in certain cases to determine quickly that Eq.(1) has no solution. However it is only a necessary and not a sufficient conditions for the existence of a solution of Eq.(1),that is S(R,B) $\neq \phi$.

When c , the maximum solution Ă = (ă$_i$, $\boldsymbol{i} \in \boldsymbol{I}_n$) of Eq.(1) is determined.

By: ă$_i$ =min σ ($r_{i,j}$, b_j) ……….(3)

Where σ ($r_{i,j}$, b_j)= $\begin{cases} \boldsymbol{b}_j & \boldsymbol{if}\ \boldsymbol{r}_{i,j} \succ \boldsymbol{b}_j \\ 1 & \boldsymbol{otherwise} \end{cases}$

When $\hat{\boldsymbol{A}}$ determined in this way does not satisfy Eq.(1), then S(R,B)= ϕ, that is the existence of the maximal solution Ă as determined by Eq.(3) , is a necessary and sufficient conditions for S(R,B)= ϕ .

So $\hat{\boldsymbol{A}}$ is the maximum solution of the equation and we next determined the set $\breve{\boldsymbol{S}}(\boldsymbol{R}, \boldsymbol{B})$ of the minimal solutions.

The method we described for determining all minimal solution of Eq.(1) is based on the assumption that the component of the vector B in Eq.(1) are ordered such that b_1>b_2>…….>b_m if the component are not initially ordered in this way we permute them appropriately and perform the same permutation on the columns of the matrix R. This procedure clearly yields an equivalent matrix equation which has exactly the same set of solutions as the original equation. When $\hat{a}_i = 0$ for same cwe may eliminate this component from $\hat{A}$ as well as the ith row from matrix R, since clearly $\hat{a}_i = 0$ implies a_i=0 implies a_i=0 for each $A \in S(R,B)$. Further more when b_j=0 for some $j \in J_m$ we may eliminate this component from B and the ith column from matrix R. This reduction is not necessary, the reduced equation is easier to deal with. When we obtain solutions of the reduced equation, we simply extend them by inserting zeros at the locations that were eliminated in the reduction step. The set $\breve{S}(R,B)$ of all minimal solutions of Eq.(1) can be determined by the following procedure :-

1- Determine the sets J_j $(\hat{A})=\{i \in I_n / \min(\hat{a}_i, r_{i,j}) = b_j\}$ for all $j \in J_m$ and then construct their cartesian product J($\hat{A} = \prod_{j \in J_m} J_j(\hat{A})$ denote elements of $J_j(\hat{A})$ by $\beta = (\beta_j / j \in J_m)$.

2- For each $\beta \in J(\hat{A})$ and each $i \in I_n$ determine the set

$$K(\beta, i) = \{j \in J_m / \beta_j = i\}$$

3- For each $\beta \in J(\hat{A})$ generate the n-tuple $g(\beta) = g_i(\beta) / i \in I_n)$ by taking

$$g_i(\beta) = \begin{cases} \max\limits_{j \in K(\beta,i)} b_j & \text{if } K(\beta,i) \neq \phi \\ 0 & \text{otherwise} \end{cases}$$

4- From all the n-tuple $g(\beta)$ generated in step (3) select all the minimal ones by pairwise comparison. The resulting set of n-tuple is the set $\breve{S}(R,B)$ of the minimal solution of Eq.(1).

Finally the solution set S(R,B) is fully characterized by the maximum and minimal solutions in the following sense:

It consist exactly of the maximum solution $\hat{\boldsymbol{A}}$, all the minimal solutions and all elements of $\mathcal{A}$ that are between $\hat{\boldsymbol{A}}$ and each of the minimal solution.

Formally $\boldsymbol{S}(\boldsymbol{R},\boldsymbol{B}) = \bigcup_{\ddot{\boldsymbol{A}}} \langle \breve{\boldsymbol{A}}, \hat{\boldsymbol{A}} \rangle$

Where the union is taken for all $\breve{\boldsymbol{A}} \in \breve{\boldsymbol{S}}(\boldsymbol{R},\boldsymbol{B})$

<u>Example :</u> Given

$$\boldsymbol{R} = \begin{bmatrix} .1 & .4 & .5 & .1 \\ .9 & .7 & .2 & 0 \\ .8 & .1 & .2 & 0 \\ .1 & .3 & .6 & 0 \end{bmatrix}, \qquad B=[.8, .7, .5, 0]$$

Determine all solutions of $\boldsymbol{A} \circ \boldsymbol{R} = \boldsymbol{B}$

<u>Sol.</u> First we determine whether S(R,B)=ϕ or not, by:

Max(.1, .9, .8, .1)=.9 > .8 = b_1

Max(.4, .7, .1, .3)= .1 > .7=b_2

Max(.5, .2, .5, 0)=.5 = .5 =b_3

Max(.1, 0, 0, 0)=.1 > 0 =b_4

Thus S(R,B)=ϕ

Now, since S(R,B)=ϕ, we determine the maximum solution $\hat{\boldsymbol{A}}$ by

$\hat{\boldsymbol{a}}_1$=min(1, 1, 1, 0) = 0

$\hat{\boldsymbol{a}}_2$=min(.8, 1, 1, 1) = .8

$\hat{\boldsymbol{a}}_3$=min(1, .7, 1, 1) = .7

$\hat{\boldsymbol{a}}_4$=min(1, 1, .5, 1) = .5

$\hat{\boldsymbol{A}}$= (0, .8, .7, .5). we can easily check that $\hat{\boldsymbol{A}} \in \boldsymbol{S}(\boldsymbol{R},\boldsymbol{B})$

Since $\hat{\boldsymbol{a}}_1$= 0 we may reduce the matrix equation by excluding a_1 and the first row of the matrix R, since b_4 = 0 we may make a further reduction by excluding b_4 and the fourth column of R, the reduced equation has the form

Where a_1, a_2 and a_3 in this reduced equation represent a_2, a_3 and a_4 of the original equation, respectively.

Next we apply the four steps of the procedure for determining the set $\breve{S}(\boldsymbol{R},\boldsymbol{B})$ of the minimal solution of this reduced matrix equation:-

1- Employing the maximum solution $\hat{\boldsymbol{A}}$ =(.8, .7, .5) of the reduced equation, we obtain $\boldsymbol{J}_1(\hat{\boldsymbol{A}}) = \{1\}, \boldsymbol{J}_2(\hat{\boldsymbol{A}}) = \{1,2\}, \boldsymbol{J}_3(\hat{\boldsymbol{A}}) = \{2,3\}$

Hence $\boldsymbol{J}(\hat{\boldsymbol{A}}) = \prod \boldsymbol{J}_j(\hat{\boldsymbol{A}}) = \{1\} \times \{1,2\} \times \{2,3\}$

2- The sets $\boldsymbol{K}(\beta,\boldsymbol{i})$ that we must determine for all $\boldsymbol{i} \in \boldsymbol{I}_n$ are listed in the following table:

	$\boldsymbol{K}(\beta,\boldsymbol{i})$			
β	t=1	2	3	$\boldsymbol{g}(\beta)$
(1, 1, 2)	{1, 2}	{3}	ϕ	(.8, .5, 0)
(1, 1, 3)	{1, 2}	ϕ	{3}	(.8, 0, .5)
(1, 2, 2)	{1}	{2, 3}	ϕ	(.8, .7, 0)
(1, 2, 3)	{1}	{2}	{3}	(.8, .7, .5)

3- for each $\beta \in \boldsymbol{J}(\hat{\boldsymbol{A}})$, we generate the triples $\boldsymbol{g}(\beta)$ which are also listed in the table above.

4- Two of the triples $\boldsymbol{g}(\beta)$ in the table above are minimal: (.8, .5, 0) and (.8, 0, .5).

Therefore comprise all the minimal solutions of the reduced matrix equation. By adding 0 as the first component to each of these triples, we obtain the minimal solution of the original matrix equation. Hence,

$$\breve{\boldsymbol{S}}(\boldsymbol{R},\boldsymbol{B}) = \{{}^{1}\breve{\boldsymbol{A}} = (0,.8,.5,0),\quad {}^{2}\breve{\boldsymbol{A}} = (0,.8,\ 0,\ .5)\}$$

The set S(R,B) of all solutions of the given matrix equation is now fully captured by the maximum solution $\hat{\boldsymbol{A}}$ =(0, .8, .7, .5) and the two minimal solutions ${}^{1}\breve{\boldsymbol{A}} = (0,.8,.5,0)$ and ${}^{2}\breve{\boldsymbol{A}} = (0,.8,\ 0,\ .5)$ so we have :

S(R,B) = $\{\mathrm{A} \in \mathcal{A} /\ {}^{1}\breve{\boldsymbol{A}} \le \boldsymbol{A} \le \breve{\boldsymbol{A}}\} \cup \{\ \mathrm{A} \in \mathcal{A} / {}^{2}\breve{\boldsymbol{A}} \le \boldsymbol{A} \le \breve{\boldsymbol{A}}\}$.

Basic Procedure to determine all solutions of the equation A o R=B ……(1)

1- If $\max_{i \in I_n} r_{i,j} < b_j$ then the equation has no solution S(R,B) $=\phi$ and the procedure terminates, otherwise proceed to step 2.

2- Determine $\hat{A}$ by procedure 1.

3- If $\hat{A}$ is not a solution of Eq.(1) , then the equation has no solution , S(R,B) $=\phi$ and the procedure terminates, otherwise proceed to step 4.

4- For each $\hat{a}_i$ = 0 and b_j = 0, exclude these component as well as the corresponding rows i and the columns j for the matrix R in Eq.(1): this results in the reduced equation $A' \circ R' = B'$ …..(2)

5- Determine all minimal solutions of the reduced equation (2) by procedure 2: this result in $\overset{\vee}{S}(R', B')$

6- Determine the solution set of the reduced equation(2) : $S(R', B') = \bigcup_{\breve{A}'} < \breve{A}', \hat{A}' >$ where the union is taken over all $\breve{A}' \in \overset{\vee}{S}(R', B')$.

7- Extend all solutions in $S(R', B')$ by the zeros that were excluded in step 4: this result in the solutions set S(R,B) of equation(1).

Procedure (1)

From the vector $\hat{A} = (\hat{a}_i,\ i \in I_n)$ in which $\hat{a}_i = \min\ \sigma(r_{i,j},\ b_j)$

Where $\sigma(r_{i,j},\ b_j) = \begin{cases} b_j & \text{if } r_{i,j} > b_j \\ 1 & \text{otherwise} \end{cases}$

Procedure (2)

1- Permute elements of β' and the corresponding columns of R' appropriately to arrange them in decreasing order.

2- Determine the sets $J_j(\hat{A}') = \{ i \in I_n / \min(\hat{a}', r') = b'_j \}$ for all $j \in J_m$ and then constructed their cartesian product $J(\hat{A}') = \prod_{j \in J_m} J_j(\hat{A}')$.

3- For each $\beta \in J(\hat{A})$ and each $i \in I_n$, determined the set,

$K(\beta, i) = \{ j \in J_m / \beta_j = i \}$

4- For each $\beta \in J(\hat{A})$ generated the n-tuple g(β) = (g$_i$(β)/ $i \in I_n$) by taking

$$g_i(\beta) = \begin{Bmatrix} \max\limits_{j \in K(\beta,i)} b_j & \text{if } K(\beta,i) \neq \phi \\ 0 & \text{otherwise} \end{Bmatrix}$$

5- From all the n-tuple g(β) generated in step (4) selected only the minimal ones this result in $\overset{\vee}{S}(R', B')$.

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المستخلص:

يتضمن البحث تعريف نوع من المعادلات الضبابية و إعطاء خوارزمية الحل حيث تم التوصل الى طريقة جديدة للحصول على مجموعة الحلول لهذه المعادلات.