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On the existence and the nonexistence of some (k,n) –arcs in $PG(2,37)$

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Abstract:

A $(k,n)-$ arc is a set of k points of a projective plane such that some n, but no n+1 of them, are collinear. The maximum size of a $(k,n)-$ arc in $PG(2\ q)$ is denoted by $m_n(2,q)$. In this paper we proved that $666 \le m_{21}(2,37) \le 741$, $703 \le m_{22}(2,37) \le 779$, $739 \le m_{23}(2,37) \le 817$, $777 \le m_{24}(2,37) \le 855$, $816 \le m_{25}(2,37) \le 893$, $854 \le m_{26}(2,37) \le 931$, $894 \le m_{27}(2,37) \le 969$, $933 \le m_{28}(2,37) \le 1007$, $970 \le m_{29}(2,37) \le 1045$, $m_{30}(2,37) \le 1083$, $m_{31}(2,37) \le 1121$, $m_{32}(2,37) \le 1159$, $m_{33}(2,37) \le 1197$, $m_{34}(2,37) \le 1235$, $m_{35}(2,37) \le 1273$ and $m_{36}(2,37) \le 1311$.

Keywords: Projective spaces, (k, n)-arcs, {l, t}-blocking set.

1- *Introduction*

Let $GF(q)$ be the Galois field of q elements and $V(3,q)$ be the vector space of row vectors in length three whose entries in $GF(q)$. Let $PG(2,q)$ be the corresponding projective plane. The points of $PG(2,q)$ are the non-zero vectors of $V(3,q)$ with the rule that $X = (x_1, x_2, x_3)$ and $Y = (\lambda x_1, \lambda x_2, \lambda x_3)$ are the same point, where $\lambda \in GF(q)\backslash\{0\}$. Since any non-zero vector has precisely $q-1$ non-zero scalar multiples, the number of points in the form $(1, x_1, x_2)$ is q^2 and the form $(1, x_1, 0)$ is q while the form $(0,0,1)$ is 1, so the number of points of $PG(2,q)$ is $\frac{q3-1}{q-1} = q^2 + q + 1$. If the point $P(X)$ is the equivalence class of the vector X, then we will say that X is a vector representing $P(X)$. A subspace of dimension one is a set of points all of them representing vectors that form a subspace of dimension two of $V(3,q)$, such subspaces are called lines. The number of lines in $PG(2,q)$ is $q^2 + q + 1$. There are $q + 1$ points on every line and any two distinct points lie exactly on one line. There are $q + 1$ lines throughout every point and any two distinct lines have exactly one common point.

In (1947) Bose [1] proved that $m_2(2,q) = q + 1$ for q odd, and $m_2(2,q) = q + 2$ for q even. In the mid of (1950s), Segre [2,3] proved that for q odd every $q + 1$ −arc is a conic, for $q = 2, q = 4$ and $q = 8$ every $q + 2$ −arc is a conic plus its nucleus [4], and for $q = 16, q = 32, q = 2^h$ $(h \geq 7)$, there exists a $q + 2$ −arc other than the conic plus its nucleus. In (1956) Barlotti [5] proved that the first of many results in the attempt to determine the value of $m_n(2,q)$, and this has been proved to be far from simple. The existence and the nonexistence of some (k, n)-arcs in PG(2,17) has been proved by Daskalov [6]. The existence and the nonexistence of some (k, n)-arcs in PG(2,31) has been proved by [7].

Definition 1.1 [7] A (k, n) −arc K is a set of k points, such that there is some n but no $(n + 1)$ are collinear.

Definition 1.2 [7] An $\{l, t\}$ −blocking set S in $PG(2,q)$ is a set of l points such that every line of $PG(2,q)$ intersects S in at least t points, and there is a line intersecting S in exactly t points.

Note that A (k, n)− arc is the complement of a $\{q^2 + q + 1 - k, q + 1 - n\}$-blocking set in a projective plane $PG(2,q)$ and conversely.

Definition 1.3 [7] Let M be a set of points in any plane. An i −secant is a line meeting M in exactly i points. Define T_i as the number of i −secants to a set M.

The T_i satisfy the next three diophantine equations in any projective plane, which are known as the standard equations [8].

Lemma 1.1 For any set of k points in $PG(2,q)$ the following hold:

1- $\sum_{i=0}^{q+1} T_i = q^2 + q + 1$

2- $\sum_{i=1}^{q+1} iT_i = k(q + 1)$

3- $\sum_{i=2}^{q+1} i(i - 1)T_i = k(k - 1)$

In [9] the next theorem is proved:

Theorem 1.1

Let K be a (k, n) −arc in $PG(2,q)$ where q is prime, then

1. If $n \leq (q + 1)/2$, then $m_n(2,q) \leq (n - 1)q + 1$.

2. If $n \geq (q + 3)/2$, then $m_n(2,q) \leq (n - 1)q + n - (q + 1)/2$.

From Theorem 1.1 the next corollary holds:

Corollary 1.1

$m_{21}(2,37) \leq 742$, $m_{22}(2,37) \leq 780$, $m_{23}(2,37) \leq 818$, $m_{24}(2,37) \leq 856$, $m_{25}(2,37) \leq 894$, $m_{26}(2,37) \leq 932$, $m_{27}(2,37) \leq 970$, $m_{28}(2,37) \leq 1008$, $m_{29}(2,37) \leq 1046$, $m_{30}(2,37) \leq 1084$, $m_{31}(2,37) \leq 1122$, $m_{32}(2,37) \leq 1160$, $m_{33}(2,37) \leq 1198$, $m_{34}(2,37) \leq 1236$, $m_{35}(2,37) \leq 1274$ and $m_{36}(2,37) \leq 1312$.

1.4 The Projective Plane $PG(2,37)$

1.4.1 The cyclic projectivity of $PG(2,37)$

The plane PG(2,37) contains 1407 points and 1407 lines, every line contains 38 points and every point passes through it 38 lines. It is convenience to use the numbers 0,1,2,3, …, 36 to be the elements of GF(37).

Let $f(x) = x^3 - 3x^2 - x - 2$ be a monic irreducible polynomial over GF(37) then the matrix $T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$ is a cyclic projectivity which is given by the right multiplication of the points of PG(2,37) and the order of T is 1407 .

1.4.2 The points of $PG(2,37)$

Let P_0 be the initial point that is represented by the vector (1 0 0) then $P_i = P_{i-1}T$, $i = 1, \ldots, 1406$ are the 1406 points of PG(2,37).

1.4.3 The lines of $PG(2,37)$

Let L_0 be the initial line which contains the points { 0 1 9 29 152 156 182 193 262 300 323 401 404 419 425 489 539 605 621 679 689 714 754 851 923 945 972 1034 1195 1229 1231 1248 1262 1308 1321 1353 1360 1365} and has the equation $X_3 = 0$ then $L_i = L_{i-1}T$, $i = 1, \ldots, 1406$ are the 1406 lines of PG(2,37).

2- The new arcs in $PG(2,37)$

Theorem 2.1 There exist a $(666,21)-$arc, $(703,22)-$arc, $(739,23)-$arc, $(777,24)-$arc, $(816,25)-$arc, $(854,26)-$arc, $(894,27)-$arc, $(933,28)-$arc and $(970,29)-$arc.

Proof: We will construct the above arcs by using the equation $x_1x_2 + 18x_2x_3 + 18x_1x_3$ which forms the conic in PG(2,37).

The set of points β_1

0 1 2 1109 8 18 48 62 70 109 115 137 1336 1280 1214 457 1091 222 1237 549 150 314 374 394 458 491 520 554 568 619 649 720 764 1051 1074 1148 1238 1300 3 4 1073 1127 156 252 782 33 407 1389 217 292 208 1005 175 1171 260 864 528 37 1040 1123 1309 1317 455 944 613 1060 673 392 87 1072 677 637 768 11 561 952 16 10 854 922 655 772 359 83 241 347 1367 273 592 419 585 964 162 516 1211 1095 490 303 899 191 250 28 507 509 1262 275 904 25 834 1172 643 1335 1298 744 291 582 573 1077 735 1242 889 1020 421 311 879 42 1047 820 750 12 135 379 330 1132 60 555 240 428 529 363 40 1329 331 1315 685 1245 1144 751 465 369 880 474 253 463 713 320 800 106 479 1404 811 943 1112 853 370 41 894 461 293 747 526 932 1143 845 1390 353 367 364 1259 336 533 1049 928 890 1244 469 100 513 174 1339 198 259 167 614 467 1054 965 857 194 886 732 79 401 1064 96 1007 942 1359 319 17 299 1294 700 590 315 1098 1000 1022 1176 1004 503 151 591 849 989 32 929 858 669 98 1357 1115 631 345 687 114 180 770 276 898 9 595 89 1088 610 838 523 941 493 1015 741 638 73 851 287 1208 876 67 26 108 215 413 1287 625 47 429 495 164 1319 546 325 583 192 451 468 396 78 719 1169 883 157 998 1334 656 958 286 124 430 197 753 780 1124 1202 201 891 1272 460 521 611 777 171 494 234 272 236 584 444 593 316 426 1297 1016 711 873 996 1267 1268 1150 1184 795 1313 94 224 1322 1370 731 185 806 659 1205 539 1105 1162 464 742 759 1304 271 624 402 143 245 959 81 447 535 630 588 409 466 69 675 949 149 1181 59 1207 704 1365 1119 909 43 829 1188 706 202 824 482 229 642 639 1379 155 417 968 57 779 884 960 896 1231 1192 1235 514 511 693 653 784 434 1094 386 501 82 322 101 342 939 231 22 566 1216 1295 267 1147 510 947 792 1122 408 163 193 701 308 661 840 21 599 804 560 641 365 738 936 1183 869 813 737 809 930 1180 980 169 1342 1374 1346 226 1118 978 1002 1161 203 775 826 1030 1358 979 90 1384 867 1096 13 54 730 378 1174 339 1001 235 116 536 1227 676 902 993 1039 66 104 850 368 1276 1393 178 634 975 1247 361 508 200 223 1229 1130 697 266 725 1292 80 1145 728 763 957 707 1323 1038 1385 1066 1093 357 204 302 478 502 1283 1248 627 362 1369 220 1199 1377 629 1279 400 173 1131 696 524 210 575 219 145 373 579 632 1353 489 848 847 310 404 793 1281 1025 1210 814 1204 389 1364 213 196 835 694 966 756 805 230 227 722 856 836 1321 600 384 1011 1179 1273 431 1289 887 988 1099 1296 462 916 190 132 905 1133 435 341 277 688 1391 411 51 53 559 415 1139 1395 825 423 1026 139 1241 515 615 852 745 348 606 52 822 686 892 92 237 425 486 882 111 110 999 476 1284 537 1167 920 766 1006 1351 823 499 321 424 1153 1201 170 1378 68 798 279 667 256 982 397 27 552 1376 877 55 1328 946 498 710 481 547 1253 895 512 789 1068 1027 970 917 1266 290 14 925 651 1070 1190 1163 205 668 285 540 776 953 1368 1264 159 49 179 375 803 885.

forms a $(666, 21) -$ arc in PG(2,37) with secant distribution
[Ti // i=0,...,21//]= [0 0 0 0 4 2 0 6 7 9 18 19 22 41 68 52 96 109 145 243 317 249]

2- A new $(703, 22) -$ arc set β_2 has been constructed by extension of the previous set β_1, by adding the next 37 points
1053 1107 1290 1151 1232 1079 927 921 391 935 86 371 211 31 1218 1010 38 672 973 578 658 323 797 1165 332 901 1258 913 304 992 126 218 283 1305 187 261 581
with secant distribution
[Ti // i=0,...,22//]= 0 0 0 0 1 4 1 3 3 10 8 18 14 29 37 54 66 84 112 164 235 326 238

3- A new $(739, 23) -$ arc set β_3 has been constructed by extension of the previous set β_2, by adding the next 36 points
333 358 360 483 1041 604 828 553 1230 141 951 567 945 1140 485 602 123 496 158 130 91 1301 1146 1189 1149 39 313 243 843 262 868 548 487 626 312 1076
with secant distribution
[Ti // i=0,...,23//]= [0 0 0 0 1 1 2 2 4 2 8 12 11 24 22 44 48 64 89 120 160 237 330 226]

4- A new $(777, 24) - arc$ set β_4 has been constructed by extension of the previous set β_3, by adding the next 38 points
301 1366 522 335 622 45 388 908 569 1125 646 195 754 295 344 812 695 865 176 601 671 977 131 289 1340 1310 1129 1142 721 690 550 818 997 500 678 1071 1024 72
with secant distribution
[Ti // i=0,...,24//]= [0 0 0 0 0 0 2 1 6 2 3 2 12 18 17 30 43 37 62 94 115 170 231 336 226]

5- A new $(816, 25) - arc$ set β_5 has been constructed by extension of the previous set β_4, by adding the next 39 points
35 1405 77 633 1056 505 976 251 1136 556 1223 961 866 587 893 983 881 1086 112 1104 545 787 129 298 1302 1090 422 1212 144 1275 294 305 995 733 346 771 329 915 1035
with secant distribution
[Ti // i=0,...,25//]= [0 0 0 0 0 0 1 1 2 4 1 3 3 12 19 15 30 31 56 50 97 100 185 229 325 243]

6- A new $(854, 26) - arc$ set β_6 has been constructed by extension of the previous set β_5, by adding the next 38 points
919 1341 1065 420 900 1282 698 740 954 574 1348 1333 1152 1354 1021 703 183 1285 64 233 119 414 681 249 1375 1168 871 326 1166 562 1271 327 242 1012 189 350 1236 1397
with secant distribution
[Ti // i=0,...,26//]= [0 0 0 0 0 0 0 0 1 3 3 0 4 4 15 13 23 25 34 37 61 84 120 174 243 322 241]

7- A new $(894, 27) - arc$ set β_7 has been constructed by extension of the previous set β_6, by adding the next 40 points
138 1243 356 551 875 152 453 76 254 839 597 1263 488 433 1260 645 796 714 1050 594 107 532 58 23 212 61 1221 557 307 1069 244 665 1155 837 284 317 598 525 398 153
with secant distribution

[Ti // i=0,...,27//]= [0 0 0 0 0 0 0 0 1 1 2 0 4 4 4 12 13 18 30 25 33 62 91 114 170 240 337 246].

8- A new $(933, 28) - arc$ set β_8 has been constructed by extension of the previous set β_7, by adding the next 39 points

120 1291 950 166 134 683 769 808 376 1120 860 450 760 75 239 445 802 160 113 906 1344 542 715 1097 1158 918 1018 441 1332 5 1044 257 46 563 1014 142 351 692 1182

with secant distribution

[Ti // i=0,...,29//]= [0 0 0 0 0 0 0 0 1 0 1 0 1 3 6 6 9 10 17 21 33 39 60 87 98 191 235 337 252].

9- A new $(970, 29) - arc$ set β_9 has been constructed by extension of the previous set β_8, by adding the next 39 points

6 427 761 1265 1355 121 739 99 50 1170 65 477 122 1159 862 232 743 794 7 1228 1331 270 263 387 209 136 833 74 1213 288 1160 663 1345 1034 264 788 934

with secant distribution

[Ti // i=0,...,29//]= [0 0 0 0 0 0 0 0 0 0 0 1 1 3 3 1 6 9 12 14 23 24 47 66 72 130 164 251 345 235].

3- The nonexistence of some arcs in $PG(2, 37)$

Theorem 3.1 [6] Let B be an $\{l, t\}$ −blocking set in $PG(2, q)$ (q-prime).

1. If $t < \frac{q}{2}$ and $q > 3$, then $l \geq t(q+1) + \frac{q+1}{2}$.
2. If $l = t(q+1) + \frac{q+1}{2}$ then each point of B has exactly $\frac{q+3}{2}$ lines through it that are not t −secants and exactly $\frac{q-1}{2}$ lines that are t −secants. So the total number of t −secants is $\frac{l(q-1)}{2t}$.

Applying this theorem the next theorem holds.

Theorem 3.2

1. There exists no $(742, 21)$ −arc and hence $m_{21}(2,37) \leq 741$.
2. There exists no $(780, 22)$ −arc and hence $m_{22}(2,37) \leq 779$.
3. There exists no $(818, 23)$ −arc and hence $m_{23}(2,37) \leq 817$.
4. There exists no $(856, 24)$ −arc and hence $m_{24}(2,37) \leq 855$.
5. There exists no $(894, 25)$ −arc and hence $m_{25}(2,37) \leq 893$.
6. There exists no $(932, 26)$ −arc and hence $m_{26}(2,37) \leq 931$.
7. There exists no $(970, 27)$ −arc and hence $m_{27}(2,37) \leq 969$.
8. There exists no $(1008, 28)$ −arc and hence $m_{28}(2,37) \leq 1007$.
9. There exists no $(1084, 30)$ −arc and hence $m_{30}(2,37) \leq 1083$.
10. There exists no $(1122, 31)$ −arc and hence $m_{31}(2,37) \leq 1121$.
11. There exists no $(1198, 33)$ −arc and hence $m_{33}(2,37) \leq 1197$.
12. There exists no $(1236, 34)$ −arc and hence $m_{34}(2,37) \leq 1235$.

Proof. 1. Finding a maximum $(742, 21)$ −arc is equivalent to find a $\{665,17\}$ −blocking set. Theorem 3.1 implies, that the total number of 17 −secants is $\frac{665.18}{17}$ which is not an integer (a contradiction).

The proof of the remaining cases is similar to the proof of the previous one.

Theorem 3.3 [10] Let B be an $\{l, t\}$ −blocking set in $PG(2, q)$ that contains a line. Then: If $(t-1, q) = 1$ then $|B| = l \geq q(t+1)$.

Theorem 3.4

1. There exists no $(1046, 29)$ −arc and hence $m_{29}(2,37) \leq 1045$.
2. There exists no $(1160, 32)$ −arc and hence $m_{32}(2,37) \leq 1159$.
3. There exists no $(1274, 35)$ −arc and hence $m_{35}(2,37) \leq 1273$.
4. There exists no $(1312, 36)$ −arc and hence $m_{36}(2,37) \leq 1311$.

Proof. 1. Finding a maximum $(1046, 29)$ −arc is equivalent to find a $\{361,9\}$ −blocking set. Theorem 3.1 implies, that the total number of 9 −secants is 722. Let r is the length of the longest secant. If r = 38, then B contains a line and Theorem 3.3 can be applied. It follows from Theorem 3.3
that $|B| = l \geq 37.10 = 370$, a contradiction.

If $29 \leq r \leq 37$ then considering lines through a point on the longest secant but not in B, B must have at least 9.37+r points, a contradiction.

Now consider the intersection of the 9 −secants with the longest secant.

If $r = 28$ then $T_6 = 28.18 + 10.37 = 874 > 722$, a contradiction.
If $r = 27$ then $T_6 = 27.18 + 11.36 = 882 > 722$, a contradiction.
If $r = 26$ then $T_6 = 26.18 + 12.35 = 888 > 722$, a contradiction.
If $r = 25$ then $T_6 = 25.18 + 13.34 = 892 > 722$, a contradiction.
If $r = 24$ then $T_6 = 24.18 + 14.33 = 894 > 722$, a contradiction.
If $r = 23$ then $T_6 = 23.18 + 15.32 = 894 > 722$, a contradiction.
If $r = 22$ then $T_6 = 22.18 + 16.31 = 892 > 722$, a contradiction.
If $r = 21$ then $T_6 = 21.18 + 17.30 = 888 > 722$, a contradiction.
If $r = 20$ then $T_6 = 20.18 + 18.29 = 882 > 722$, a contradiction.
If $r = 19$ then $T_6 = 19.18 + 19.28 = 874 > 722$, a contradiction.
If $r = 18$ then $T_6 = 18.18 + 20.27 = 864 > 722$, a contradiction.
If $r = 17$ then $T_6 = 17.18 + 21.26 = 852 > 722$, a contradiction.
If $r = 16$ then $T_6 = 16.18 + 22.25 = 838 > 722$, a contradiction.
If $r = 15$ then $T_6 = 15.18 + 23.24 = 822 > 722$, a contradiction.
If $r = 14$ then $T_6 = 14.18 + 24.23 = 804 > 722$, a contradiction.
If $r = 13$ then $T_6 = 13.18 + 25.22 = 784 > 722$, a contradiction.
If $r = 12$ then $T_6 = 12.18 + 26.21 = 762 > 722$, a contradiction.
If $r = 11$ then the standard equations for the set B are:

$$T_9 + T_{10} + T_{11} = 1407$$
$$9T_9 + 10T_{10} + 11T_{11} = 13718$$
$$72T_9 + 90T_{10} + 110T_{11} = 129960$$

The unique solution of this system is $T_9 = 5185,\ T_{10} = -8611,\ T_{11} = 4833$, a contradiction.

If $r = 10$ then the standard equations for the set B are:

$$T_9 + T_{10} = 1407$$
$$9T_9 + 10T_{10} = 13718$$
$$72T_9 + 90T_{10} = 129960$$

From the first two equations we obtain $T_9 = 352$ *and* $T_{10} = 1055$. But $72.352 + 90.1055 = 120294$ and we have a contradiction again. This completes the proof.
The proof of the remaining cases is similar to the proof of the previous one.

Now we can summarize the results from this paper in the next Table I.

n	21	22	23	24	25	26	27
k	666…741	703…779	739…817	777…855	816…893	854…931	894…969

n	28	29	30	31	32	33	34	35	36
k	933…1007	970…1045	…1083	…1121	…1159	…1197	…1235	…1273	…1311

References

[1] Bose R. C., "Mathematical theory of the symmetrical factorial design", Sankyha, **8**, (1947), 107-166.
[2] Segre B., "Sulle ovali nei piani lineari finiti", Atti Accad. Naz. Lincei Rend, **17**, (1954), 1-2.
[3] Segre B., "Ovals in a finite projective plane", Canad. J. Math., **7**, (1955), 414-416.
[4] Segre B., "Sui k-archi nei piani finiti di caratteristica due", Rev. Math. Pures Appl., **2**, (1957), 289-300.
[5] Barlotti A., "Su {k; n}-archi di un piano lineare finito", Boll. Un. Mat. Ital., **11**, (1956), 553-556.
[6] Daskalov R.N., "On the existence and the nonexistence of some (k, n)-arcs in PG(2,17) ", University of Gabrovo, Bulgaria, (2004).
[7] Najem H. S., "The maximum Values for a (k,r) −arcs and the Minimum Values for a complete (k,r) −arcs in $PG(2{,}31)$ and a minimal $\{l,t\}$ − blocking Sets in $PG(2,q)$", M. Sc. thesis, University of Mosul, Iraq, (2010).
[8] Tallini Scafati M., Sui (k, n)-archi di un piano grafico finito, Rend. Naz. Lincei **(8) 49**, (1966), 1-6
[9] Ball S., On sets of points in finite planes, Ph.D. Thesis, University of Sussex, (1994).
[10] Ball S., Multiple blocking sets and arcs in ¯nite planes, J. London Math Soc. **54**, (1996), 427-435.

حول الوجود وعدم الوجود لبعض الأقواس − (k,n) في$\mathbf{PG(2,37)}$

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الملخص

القوس − (k,n) هو مجموعة k من النقاط في المستوي الاسقاطي بحيث يوجد n ولا يوجد n+1 من هذه النقاط تقع على إستقامة واحدة. الحجم الأعظم للأقواس − (k,n) في المستوي الإسقاطي $PG(2,q)$ يرمز له بالرمز $m_n(2,q)$ في هذا البحث برهنا

$666 \le m_{21}(2{,}37) \le 741$, $703 \le m_{22}(2{,}37) \le 779$, $739 \le m_{23}(2{,}37) \le 817$, $777 \le m_{24}(2{,}37) \le 855$, $816 \le m_{25}(2{,}37) \le 893$, $854 \le m_{26}(2{,}37) \le 931$, $894 \le m_{27}(2{,}37) \le 969$, $933 \le m_{28}(2{,}37) \le 1007$, $970 \le m_{29}(2{,}37) \le 1045$, $m_{30}(2{,}37) \le 1083$, $m_{31}(2{,}37) \le 1121$, $m_{32}(2{,}37) \le 1159$, $m_{33}(2{,}37) \le 1197$, $m_{34}(2{,}37) \le 1235$, $m_{35}(2{,}37) \le 1273$ and $m_{36}(2{,}37) \le 1311$.