2024, 21(5): 1639-1648
https://dx.doi.org/10.21123/bsj.2023.8378
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
Baghdad Science Journal

On Partition Dimension and Domination of Abid-Waheed $(AW)_r^4$ Graph

Jalal Hatem Hussein Bayati [1], ***Abid Mahboob**** [*2], ***Muhammad Waheed Rasheed*** [2], ***Dur e Najaf*** [2]

[1]Department of Mathematics, College of Science for Women, University of Baghdad, Baghdad, Iraq.
[2]Department of Mathematics, Division of Science and Technology, University of Education, Lahore, Pakistan.
*Corresponding Author.

Received 16/01/2023, Revised 19/05/2023, Accepted 21/05/2023, Published Online First 20/10/2023, Published 01/05/2024

Abstract

A graph denoted by H, which has a simple link between its vertices, possesses the set of vertices V(H) = $\{x, x_1, x_2, \ldots, x_m\}$. Given a graph, $H = (V, E)$, S a set that is dominant, is a subset of vertex set V such that any vertex outside of S is close to at least one vertex inside of S. The smallest size of S for the dominating set is known as the graph's domination number. When a linked graph H has a vertex x and a subset S of the vertex set V (H), the separation between x and S is given by $d(x,S) = min\{d(x,u)|u;\ S\}$. Pertaining to an ordered k-partition $\pi = \{S_1, S_2, \ldots, S_k\}$ of $V(H)$, the illustration of x in relation to Π is to be the k-vector $r(x/\Pi) = (d(x,S_1), d(x,S_2),, d(x,S_k))$. Abid-Waheed graph $(AW)_r^s$ is a simply connected graph which contains $rs+1$ vertices and $r(s+1)$ edges for all $r \geq 1$ and $s \geq 3$. In this paper, we studied some results on the domination number, independent and restrained domination number denoted by $\gamma(H)$, $\gamma_i(H)$ and $\gamma_r(H)$ respectively in the Abid-Waheed graphs $(AW)_r^4$ and the relation between domination number, independent and restrained domination number. Also, the objective of this paper is to generate a partition dimension of $(AW)_r^s$.

Keywords: Abid-Waheed Graph, Domination, Independent Domination, Partition Dimension, Restrained Domination.

Introduction

Graph theory is one of the mathematical sub-fields. It is commonly known that an interconnection network can be represented by a graph with nodes corresponding to network sites and edges corresponding to connections between network nodes. Mathematics and computer science both have a component that deals with the analysis of graphs, or representations using points and lines that graphically demonstrate mathematical concepts. It can be applied in a number of circumstances and the use of graph theory has expanded quickly. It is helpful to comprehend how computers compute, as well as how communications work, how information is managed, and how Google Street View works. For a very long time, literature has explored the ideas of dominance and independence [1]. The first people to discuss these ideas are Ore [2], who used the name 'Domination Number' in the space of coefficient of External Stability. Berge was the first to present the fundamental definition of domination in graph theory [3]. Next, it was suggested to use edge-subdivision and bondage number notations in the description and modeling of transportation networks [4]. An area of graph theory that has received a lot of investigation is dominance in networking. Because of its numerous applications to discrete optimization

2024, 21(5): 1639-1648
https://dx.doi.org/10.21123/bsj.2023.8378
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
Baghdad Science Journal

issues, combinatorial problems, and classical algebraic problems, graph theory has experienced phenomenal growth during the past 30 years. The notion of domination has been the focal point of contemporary graph theoretical research, numerous fields including engineering, the physical, social, and biological sciences, languages, etc, employ it for a wide range of purposes. This is mostly because of the numerous new criteria that can be created from the fundamental definition of domination [5]. Around 1960, graph theory's thorough examination of dominating sets began not standing with the fact that the topic has historical roots that date back to 1862 when the Jaenisch researched the issues of figuring out the bare minimum of queens required to cover or control a $n * n$ chessboard [6]. The concept of independent domination first appeared in chessboard puzzles [7]. A recent book [8] on domination has sparked enough motivation to fuel the enormous expansion of this field of study. Vasumathi and Vangipuram [9] and Vijayasaradhi and Vangipuram [10] both discovered the domination parameters of an arithmetic graph and came up with a refined technique for creating an arithmetic graph using the determined domination parameter. Apply this to many applications such as eradicating pests in Agriculture, controlling viruses that produce diseases in an epidemic form, to maintain confidentiality when transferring information, and it is especially very useful in the Defense sector [11]. Problems involving discovering sets of representatives, observing communication or electrical networks, and other situations involving control also use concepts of domination. Studies on the independent dominance number in several families of graphs, including planar graphs, triangle-free graphs, and graphs with constrained diameters, as well as structural results on domination-perfect graphs were presented. In a graph, the domination problem is to identify a minimal-sized vertex collection. Over the past 20 years, substantial research on dominance and its variations has been conducted. Edge domination is one variation that is examined in this study from an algorithmic perspective [12-14]. In a graph, an edge-dominating set is a set S of edges where each edge outside of S is connected to at least one edge inside S it is nominated as $\gamma'(H)$. When there are no two different edges that are contiguous, an edge-dominating set is said to be independent. Finding the smallest set of edges that dominate a graph independently is known as the (independent) edge dominance problem and is denoted as $\gamma_i(H)$. The edge domination problem in trees was addressed by a linear time approach by Mitchell and Hedetniemi [15-17]. If every edge outside of S is incident to both an edge inside of S and an edge inside of $E - S$, then this is a restrained edge dominant set. Restrained edge domination is the term used to describe the smallest cardinality of a set of H that dominates its edges, it is nominated by $\gamma_r(H)$ [18]. Every vertex that is not in S is next to a vertex that is in S, and this is known as a dominant set for graph H. A dominating set must be at least as big as its domination number, which is represented by $\gamma(H)$. If no two of its vertices are next to one another, the set is independent (or unconnected). In a graph, a set that is both dominating and independent is known as an independent dominating set. The size of an independent dominating set's minimal independent dominance number is indicated by the symbol $\gamma_i(H)$. Every vertex in a set $V - S$ that is adjacent to both a vertex in S and another vertex in $V - S$ is referred to as being in a restrained dominating set. The minimum cardinality of a restrained dominating set of H is the restrained domination number of H, indicated by $\gamma_r(H)$ [19]. When a linked graph H has a vertex x and a subset of vertex V (H), the separation between x and S is given by $d(x,S) = min\{d(x,u)|u;\ S\}$. Pertaining to an ordered k-partition $\pi = \{S_1, S_2, , S_k\}$ of $V\,(H)$, the illustration of x in relation to π is to be the k-vector $r(x/\Pi) = (d(x,S_1), d(x,S_2), , d(x,S_k))$ [20-22]. The smallest number of sets in any resolving partition of G is known as the partition dimension of G, indicated by the symbol pd(G). Chartrand, Salehi, and Zhang examined the partition dimension first in [23,24], followed by [25] by Chappell, Gimbel, and Hartman, to learn more about the 'metric dimension', a different graph parameter. Applications of this invariant to robotic network navigation are covered in [26], while applications to chemistry are provided in [27].

Materials and Methods

Definition of Abid-Waheed $(AW)_r^s$ graph for $s \geq 3$ and $r \geq 1$

This section discusses the planning and creation of our ground-breaking connected the Abid-Waheed $(\mathrm{AW})_r^s$ graph, that is shown in Fig 1; Abid-Waheed graph $(\mathrm{AW})_r^s$ contains $rs + 1$ vertices and $r(s + 1)$

2024, 21(5): 1639-1648
https://dx.doi.org/10.21123/bsj.2023.8378
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
Baghdad Science Journal

edges for all $r \geq 1$ and $s \geq 3$, i.e, a graph is generated by r-cycles (with each of order s), meeting at an external vertex of degree r. It is represented by $(\mathrm{AW})_r^s$ for $s \geq 3$ and $r \geq 1$ [28].

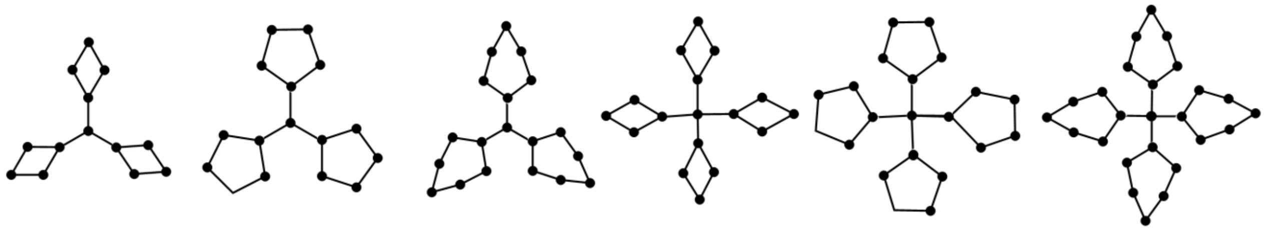

Figure 1. Abid-Waheed $(AW)_r^s$

Results and Discussion

Major results

Theorem 1: Suppose that $(\mathrm{AW})_r^s$ is the Abid-Waheed graph with $r \geq 1$ and $s = 4$ then the domination number of $(\mathrm{AW})_r^s$ is given by;

$$\Upsilon(AW)_r^4 = r + 1$$

Proof: Let H be a $(AW)_r^4$ graph and its vertex set $V(AW)_r^4 = \{\mathrm{x}, x_1, x_2, \dots, x_m\}$ and x is the central vertex of the graph. Each vertex in $(\mathrm{AW})_r^4$ has been clearly manifested so, it is necessary to demonstrate for every cycle 'r' may it contains an even or odd number of vertices, it shares the same dominating number, which is $r + 1$. Initially, analyzed for $r = 1,2,3$ then demonstrated for the general case when $r = m$. To begin the proof consider the case where $r = 1$ and $s = 4$. Predict for cycle r_1 and $s = 4$ the vertex set is $V(AW)_1^4 = \{\mathrm{x}, x_1, x_2, x_3, x_4\}$ then examine the structure of H and determine the dominating number by varying $'r'$ while keeping $'s'$ constant.

It is evident that $S_0 = \{x_1, x_3\}$ serves as a dominating set for $(\mathrm{AW})_1^4$. Because (x, x_2) is adjacent to x_1 and x_3 is directly linked to x_2 so $S_0 = \{x_1, x_3\}$ is dominating set for $(\mathrm{AW})_1^4$ and its domination number is 2 which is $r + 1$ as $r = 1$. Now for the case when $r = 2$ and $s = 4$ let the vertex set of $(\mathrm{AW})_2^4$ is $V(AW)_2^4 = \{\mathrm{x}, x_1, x_2, x_3, \dots, x_8\}$. As there are two cycles and each cycle has four vertices one vertex has a degree of three and the other vertices have a degree of two and the central vertex has a degree equal to 'r'. So, a minimum of three vertices are needed to dominate the whole of the vertices of the graph. Let $S_1 = \{x, x_3, x_7\}$ be the dominating set for $(\mathrm{AW})_2^4$. As two vertices are in the neighborhood of x, one from each cycle and two vertices of the graph are in the neighborhood of each vertex (x_3, x_7). Hence its domination number is 3.

Moreover, for the case when $r = 3$ and $s = 4$ suppose the vertex set of $(\mathrm{AW})_3^4$ is $V(AW)_3^4 = \{\mathrm{x}, x_1, x_2, x_3, \dots, x_{12}\}$. Now there are three cycles and only four minimum vertices are needed for dominating all the vertices of the graph according to the degree of the vertices. Let $S_2 = \{x, x_3, x_7, x_{11}\}$ be the dominating set for $(\mathrm{AW})_3^4$, three vertices are directly linked to x, and (x_3, x_7, x_{11}) dominate those vertices which are directly linked with them. So, $S_2 = \{x, x_3, x_7, x_{11}\}$ is the set that dominates all the vertices of $(\mathrm{AW})_3^4$ and hence its dominating number is 4. From the above observations for $r = 1,2,3$ the dominating set consists of one central vertex and one middle vertex of $'s'$ which has a degree of two.

Furthermore, in general when $r = m$ and $s = 4$ let the vertex set of $(\mathrm{AW})_m^4$ is $V(AW)_m^4 = \{\mathrm{x}, x_1, x_2, x_3, \dots, x_{m-1}, x_m\}$. In this case, the subset of $m + 1$ vertices is needed to dominate every vertex of the graph. Let $S_m = \{x, x_3, x_7, \dots, x_{m-1}\}$ be the dominating set for $(\mathrm{AW})_m^4$. $(x_1, x_5, x_9, \dots, x_{m-3})$ in the neighborhood of 'x', (x_2, x_4) is neighboring x_3, (x_6, x_8) is neighboring x_7, (x_{10}, x_{12}) is neighboring x_{11} and that is continued as (x_{m-2}, x_m) is neighboring x_{m-1}. So, $S_m = \{x, x_3, x_7, \dots, x_{m-1}\}$ is the set that dominates all the vertices of $(\mathrm{AW})_m^4$ and hence its dominating number is $m + 1$. The statement of the theorem is verified when the Abid-Waheed graph has $r = m$ and $s = 4$ then its domination number is $r + 1$.//

Theorem 2: Let $(\mathrm{AW})_r^4$ be a graph, for every integer $r \geq 2$ and $s = 4$ then the independent domination number of a graph is stated as;

$$\Upsilon_i(AW)_r^4 = r + 1$$

2024, 21(5): 1639-1648
https://dx.doi.org/10.21123/bsj.2023.8378
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
Baghdad Science Journal

Proof: Let $H = (\mathrm{AW})_r^4$ graph and its vertex set $V(AW)_r^4 = \{\mathrm{x}, x_1, x_2, \ldots, x_m\}$. Each vertex in $(AW)_r^4$ has a distinctive depiction to demonstrate that a graph with any number of cycles has the same independent domination number, which is $r+1$. First, demonstrate the cases where $r = 1,2,3$ and then prove for general case when $r = m$. Consider the structure of graph H and find independent domination by varying $'r'$ while $'s'$ remains constant. Let the case where $r = 1$ and $s = 4$. Since, the domination number of $(AW)_1^4$ graph is two, let $S_0 = \{x_1, x_3\}$ be the dominating set as no one vertex of S_0 is in the neighborhood of each other, it is also an independent dominating set and its independent domination number is 2. In the case when $s = 4$ and $r = 2$ the graph $(\mathrm{AW})_2^4$ has a domination number of three. Let $S_1 = \{x, x_3, x_7\}$ be a dominating set for $(\mathrm{AW})_2^4$ as no one vertex of S_1 share's common edge so, it is an independent dominating set for the given graph.

Since, for $r = 3$ and $s = 4$ the graph $(AW)_3^4$ has a domination number is four. Let $S_2 = \{x, x_3, x_7, x_9\}$ be the dominating set for $(\mathrm{AW})_3^4$ and no one vertex of S_2 have a common edge or directly linked. Similarly for the general case where $r = m$ and $s = 4$ then the dominating set $S_m = \{x, x_3, x_7, \ldots, x_{m-1}\}$ is the independent dominating set for $(\mathrm{AW})_m^4$ as no one vertex is adjacent to each other. Hence our claim is true that is for $(\mathrm{AW})_m^4$ the independent domination is $r+1$.

Theorem 3: If $H = (\mathrm{AW})_r^4$ is the Abid-Waheed graph, for every integer $r \geq 2$ and $s = 4$ then the restrained domination number of the graph is given by;

$$\Upsilon_r(AW)_r^4 = r+1$$

Proof: All the cycles and vertices of the Abid-Waheed graph have unique positions and there is not any parallel edge between the vertices of the graph. Let H be a $(\mathrm{AW})_r^4$ graph with vertex set $V(AW)_r^4 = \{\mathrm{x}, x_1, x_2, \ldots, x_m\}$, demonstrating that the graph has $r+1$ restrained domination number. Initiating proof for case where $r = 1$ and $s = 4$, let the vertex set for $V(AW)_1^4 = \{\mathrm{x}, x_1, x_2, x_3, x_4\}$. Since, the dominating set for $(AW)_1^4$ is $S_0 = \{x, x_3\}$ and $V - S_0 = \{x_1, x_2, x_4\}$ is a set in which all the vertices are linked with each other directly by an edge. So, S_0 is a restrained dominating set for $(AW)_1^4$ and its restrained domination is 2.

Additionally for the case where $r = 2$ and $s = 4$ let the vertex set of $(AW)_2^4$ is $V(AW)_2^4 = \{\mathrm{x}, x_1, x_2, x_3, \ldots, x_8\}$. Since, $S_1 = \{x, x_3, x_7\}$ is dominating set with minimum cardinality for $(AW)_2^4$ and all the elements in $V - S_1$ are in the neighborhood of each other so, $S_1 = \{x, x_3, x_7\}$ is a restrained dominating set for $(AW)_2^4$ and hence its restrained domination is 3.

Moreover, for cases where $r = 1,2,3$ the restrained dominating set consists of one central external vertex graph and one middle vertex of $'s'$. Besides that, check for general cases when $r = m$ and $s = 4$ let the vertex set of $(\mathrm{AW})_m^4$ is $V(AW)_m^4 = \{\mathrm{x}, x_1, x_2, x_3, \ldots, x_{m-1}, x_m\}$. Let $S_m = \{x, x_3, x_7, \ldots, x_{m-1}\}$ is the dominating set for $(\mathrm{AW})_m^4$. Then in the subset $V - S = \{x_1, x_2, x_4, x_5, x_6, \ldots, x_{m-2}, x_{m-3}, x_m\}$ all the elements are adjacent to each other so, $S_m = \{x, x_3, x_7, \ldots, x_{m-1}\}$ is the restrained dominating set of $(\mathrm{AW})_m^4$ and hence, its restrained domination is $r+1$, and hence, verifies that when the Abid-Waheed graph has $r = m$ and $s = 4$ then its restrained domination is $r+1$.

Theorem 4: Let $(\mathrm{AW})_r^s$ be a graph, for every integer $r \geq 1$ and $s = 4$ then the edge domination number of a graph is presented by;

$$\Upsilon'(AW)_r^4 = 2r$$

Proof: Let $H = (AW)_r^4$ be the Abid-Waheed graph with an edge set $E(AW)_r^4 = \{e_1, e_2, e_3, \ldots, e_m\}$. All the edges in $(AW)_r^4$ are accurately outlined so, every cycle demonstrates that the domination number is identical, that is equivalent to $2r$. To begin our proof for the case where $r = 1$ and $s = 4$ and foresee cycle r_1 the edge set is $E(AW)_1^4 = \{e_1, e_2, e_3, e_4, e_5\}$. The structure of the graph H is being examined and finding the domination number by changing $'r'$ and $'s'$ remains the same. As in $(AW)_1^4$ graph degree of two edges is 3 and the degree of three edges is 2. So, $S_1 = \{e_1, e_4\}$ is the edge-dominating set that has minimum cardinality for $(AW)_1^4$. Review that (e_2, e_5) is the neighbor of e_1 and e_3 is the neighbor of e_4 so $S_1 = \{e_1, e_4\}$ is edge dominating set for $(AW)_1^4$ and its domination number is 2.

Currently examine the case where $r = 2$ and $s = 4$, let the edge set of $(AW)_2^4$ be $E(AW)_2^4 = \{e_1, e_2, e_3, \ldots, e_{10}\}$. Here the degree of four edges is 3 and the degree of the remaining edges is 2. Let $S_2 = \{e_1, e_4, e_6, e_9\}$ be an edge-dominating set for $(AW)_2^4$

2024, 21(5): 1639-1648
https://dx.doi.org/10.21123/bsj.2023.8378
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
Baghdad Science Journal

graph, (e_2, e_5) are neighbors of e_1, e_5 is a neighbor of e_4 and (e_7, e_{10}) are in the neighborhood of e_6 and e_8 is a neighbor of e_9. So, $S_2 = \{e_1, e_4, e_6, e_9\}$ is edge dominating set for $(AW)_2^4$ and hence, its dominating number is 4.

Now let us move on to the case where $r = 3$ and $s = 4$ let an edge set of $(AW)_3^4$ graph be $E(AW)_3^4 = \{e_1, e_2, e_3, \dots, e_{15}\}$. Let $S_3 = \{e_1, e_4, e_6, e_9, e_{11}, e_{14}\}$ be the edge-dominating set for $(AW)_3^4$ by considering the degree of vertices. As (e_2, e_5) is directly linked to e_1 and e_3 is a neighbor of e_4, (e_7, e_{10}) are neighbors of e_6 and e_8 is a neighbor of e_9, (e_{12}, e_{15}) are neighbors of e_{11} and e_{14} is a neighbor of e13. So, $S_3 = \{e_1, e_4, e_6, e_9, e_{11}, e_{14}\}$ is the edge-dominating set for $(AW)_3^4$ and hence, its edge-dominating number is 6.

The edge dominating set consists of one rooted edge of cycle $'s'$ and one edge from the cycle of $'s'$ for the case where $r = 1,2,3$. Now to demonstrate for a general case where $r = m$ and $s = 4$ let the edge set of $(AW)_m^4$ be $E(AW)_m^4 = \{e_1, e_2, e_3, \dots, e_{m-1}, e_m\}$. Let $S_m = \{e_1, e_4, e_6, \dots, e_{m-1}\}$ be the edge-dominating set for $(AW)_m^4$ and observe that $(e_1, e_6, e_{11}, \dots, e_{m-4})$ are in the neighborhood of each other, (e_2, e_5) are neighbors of e_1, (e_7, e_{10}) are neighbors of e_6 and likewise, that is continued. So, the set $S_m = \{e_1, e_4, e_6, \dots, e_{m-1}\}$ is set with minimum cardinality that dominates all the edges of $(AW)_m^4$ and hence, its edge domination number is $2r$ so, it is verified that when the Abid-Waheed graph has $r = m$ and $s = 4$ then its edge domination number is $2r$.

Theorem 5: Let $(\mathrm{AW})_r^s$ be a graph, for every integer $r \geq 1$ and $s = 4$ then the independent edge domination number of a graph is presented by;

$$\Upsilon_i'(AW)_r^4 = 2r; r \geq 1$$

Proof: Let $H = (AW)_r^4$ be the Abid-Waheed graph with an edge set $E(AW)_r^4 = \{e_1, e_2, e_3, \dots, e_m\}$. Every edge in $(AW)_r^4$ has a different representation for each cycle, verified as the independent edge domination number, which is $2r$ for all cycles. First conduct a verification process for $r = 1,2$, and 3, and subsequently establish a proof for the general case when $r = m$. Examine the structure of graph H and find independent edge domination by varying $'r'$ while $'s'$ remains the same. Let the case where $r = 1$ and $s = 4$. As in the graph $(AW)_1^4$ the degree of two edges is 3 and the degree of the remaining edges is 2, by considering the degree of the edges let $S_0 = \{e_2, e_4\}$ be the edge-dominating set for $(AW)_1^4$ it is also the independent edge dominating set as no one edge of S_0 is in the neighborhood of each other and its independent edge domination number is 2.

Furthermore, for the case where $s = 4$ and $r = 2$ in the graph $(AW)_2^4$ let $S_1 = \{e_2, e_4, e_7, e_9\}$ be edge dominating set for $(AW)_2^4$ also, an independent edge dominating set for $(AW)_2^4$ as no one edge of S_1 is in the neighborhood of each other so it is the independent edge dominating set. Now for the case where $r = 3$ and $s = 4$ by examining the structure and degree of edges let $S_2 = \{e_2, e_4, e_7, e_9, e_{11}, e_{14}\}$ is an independent edge-dominating set for $(AW)_3^4$ because no edge is adjacent to each other in S_2. Similarly for general cases when $r = m$ and $s = 4$ then $S_m = \{e_2, e_4, e_7, \dots, e_{m-1}\}$ is an independent edge-dominating set for $(AW)_m^4$ as no one edge is adjacent to each other. Concluding that when r varies and s remains constant then the graph will have the same independent edge domination with respect to r which is $2r$. Hence, our claim is true that for $(AW)_m^4$ the independent edge domination is $2r$.

Theorem 6: If $H = (\mathrm{AW})_r^s$ be the Abid-Waheed graph with $r \geq 1$ and $s = 4$ then restrained edge domination number of $(\mathrm{AW})_r^s$ is given by;

$$\Upsilon_r'(AW)_r^4 = 2r; \ r \geq 1$$

Proof: Let H be an $(AW)_r^4$ graph and $E(H)$ is the edge set of the graph whose elements are represented as $(e_1, e_2, e_3, \dots, e_m)$. Edges in $(AW)_r^4$ graph are differently outlined so, it demonstrates that for any number of r-cycles, the restrained edge domination is equivalent to $2r$. The structure of the graph is examined to verify the restrained edge domination of H by changing $'r'$ and $'s'$ to remain constant. Let the case when $r = 1$ and $s = 4$ foresee a cycle r_1 the edge set is $E(AW)_1^4 = \{e_1, e_2, e_3, e_4, e_5\}$. Since the edge dominating set for graph $(AW)_1^4$ is $S_1 = \{e_1, e_4\}$ and in the subset $E - S_1 = \{e_2, e_3, e_5\}$ each edge has a neighbor edge present in the same subset set so, it is the restrained edge dominating set and its restrained edge domination is 2.

Consequently, the in case where $r = 2$ and $s = 4$ let the edge set of $(AW)_2^4$ be $E(AW)_2^4 = \{e_1, e_2, e_3, \dots, e_{10}\}$. Since, $S_2 = \{e_1, e_4, e_6, e_9\}$ is edge dominating set for $(AW)_2^4$ and $E - S_2 = \{e_2, e_3, e_5, e_7, e_8, e_{10}\}$ is the subset of vertices where each edge of $E - S_2$ has a neighboring edge present in $E - S_2$ subset. So, S_2 is restrained dominating set for the case when $s = 4$ and $r = 2$ and restrained edge domination is 4. Now for the case when $r = 3$ and $s = 4$ let the edge set of $(AW)_3^4$ graph be $E(AW)_3^4 = \{e_1, e_2, e_3, \dots, e_{15}\}$. As $S_3 = \{e_1, e_4, e_6, e_9, e_{11}, e_{14}\}$ is the edge-dominating set for $(AW)_3^4$ and the subset $E - S_3$ has neighbors present in $E - S_3$, so S_3 is restrained edge dominating set for $(AW)_3^4$ hence, its restrained edge domination is 6 which is $2r$ as $r = 3$.

As, examine for the cases where $r = 1,2,3$ the restrained edge dominating set consists of two edges of cycle 's', the same is the case for m-edges. Now proved for general case when $r = m$ and $s = 4$ let the edge set of $(AW)_m^4$ be $E(AW)_m^4 = \{e_1, e_2, e_3, \dots, e_{m-1}, e_m\}$. In $E - S_m$ all the edges have adjacent edges present in their own set $E - S_m$ so, for $(AW)_m^4$ the set S_m is restrained edge dominating set hence, its restrained edge domination is $2r$, so the theorem is proved that when the Abid-Waheed graph has $r = m$ and $s = 4$ then its restrained edge domination is $2r$.

Theorem 7: Let $(\mathrm{AW})_r^s$ be a graph then for every integer $r \geq 1$ and $s \geq 3$ the partition dimension of the graph is;

$$PD(AW)_r^s = \begin{cases} 3, & for\ r = 1,2\ and\ s \geq 3 \\ r+1, & for\ r \geq 3, s \geq 3 \end{cases}$$

Proof: Let $H = (\mathrm{AW})_r^s$ be the Abid-Waheed graph with a basis set $V(AW)_r^4 = \{\mathrm{x}, x_1, x_2, \dots, x_m\}$ and x is the universal vertex of a graph by which every cycle is linked to each other. Based on the number of cycles of the graph following cases are formed separately, establishing both claims.

Case-I: If $r = 1$ and 2 for $s \geq 3$ then consider the Partition dimension of $(AW)_r^s$ is 3. Initiate for the case where $r = 1$ and $s = 3$ let basis set $V(AW)_1^3 = \{x, x_1, x_2, x_3\}$ and π_1 be the partition set of H for $s = 3$ and $r = 1$ which consist of the following subsets $\pi_1 = \{S_1, S_2, S_3\}$, by examining the structure of the graph only 3 minimum subsets are formed. Elements of the subsets are arranged in forms like, $S_1 = \{x, x_1\}$, $S_2 = \{x_2\}$ and $S_3 = \{x_3\}$. Since, $(x/\pi_1) = (0,2,2)$, $r(x_1/\pi_1) = (0,1,1)$, $r(x_2/\pi_1) = (1,0,1)$, $r(x_3/\pi_1) = (1,1,0)$. So, its $PD(AW)_1^3 = 3$.

Currently, for cases $r = 1$ and $s = 4$ let the basis set $V(AW)_1^4 = \{x, x_1, x_2, x_3, x_4\}$. Since the highest degree of one vertex is three then necessarily the minimum cardinality of the partition set is three. Let $\pi_2 = \{S_1, S_2, S_3\}$ be the partition of $(AW)_1^4$ with $S_1 = \{x, x_1\}$, $S_2 = \{x_3, x_4\}$ and $S_3 = \{x_2\}$. Since $(x/\pi_2) = (0,2,2)$, $r(x_1/\pi_2) = (0,1,1)$, $r(x_2/\pi_2) = (1,1,0)$, $r(x_3/\pi_2) = (2,0,1)$ and $r(x_4/\pi_2) = (1,1,0)$. All the dimensions are different so, its $PD(AW)_1^4 = 3$.

Now the case where $r = 1$ and $s = 5$ and let the basis set $V(AW)_1^5 = \{x, x_1, x_2, \dots, x_5\}$ and let $\pi_3 = \{S_1, S_2, S_3\}$ be the partition of $(AW)_1^5$ with $S_1 = \{x, x_1\}$, $S_2 = \{x_5, x_4\}$ and $S_3 = \{x_2, x_3\}$. Since $(x/\pi_3) = (0,2,2)$, $r(x_1/\pi_3) = (0,1,1)$, $r(x_2/\pi_3) = (1,1,0)$, $r(x_3/\pi_3) = (3,1,0)$ $r(x_4/\pi_3) = (2,0,1)$ and $r(x_5/\pi_3) = (1,0,2)$, all dimensions are different with respect to π_3 so, its $PD(AW)_1^5 = 3$.

Assume for $r = 1$ and $s = m$ the basis set $V(AW)_1^m = \{x, x_1, x_2, \dots, x_s, \dots, x_n\}$ and let $\pi_m = \{S_1, S_2, S_3\}$ be the partition of $(AW)_1^m$ with $S_1 = \{x, x_1\}$, $S_2 = \{x_2, x_3, \dots, x_s\}$ and $S_3 = \{x_{s+1}, x_{s+2}, \dots, x_m\}$. Since $r(x/\pi_n) = (0,2,2)$, $r(x_1/\pi_n) = (0,1,1)$, $r(x_2/\pi_n) = (1,0,2), \dots$, $r(x_{m-1}/\pi_n) = (2,3,0)$ $r(x_m/\pi_n) = (1,2,0)$, because all the dimensions are different so, its $PD(AW)_1^m = 3$.

Furthermore, suppose for $r = 2$ and $s = 3$ the vertex set $V(AW)_2^3 = \{x, x_1, x_2, \dots, x_6\}$ and let $\pi_1 = \{S_1, S_2, S_3\}$ be the partition of $(AW)_2^3$ with $S_1 = \{x, x_3, x_6\}$, $S_2 = \{x_2, x_1\}$ and $S_3 = \{x_5, x_4\}$. Since $(x/\pi_1) = (0,1,1)$, $r(x_1/\pi_1) = (1,0,2)$, $r(x_2/\pi_1) = (1,0,3)$, $r(x_3/\pi_1) = (0,1,3),)$, $r(x_4/\pi_1) = (1,2,0)$, $r(x_5/\pi_1) = (1,3,0)$, $r(x_6/\pi_1) = (0,3,1)$. As all the dimensions are different so, its $PD(AW)_2^3 = 3$.

After that let for $r = 2$ and $s = 4$ the vertex set $V(AW)_2^4 = \{x, x_1, x_2, \dots, x_8\}$ and let subsets $\pi_2 = \{S_1, S_2, S_3\}$ be the partition of $(AW)_2^4$ with $S_1 = \{x, x_1, x_2, x_3\}$, $S_2 = \{x_5, x_6, x_7\}$ and $S_3 = \{x_8, x_4\}$. Since $(x/\pi_2) = (0,1,1)$, $r(x_1/\pi_2) = (0,3,2)$, $r(x_2/\pi_2) = (1,0,3)$, $r(x_3/\pi_2) = (0,4,1)$, $r(x_4/\pi_2) = (1,3,0)$, $r(x_5/\pi_2) = (1,0,1)$, $r(x_6/\pi_2) = (2,0,2)$, $r(x_7/\pi_2) = (3,0,1)$, $r(x_8/\pi_2) = (2,1,0)$. So, its $PD(AW)_2^4 = 3$.

Finally, take for the case where $r = 2$ and $s = 5$ the vertex set $V(AW)_2^5 = \{x, x_1, x_2, \dots, x_{10}\}$ and let $\pi_3 = \{S_1, S_2, S_3\}$ be the partition of $(AW)_2^5$ with $S_1 = \{x, x_4, x_5, x_9, x_{10}\}$, $S_2 = \{x_1, x_2, x_3\}$ and $S_3 = \{x_8, x_6, x_7\}$. Since, $(x/\pi_3) = (0,1,1)$, $r(x_1/$

2024, 21(5): 1639-1648
https://dx.doi.org/10.21123/bsj.2023.8378
P-ISSN: 2078-8665 - E-ISSN: 2411-7986

$\pi_3) = (2,3,0)$, $r(x_2/\pi_3) = (1,0,3)$, $r(x_3/\pi_3) = (1,0,4)$, $r(x_4/\pi_3) = (0,1,4)$, $r(x_5/\pi_3) = (0,1,3)$, $r(x_6/\pi_3) = (1,2,0)$, $r(x_7/\pi_3) = (2,3,0), r(x_8/\pi_3) = (1,4,0)$, $r(x_9/\pi_3) = (0,4,1), r(x_{10}/\pi_3) = (0,3,1)$. Hence its $PD(AW)_2^5 = 3$. Similarly for 'm' vertices of cycle 's' and $r = 2$ the dimension is 3. Hence, our first condition that is for $r = 1$, $r = 2$ and $s \geq 3$ the $PD(AW)_r^s = 3$ here our first assumption is verified.

Case-II: In this case, verify that the partition dimension of (Abid-Waheed) $(AW)_r^s$ graph is '$r + 1$' for $r \geq 3$. Proving this case by varying 'r' and 's' in the graph and showing that their partition dimension is $r + 1$. First, let a graph with $r = 3$ and $s = 3$ have vertex set $V(AW)_3^3 = \{x, x_1, x_2, \dots, x_9\}$ and $\pi_1 = \{S_1, S_2, S_3\}$ be the partition of $(AW)_3^3$. The partition sets consist of all the vertices of the graph as partitioned subsets which are $S_1 = \{x, x_4, x_5\}$, $S_2 = \{x_1, x_2, x_8\}$ and $S_3 = \{x_3, x_6, x_7, x_9\}$. Since the partition dimension for $(AW)_3^3$ is represented as $r(x/\pi_1) = (0,1,1)$, $r(x_1/\pi_1) = (1,0,1), r(x_2/\pi_1) = (2,0,1)$, $r(x_3/\pi_1) = (2,1,0), r(x_5/\pi_1) = (0,3,1), r(x_6/\pi_1) = (1,3,0), r(x_7/\pi_1) = (2,0,1)$, $r(x_8/\pi_1) = (1,4,0)$, $r(x_9/\pi_1) = (2,1,0)$. Since each vertex of $(AW)_3^3$ has no distinct representation with respect to π_1 as $r(x_3/\pi_1) = r(x_9/\pi_1)$.

Then take another partitioned set $\pi_2 = \{S_1, S_2, S_3, S_4\}$ to be the partition of $V(AW)_3^3$ with $S_1 = \{x, x_6, x_3, x_9\}$, $S_2 = \{x_1, x_2\}, S_3 = \{x_4, x_5\}$ and $S_4 = \{x_7, x_9\}$. Since the partition dimension for $(AW)_3^3$ is represented as $r(x/\pi_2) = (0,1,1,1)$, $r(x_1/\pi_2) = (1,0,2,2)$, $r(x_2/\pi_2) = (1,0,3,3)$, $r(x_3/\pi_2) = (0,1,3,3)$, $r(x_4/\pi_2) = (1,2,0,2)$, $r(x_5/\pi_2) = (1,3,0,3)$, $r(x_6/\pi_2) = (1,3,3,0)$, $), r(x_7/\pi_1) = (2,0,1)$, $r(x_8/\pi_2) = (1,3,3,0)$, $r(x_9/\pi_2) = (0,3,3,1)$. As each vertex of $(AW)_3^3$ has a distinct representation with respect to π_2, hence π_2 is a resolving partition of $(AW)_3^3$. Hence, partition dimension of $(AW)_3^3 = 4$.

Furthermore, let the case when $s = 4$ and $r = 3$ graph have basis set $V(AW)_3^4 = \{x, x_1, x_2, \dots, x_{12}\}$. Take a partition set π_3 containing all the vertices of $(AW)_3^4$ in different subsets. Let $\pi_3 = \{S_1, S_2, S_3, S_4\}$ be the partition of $(AW)_3^4$ with $S_1 = \{x, x_1, x_5, x_9\}$, $S_2 = \{x_3, x_2, x_8\}, S_3 = \{x_6, x_7, x_{12}\}$ and $S_4 = \{x_4, x_{10}, x_{11}\}$. Since the partition dimension is represented as $r(x/\pi_3) = (0,2,2,2)$, $r(x_1/\pi_3) = (0,1,3,1)$, $r(x_2/\pi_3) = (1,0,4,1)$, $r(x_3/\pi_3) = (2,0,5,1)$, $r(x_4/\pi_3) = (1,1,4,0)$, $r(x_5/\pi_3) = (0,3,1,3,)$, $r(x_6/\pi_3) = (1,2,0,4)$, $), r(x_7/\pi_3) = (2,1,0,5)$, $r(x_8/\pi_3) = (2,0,2,4)$, $r(x_9/\pi_3) = (0,3,3,1)$, $r(x_{10}/\pi_3) = (1,4,2,0)$, $r(x_{11}/\pi_3) = (2,5,1,0)$, $r(x_{12}/\pi_3) = (2,4,0,1)$. As each vertex of $(AW)_3^4$ has a distinct representation with respect to π_3, then π_3 is a resolving partition of $(AW)_3^4$. Hence, the partition dimension of $(AW)_3^4$ =4.

Moreover, check for the case where $s = 5$ and $r = 3$ the Abid-Waheed graph with vertex set $V(AW)_3^5 = \{x, x_1, x_2, \dots, x_{15}\}$ and take a partitioned set consisting of different subsets, let $\pi_4 = \{S_1, S_2, S_3, S_4\}$ be the partition of $(AW)_3^5$ with $S_1 = \{x, x_1, x_6, x_{11}\}$, $S_2 = \{x_2, x_3, x_4, x_{10}\}$, $S_3 = \{x_5, x_7, x_8, x_{15}\}$, and $S_4 = \{x_{14}, x_9, x_{12}, x_{13}\}$. Since $r(x/\pi_4) = (0,1,2,2)$, $r(x_1/\pi_4) = (0,1,1,3)$, $r(x_2/\pi_4) = (1,0,2,4)$, $r(x_3/\pi_4) = (2,0,2,5)$, $r(x_4/\pi_4) = (2,0,1,5)$, $r(x_5/\pi_4) = (1,1,0,4)$, $r(x_6/\pi_4) = (0,1,1,2)$, $), r(x_7/\pi_4) = (1,2,0,2)$, $r(x_8/\pi_4) = (2,2,0,1)$, $r(x_9/\pi_4) = (0,3,1,1)$, $r(x_{10}/\pi_4) = (1,0,2,1)$, $r(x_{11}/\pi_4) = (0,3,1,1)$, $r(x_{12}/\pi_4) = (2,4,0,1), r(x_{13}/\pi_4) = (2,5,2,0)$, $r(x_{14}/\pi_4) = (2,5,1,0)$, $r(x_{15}/\pi_4) = (1,4,0,1)$. As each vertex of $(AW)_3^5$ has a distinct representation with respect to π_4, then π_4 is a resolving partition of $(AW)_3^5$. Hence, the partition dimension of $(AW)_3^5 = 4$ which is $r + 1$. So, for all $r = 3$ and $s \geq 3$ the partition dimension is $r + 1$.

Now verify for $r = 4$ and $s \geq 3$ and let the graph H with $r = 4$ and $s = 3$ and the vertex set of $(AW)_4^3$ is $V(AW)_4^3 = \{x, x_1, x_2, \dots, x_{12}\}$. Take the partitioned set π_1 of graph $(AW)_4^3$ which contained all the vertices in five subsets that are $\pi_1 = \{S_1, S_2, S_3, S_4, S_5\}$ be the partition of $\mathrm{V}(AW)_4^3$ with $S_1 = \{x, x_3, x_6, x_9, x_{12}\}$, $S_2 = \{x_1, x_2\}$, $S_3 = \{x_5, x_4\}, S_4 = \{x_7, x_8\}$ $S_5 = \{x_{10}, x_{11}\}$. Since the partition resolving representation is $r(x/\pi_1) = (0, 1, 1, 1, 1)$, $r(x_1/\pi_1) = (1, 0, 2, 2, 2)$, $r(x_2/\pi_1) = (1, 0, 3, 3, 3)$, $r(x_3/\pi_1) = (0, 1, 3, 3, 3)$, $r(x_4/\pi_1) = (1, 2, 0, 2, 2)$, $r(x_5/\pi_1) = (1, 3, 0, 3, 3)$, $r(x_6/\pi_1) = (0, 3, 1, 3, 3)$, $r(x_7/\pi_1) = (1, 2, 2, 0, 2)$, $r(x_8/\pi_1) = (1, 3, 3, 0, 3)$, $r(x_9/\pi_1) = (0, 3, 3, 1, 3)$, $r(x_{10}/\pi_1) = (1, 2, 2, 2, 0)$, $r(x_{11}/\pi_1) = (1, 3, 3, 3, 0)$, $r(x_{12}/\pi_1) = (0, 3, 3, 3, 1)$. As each vertex of $(AW)_4^3$ has a distinct representation with respect to π_1, then π_1 is a resolving partition of $(AW)_4^3$.

Consequently, check for $r = 4$ and $s = 4$ the partition dimension of $(AW)_4^4$. Let the vertex set be

2024, 21(5): 1639-1648
https://dx.doi.org/10.21123/bsj.2023.8378
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
Baghdad Science Journal

$V(AW)_4^4 = \{x, x_1, x_2, \ldots, x_{16}\}$. Let π_2 be the partition of $(AW)_4^4$ for $s = 4$ and $r = 4$. The partition set consists of the following vertices set $\pi_2 = \{S_1, S_2, S_3, S_4, S_5\}$ be the partition of $(AW)_4^4$ with $S_1 = \{x, x_1, x_5, x_9, x_{13}\}$, $S_2 = \{x_3, x_2, x_{16}\}$, $S_3 = \{x_4, x_6, x_7\}$, $S_4 = \{x_{10}, x_{11}, x_{15}\}$, $S_5 = \{x_8, x_{14}, x_{12}\}$. Since $r(x/\pi_2) = (0,2,2,2,2), r(x_1/\pi_2) = (0,1,1,3,3), r(x_2/\pi_2) = (1,0,1,4,4), r(x_3/\pi_2) = (2,0,1,4,4), r(x_4/\pi_2) = (1,2,0,4,4), r(x_5/\pi_2) = (0,3,1,3,3), r(x_6/\pi_2) = (1,4,0,4,4)$, $r(x_7/\pi_2) = (2,5,0,5,5), r(x_8/\pi_2) = (1,4,0,4,4), r(x_9/\pi_2) = (0,3,3,1,1), r(x_{10}/\pi_2) = (1,4,4,0,2)$, $rx_{11}(/\pi_2) = (2,5,5,0,5)$, $r(x_{12}/\pi_2) = (1,4,4,1,0)$, $r(x_{13}/\pi_2) = (0,1,3,2,1)$, $r(x_{14}/\pi_2) = (1,2,4,1,0)$, $r(x_{15}/\pi_2) = (2,1,5,0,1)\ and\ r(x_{16}/\pi_2) = (1,0,4,1,2)$.

As all vertices of $(AW)_4^4$ have a distinct representation with respect to π_2, then π_2 is a resolving partition of $(AW)_4^4$. Hence, the partition dimension of $(AW)_4^4 = r + 1 = 4 + 1 = 5$. The hypothesis assumed that for all $r \geq 3$ and $s \geq 3$ each vertex of $(AW)_4^4$ has a distinct representation with respect to π, then π is a resolving partition of $(AW)_4^4$. Hence, our second assumption is true and our theorem is proved.

Lemma 1: Let $(AW)_r^s$ be a graph for every integer $r \geq 1$ and $s = 4$ then the relation between domination number and independent domination is presented as:

$$\Upsilon_i(AW)_r^4 = \Upsilon(AW)_r^4 = r + 1$$

Lemma 2: Let $(AW)_r^s$ be a graph for every integer $r \geq 1$ and $s = 4$ then the relation between independent domination and restrained domination is given by;

$$\Upsilon_i(AW)_r^4 = \Upsilon_r(AW)_r^4 = r + 1$$

Conclusion

This paper discussed the vertex and edge domination, independent domination, restrained domination, and relations between them of the Abid-Waheed graph $(AW)_r^s$ and partition dimension of the $(AW)_r^s$ graph, which helps in solving many problems like the networking problem related to the location, aviation, chemical theory, and computer networking in the structures and networks related to the Abid-Waheed graph.

Data Availability

In this article, no data were utilized.

Funding statement

This research received no funding.

Acknowledgment

The authors would like to thank the anonymous referees for their valuable comments and suggestions, which led to considerable improvement of the article.

Authors' Declaration

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are ours. Furthermore, any Figures and images, that are not ours, have been included with the necessary permission for re-publication, which is attached to the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in University of Education, Lahore, Pakistan.

Authors' Contribution Statement

All authors equally contributed to this work. All authors read and approved the final manuscript. Conceptualization; A. M. and M. W. R., Methodology; M. W. R. and A. M., Investigation; D.

N. and M. W.R., Writing original draft preparation; D. N., Writing review and editing; M. W. R and D. N., Supervision; A. M. and j. H. H. B. All authors have read and agreed to the published version of the manuscript.

References

1. Laskar RC, Walikar HB. On Domination Related Concepts in Graph Theory. *J Comb Math Comb Comput*. 1981; 855(1): 308-320. https://dx.doi.org/10.1007/BFb0092276
2. Galleros DH., Enriquez EL. Fair Restrained Dominating Set in the Corona of Graphs. Int J Eng. 2020; 10(3): 110-4. https://dx.doi.org/10.31033/ijemr.10.3.17
3. Al-Harere MN, Bakhash KPA. Tadpole Domination in Graphs. Baghdad Sci J. 2018 Dec 9; 15(4): 466-471. https://dx.doi.org/10.21123/bsj.2018.15.4.0466
4. Omran AA., Oda HH. Hn Domination in Graphs. Baghdad Sci J. 2019; 16(1(Suppl.)): 242-247. https://dx.doi.org/10.21123/bsj.2019.16.1(Suppl.).0242
5. Gupta P. Domination in Graph with Application. Indian J Sci Res. 2013; 2(3): 115-117.
6. Omran AA., Swadi T. Observer Domination Number in Graphs. J Dyn Control Syst. 2019; 11(01-Special Issue).
7. Goddard W, Henning MA. Independent Domination in Graphs: A Survey and Recent Results. Discret *Appl. Math.* 2013 Apr 6; 313(7): 839-54. https://dx.doi.org/10.1016/j.disc.2012.11.031
8. Wen T, Cheong KH. The Fractal Dimension of Complex Networks: A Review. Inf Fusion. 2021; 73: 87-102. DOI: https://doi.org/10.1016/j.inffus.2021.02.001
9. Haynes TW, Hedetniemi ST, Henning MA, editors. Topics in Domination in Graphs. J Comb Math Comb Comput. 2020 Oct 19; 64. https://dx.doi.org/10.1007/978-3-030-51117-3
10. Rehman SU, Imran M, Javaid I. On the Metric Dimension of Arithmetic Graph of a Composite Number. Symmetry. 2020; 12(4); 607. https://dx.doi.org/10.3390/sym12040607
11. Ikhlaq HM, Hayat S, Siddiqui HM. Unique Identification and Domination of Edges in a Graph: The Vertex-Edge Dominant Edge Metric Dimension. ArXiv org. 2022 Nov 17; 2211(09327). https://dx.doi.org/10.48550/arXiv.2211.09327
12. Hwang SF, Chang G. The Edge Domination Problem Discuss. Math Graph Theory. 1995; 15(1): 51-57.
13. Lu CL, Ko MT, Tang CY. Perfect Edge Domination and Efficient Edge Domination in Graphs. Discret *Appl Math.* 2002; 119(3): 227-25. https://dx.doi.org/10.1016/S0166-218X(01)00198-6
14. Panda BS, Chaudhary J. Acyclic Matching in Some Subclasses of Graphs. Theor Comput Sci. 2020; 12126: 409–421. https://dx.doi.org/10.1016/j.tcs.2022.12.008
15. Żyliński P. Vertex-Edge Domination in Graphs. Aequ Math. 2019 Aug; 93(4): 735-42.
16. Dong F, Ge J, Yang Y. Upper Bounds on the Signed Edge Domination Number of a Graph. Discret Appl *Math.* 2021 Feb 1; 344(2): 112201. https://dx.doi.org/10.1016/j.disc.2020.112201
17. Grinstead DL, Slater PJ, Sherwani NA, Holmes ND. Efficient Edge Domination Problems in Graphs. Inf Process Lett. 1993; 48(5): 221-228. https://dx.doi.org/10.1016/0020-0190(93)90084-M
18. Vaidya SK., Ajani PD. Restrained Edge Domination Number of Some Path Related Graphs. J Sci Res. 2021; 13(1): 145-151. https://dx.doi.org/10.3329/jsr.v13i1.48520
19. Al-Harere MN, Metlif RJ, Sadiq FA. Variant Domination Types for a Complete h-ary Tree. Baghdad Sci J. 2021; 18(1): 797-802. https://dx.doi.org/10.21123/bsj.2021.18.1(Suppl.).0797
20. Raj RN, Raj FS. On the Partition Dimension of Honey Comb, Hexagonal Cage Networks and Quartz Network Ann. Romanian Soc. Cell Biol. 2021 Apr 11; 2811-7.
21. Tomescu I, Javaid I, Slamin I. On the Partition Dimension and Connected Partition Dimension of Wheels. Ars Comb. 2007; 84: 311-318.
22. Singh P, Sharma S, Sharma SK, Bhat VK. Metric Dimension and Edge Metric Dimension of Windmill Graphs. AIMS Mathematics. 2021 Jan 1; 6(9): 9138-53. https://dx.doi.org/10.3934/math.2021531
23. Wei C, Nadeem MF, Siddiqui HMF, Azeem M, Liu JB, Khalil A. On Partition Dimension of Some Cycle-Related Graphs. Math Probl Eng. 2021; 2021: 1-8. https://dx.doi.org/10.1155/2021/4046909
24. Chu YM, Nadeem MF, Azeem M, Siddiqui MK. On Sharp Bounds on Partition Dimension of Convex Polytopes. IEEE Access. 2020 Dec 14; 8: 224781-90. https://dx.doi.org/10.1109/ACCESS.2020.3044498
25. Chartrand G, Salehi E, Zhang P. On the Partition Dimension of a Graph. Congr Numer. 1998; 130: 157168.
26. Goddard W., Henning MA. Independent Domination in Outerplanar Graphs. Discret Appl *Math.* 2023 Jan

2024, 21(5): 1639-1648
https://dx.doi.org/10.21123/bsj.2023.8378
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
Baghdad Science Journal

30; 325: 52-7. https://dx.doi.org/10.1016/j.dam.2022.10.003

27. Idrees M, Ma H, Wu M, Nizami AR, Munir M, Ali S. Metric Dimension of Generalized Möbius Ladder and its Application to WSN Localization. J *Adv Comput* Intell *Intell Inform*. 2020 Jan 20; 24(1): 3-11. https://dx.doi.org/10.20965/jaciii.2020.p0003
28. Mahboob A, Rasheed MW. Hosaya Polynomial and Weiner Index of Abid-Waheed Graph $(AW)_p^6$. Appl Comput Math. 2022; 10(3): 89-94. https://dx.doi.org/10.11648/j.ijebo.20221003.13

حول بعد التقسيم وهيمنة الرسم البياني لعابد - وحيد $(AW)_r^4$

جلال حاتم حسين البياتي [1]، عابد محبوب * [2]، محمد وحيد رشيد[2]، دور النجف [2]

[1] قسم الرياضيات، الكلية العلوم للبنات، جامعة بغداد، بغداد، العراق
[2] قسم الرياضيات، شعبة العلم والتكنولوجيا، جامعة التربية، لاهور، باكستان

الخلاصة

يحتوي الرسم البياني H المرتبط ببساطة على مجموعة رؤوس $V(H) = \{x, x_1, x_2, \dots, x_m\}$ بالنظر إلى الرسم البياني $H = (V, E)$ ، فإن المجموعة المسيطرة S هي مجموعة فرعية من مجموعة الرؤوس V بحيث يكون أي رأس خارج S قريبًا من رأس واحد على الأقل داخل S. أصغر حجم من H هو المسيطر على المجموعات هو المعروف برقم سيطرة الرسم البياني .عندما يكون للرسم البياني H رأس x ومجموعة فرعية من الرأس V (H) ، يكون الفصل $d(x, S) = min\{d(x, u)|u; S\}$. فيما يتعلق بالقسم k المرتب= π $\{S_1, S_2, , S_k\}$ من V (H) ، فإن الرسم التوضيحي لـ x بالنسبة إلى Π هو المتجه$(r(x/\Pi) = (d(x, S_1), d(x, S_2), , d(x, S_k))$ في هذا البحث ، قمنا بدراسة بعض النتائج على رقم الهيمنة ورقم السيطرة المستقل والمقيد والمشار إليه بـ $\gamma(H)$, $\gamma_i(H)$ and $\gamma_r(H)$ على التوالي في الرسوم البيانية Abid-Waheed $(AW)_r^4$ والعلاقة بين رقم الهيمنة ورقم الهيمنة المستقلة والمقيدة .كما أن الهدف من هذه الورقة هو إنشاء أبعاد التقسيم لـ $(AW)_r^4$.

الكلمات المفتاحية: رسم عابد وحيد، الهيمنة، الهيمنة المستقلة، بعد التقسيم، الهيمنة المقيدة