

Certain results on a class of multivalent functions of power order defined by a linear integral operator

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Abstract .

By using a linear integral operator , a class of multivalent functions of power order is introduced. Some important results of this class such as coefficient estimates , integral means inequalities and other property are found.

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1. Introduction

Let $\Sigma_{p,\eta}$ be the class of functions f of the form:

$$f(z) = z^{p+\eta} + \sum_{k=2}^{\infty} a_k z^{k+p+\eta}, \quad (0 \leq \eta < 1, a_k \geq 0) \quad (1)$$

that are analytic in the unit disk

$$U = \{z \in \mathbb{C}: |z| < 1\}.$$

Let

$$g(z) = z^{p+\eta} + \sum_{k=2}^{\infty} b_k z^{k+p+\eta}, \quad (0 \leq \eta < 1, b_k \geq 0, z \in U). \quad (2)$$

A convolution (orHadamard product) of two power series f of the form (1) and a function g of the form (2) is defined by :

$$(f * g)(z) = z^{p+\eta} + \sum_{k=2}^{\infty} a_k b_k z^{k+p+\eta} = (g * f)(z), \quad (z \in U).$$

Note that the authors defined and studied some classes of analytic functions take the form (1) in [1].

In this paper , we need to introduce a generalized integral operator such that class can be defined by means of this integral operator. For a function $f \in \Sigma_{p,\eta}$ given by (1), we define the integral operator $K_{c,p,\eta}^{\delta}$ ($c > -(p + \eta)$) and ($0 \leq \eta < 1, p \in \mathbb{N}$) by:

$$\begin{aligned} & K_{c,p,\eta}^{\delta} f(z) \\ &= \frac{(c + p + \eta)^{\delta}}{\Gamma(\delta) z^c} \int_0^z t^{c-1} \left(\log \frac{z}{t}\right)^{\delta-1} f(t) dt \\ &= z^{p+\eta} + \sum_{k=2}^{\infty} \left[\frac{c + p + \eta}{c + k + p + \eta} \right]^{\delta} a_k z^{k+p+\eta}, \quad (3) \end{aligned}$$

when $\eta = 0$ the operator $K_{c,p,0}^{\delta}$ was introduced by Komatu [7].

Clearly , (3) yields :

$$f \in \Sigma_{p,\eta} \Rightarrow K_{c,p,\eta}^\delta f \in \Sigma_{p,\eta}.$$

A function $f \in \Sigma_{p,\eta}$ is said to be $(p + \eta)$ -valent starlike of order ρ , $(0 \leq \rho < p + \eta)$ if and only if :

$$Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \rho, \quad (z \in U).$$

Let $S_{p,\eta}(\rho)$ denote the class of all those functions.

A function $f \in \Sigma_{p,\eta}$ is said to be $(p + \eta)$ -valent convex of order ρ , $(0 \leq \rho < p + \eta)$ if and only if :

$$Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \rho, \quad (z \in U).$$

Let $C_{p,\eta}(\rho)$ denote the class of all those functions.

A function $f \in \Sigma_{p,\eta}$ is said to be $(p + \eta)$ -valent close-to-convex of order ρ , $(0 \leq \rho < p + \eta)$ if and only if :

$$Re \left\{ \frac{f'(z)}{z^{(p+\eta)-1}} \right\} > \rho, \quad (z \in U).$$

Let $K_{p,\eta}(\rho)$ denote the class of all those functions.

We suppose $L(p, \eta, \delta)$ denote the subclass of $\Sigma_{p,\eta}$ consisting of functions f which satisfy :

$$\left| \frac{1}{\beta} \left[\frac{\mu z \left(K_{c,p,\eta}^\delta f'(z) \right)' + (1 - \mu) \left(K_{c,p,\eta}^\delta f(z) \right)'}{\mu \left(K_{c,p,\eta}^\delta f'(z) \right) + (1 - \mu) z^{(p+\eta)-1}} - (p + \eta) \right] \right| < \gamma, \quad (4)$$

where $\delta > 0, c > -(p + \eta), 0 \leq \eta < 1, \beta \in \mathbb{C} \setminus \{0\} = \mathbb{C}^*, 0 < \gamma \leq p + \eta$ and $K_{c,p,\eta}^\delta f$ given by (3).

2. Coefficient estimates

Theorem 1.

$$\left| \frac{1}{\beta} \left[\frac{\mu z \left(K_{c,p,\eta}^\delta f'(z) \right)' + (1 - \mu) \left(K_{c,p,\eta}^\delta f(z) \right)'}{\mu \left(K_{c,p,\eta}^\delta f'(z) \right) + (1 - \mu) z^{(p+\eta)-1}} - (p + \eta) \right] \right| < \gamma, \quad (z \in U).$$

Let the function $f \in \Sigma_{p,\eta}$ be defined by (1).

Then $f \in L(p, \eta, \delta)$ if and only if

$$\sum_{k=2}^{\infty} a_k \frac{(k + p + \eta)[1 + \mu(k + \gamma|\beta| - 2)]}{\gamma|\beta|(1 - \mu) + (\gamma|\beta| - 1)\mu(p + \eta)} \left(\frac{c + p + \eta}{c + k + p + \eta} \right)^\delta \leq 1, \quad (5)$$

where $\delta > 0, c > -(p + \eta), 0 \leq \eta < 1, \beta \in \mathbb{C} \setminus \{0\} = \mathbb{C}^*$ and $0 < \gamma \leq p + \eta$.

Proof. Assume that inequality (5) holds true and $|z| = 1$. Then, we obtain

$$\begin{aligned} & \left| \mu z \left(K_{c,p,\eta}^\delta f'(z) \right)' + (1 - \mu) \left(K_{c,p,\eta}^\delta f(z) \right)' \right. \\ & \left. - (p + \eta) \mu \left(K_{c,p,\eta}^\delta f'(z) \right) - (1 - \mu)(p + \eta)z^{(p+\eta)-1} \right| \\ & \quad - \gamma|\beta| \left| \mu \left(K_{c,p,\eta}^\delta f'(z) \right) + (1 - \mu)z^{(p+\eta)-1} \right| \\ & = \left| -\mu(p + \eta)z^{(p+\eta)-1} \right. \\ & \quad \left. + \sum_{k=2}^{\infty} (k + p + \eta) \left(\frac{c + p + \eta}{c + k + p + \eta} \right)^\delta [\mu(k - 1) \right. \right. \\ & \quad \left. \left. + (1 - \mu)] a_k z^{k+p+\eta-1} \right| \\ & = \gamma|\beta| \left| [\mu(p + \eta) + (1 - \mu)z^{(p+\eta)-1}] \right. \\ & \quad \left. + \sum_{k=2}^{\infty} \mu(k + p + \eta) \left(\frac{c + p + \eta}{c + k + p + \eta} \right)^\delta a_k z^{k+p+\eta-1} \right| \\ & \leq \sum_{k=2}^{\infty} (k + p + \eta)[1 + \mu(k + \gamma|\beta| - 2)] \\ & \quad \times \left(\frac{c + p + \eta}{c + k + p + \eta} \right)^\delta a_k \end{aligned}$$

$$\leq \gamma|\beta|(1 - \mu) + (\gamma|\beta| - 1)\mu(p + \eta),$$

by hypothesis.

Hence, by maximum modulus principle, we have $f \in L(p, \eta, \delta)$. Conversely, let $f \in L(p, \eta, \delta)$. Then

That is

$$= \left| -\mu(p + \eta)z^{(p+\eta)-1} \right|$$

$$\begin{aligned}
 & + \sum_{k=2}^{\infty} (k+p+\eta) \left(\frac{c+p+\eta}{c+k+p+\eta}\right)^{\delta} [\mu(k-1) \\
 & \quad + (1-\mu)] a_k z^{k+p+\eta-1} \\
 & \div [(\mu(p+\eta) + (1-\mu))z^{(p+\eta)-1} \\
 & \quad + \sum_{k=2}^{\infty} \mu(k+p \\
 & \quad + \eta) \left(\frac{c+p+\eta}{c+k+p+\eta}\right)^{\delta} a_k z^{k+p+\eta-1}] \\
 & < \gamma|\beta| , \tag{6}
 \end{aligned}$$

since $|Ref(z)| \leq |f(z)|$ for all z , we have

$$\begin{aligned}
 & |Re\{-\mu(p+\eta)z^{(p+\eta)-1} \\
 & + \sum_{k=2}^{\infty} (k+p+\eta) \left(\frac{c+p+\eta}{c+k+p+\eta}\right)^{\delta} [\mu(k-1) + (1-\mu)] \\
 & \quad \times a_k z^{k+p+\eta-1}\} \\
 & \div [(\mu(p+\eta) + (1-\mu))z^{(p+\eta)-1} \\
 & + \sum_{k=2}^{\infty} \mu(k+p+\eta) \left(\frac{c+p+\eta}{c+k+p+\eta}\right)^{\delta} a_k z^{k+p+\eta-1}] \} \\
 & < \gamma|\beta| , \tag{7}
 \end{aligned}$$

choosing z on real axis and allowing $z \rightarrow 1$, we have

$$\begin{aligned}
 & \frac{\mu(p+\eta) + \sum_{k=2}^{\infty} (k+p+\eta) \left(\frac{c+p+\eta}{c+k+p+\eta}\right)^{\delta} [\mu(k-1) + (1-\mu)] a_k}{(\mu(p+\eta) + (1-\mu)) - \sum_{k=2}^{\infty} \mu(k+p+\eta) \left(\frac{c+p+\eta}{c+k+p+\eta}\right)^{\delta} a_k} \\
 & \leq \gamma|\beta|
 \end{aligned}$$

which gives (5).

Finally, the result is sharp with extremal function f given by :

$$\begin{aligned}
 f(z) \\
 = z^{p+\eta}
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{\gamma|\beta|(1-\mu) + (\gamma|\beta| - 1)\mu(p+\eta)}{(k+p+\eta)[1 + \mu(k + \gamma|\beta| - 2)] \left(\frac{c+p+\eta}{c+k+p+\eta}\right)^{\delta}} z^{k+p+\eta}, \tag{8} \\
 & k \geq 2.
 \end{aligned}$$

Corollary 1. Let the function f defined by (1) be in the class $L(p, \eta, \delta)$. Then

$$\begin{aligned}
 & a_k \\
 & \leq \frac{\gamma|\beta|(1-\mu) + (\gamma|\beta| - 1)\mu(p+\eta)}{(k+p+\eta)[1 + \mu(k + \gamma|\beta| - 2)] \left(\frac{c+p+\eta}{c+k+p+\eta}\right)^{\delta}}, \\
 & k \geq 2. \tag{9}
 \end{aligned}$$

The equality in (9) is attained for the function f given by (8).

Definition 1. Let $0 \leq \lambda < 1$ and $\xi, \zeta \in \mathbb{R}$. Then in terms of Gauss's hypergeometric function ${}_2F_1$, the generalized fractional derivative operator of order λ of the function f is defined by:

$$\begin{aligned}
 & D_{0,z}^{\lambda, \xi, \zeta} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \\
 & \times \left\{ z^{\lambda-\xi} \int_0^z (z-t)^{-\lambda} {}_2F_1 \left(\xi - \lambda, 1 - \zeta; 1 - \lambda; 1 - \frac{t}{z} \right) f(t) dt \right\}, \tag{10}
 \end{aligned}$$

where the function f is analytic in a simply connected region of the z -plane containing the origin, with the order

$f(z) = O(|z|^\epsilon)$, ($z \rightarrow 0$) for $\epsilon > \max\{0, \xi - \zeta\} - 1$ and the multiplicity of $(z-t)^{-\lambda}$ is removed by requiring $\log(z-t)$ to be real when $(z-t) > 0$.

Note that

$$D_{0,z}^{\lambda, \lambda, \zeta} f(z) = D_z^\lambda f(z), \quad (0 \leq \lambda < 1), \tag{11}$$

where

$$\begin{aligned}
 D_z^\lambda f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^\lambda} dt, \\
 (0 \leq \lambda < 1) \tag{12}
 \end{aligned}$$

is the Riemann – Liouville fractional derivative of order λ .

Definition 2. Under the hypothesis of Definition 1 , the fractional derivative of order $n + \lambda$ is defined , for a function f , by

$$D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} D_z^\lambda f(z), \quad (0 \leq \lambda < 1; n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).$$

It readily follows from Definition 1 that

$$D_z^\lambda z^{k+p+\eta} = \frac{\Gamma(k+p+\eta+1)}{\Gamma(k+p+\eta-\lambda+1)} z^{k+p+\eta-\lambda}, \quad (0 \leq \lambda < 1), \tag{13}$$

we shall need the concept of subordination between analytic functions and subordination theorem of Littlewood [2].

Definition 3. If the functions f and g are analytic in the open unit disk U , then f is said to be subordinate to g in U if there exists a function w analytic in U with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z)) (z \in U)$. We denote this subordination by $f < g$.

Theorem 2.

If the function f and g are analytic in U with $f < g$, then

$$\int_0^{2\pi} |f(re^{i\vartheta})|^\tau d\vartheta \leq \int_0^{2\pi} |g(re^{i\vartheta})|^\tau d\vartheta, \quad (\tau > 0; 0 < r < 1). \tag{14}$$

3. Integral means inequalities

Theorem 3.

Let g be of the form (2) and $f \in L(p, \eta, \delta)$ be of the form (1) and let for some $i \in \mathbb{N}$,

$$\frac{\varphi_i}{b_i} = \min_{k \geq 2} \frac{\varphi_k}{b_k},$$

where

$$\varphi_k = \frac{(k+p+\eta)[1+\mu(k+\gamma|\beta|-2)]}{\gamma|\beta|(1-\mu) + (\gamma|\beta|-1)\mu(p+\eta)} \left(\frac{c+p+\eta}{c+k+p+\eta} \right)^\delta.$$

Also , let for such $i \in \mathbb{N}$, the functions f_i and g_i be defined respectively by

$$f_i(z) = z^{p+\eta} + \frac{\gamma|\beta|(1-\mu) + (\gamma|\beta|-1)\mu(p+\eta)}{(i+p+\eta)[1+\mu(i+\gamma|\beta|-2)] \left(\frac{c+p+\eta}{c+i+p+\eta} \right)^\delta} z^{i+p+\eta},$$

$$g_i(z) = z^{p+\eta} + b_i z^{i+p+\eta}. \tag{15}$$

If there exists an analytic function w defined by

$$\{w(z)\}^i = \frac{(i+p+\eta)[1+\mu(i+\gamma|\beta|-2)] \left(\frac{c+p+\eta}{c+i+p+\eta} \right)^\delta}{[\gamma|\beta|(1-\mu) + (\gamma|\beta|-1)\mu(p+\eta)] b_i} \sum_{k=2}^{\infty} a_k b_k z^k,$$

then , for $\tau > 0$ and $z = re^{i\vartheta} (0 < r < 1)$,

$$\int_0^{2\pi} |(f * g)(z)|^\tau d\vartheta \leq \int_0^{2\pi} |(f_i * g_i)(z)|^\tau d\vartheta,$$

($\tau > 0$).

Proof. Convolution of f and g is defined as:

$$(f * g)(z) = z^{p+\eta} + \sum_{k=2}^{\infty} a_k b_k z^{k+p+\eta} = z^{p+\eta} \left(1 + \sum_{k=2}^{\infty} a_k b_k z^k \right).$$

Similarly , from (15) , we obtain

$$(f_i * g_i)(z) = z^{p+\eta} + \frac{[\gamma|\beta|(1-\mu) + (\gamma|\beta|-1)\mu(p+\eta)] b_i}{(i+p+\eta)[1+\mu(i+\gamma|\beta|-2)] \left(\frac{c+p+\eta}{c+i+p+\eta} \right)^\delta} z^{i+p+\eta} = z^{p+\eta} \left(1 + \frac{[\gamma|\beta|(1-\mu) + (\gamma|\beta|-1)\mu(p+\eta)] b_i}{(i+p+\eta)[1+\mu(i+\gamma|\beta|-2)] \left(\frac{c+p+\eta}{c+i+p+\eta} \right)^\delta} z^i \right).$$

To prove the theorem , we must show that for $\tau > 0$ and $z = re^{i\theta}$ ($0 < r < 1$),

$$\int_0^{2\pi} \left| 1 + \sum_{k=2}^{\infty} a_k b_k z^k \right|^\tau d\theta$$

$$\leq \int_0^{2\pi} |1 + \frac{[\gamma|\beta|(1-\mu) + (\gamma|\beta| - 1)\mu(p+\eta)] b_i}{(i+p+\eta)[1 + \mu(i + \gamma|\beta| - 2)] \left(\frac{c+p+\eta}{c+i+p+\eta}\right)^\delta} z^i|^\tau d\theta.$$

Thus , by applying Theorem 2, it would suffice to show that

$$1 + \sum_{k=2}^{\infty} a_k b_k z^k < 1 + \frac{[\gamma|\beta|(1-\mu) + (\gamma|\beta| - 1)\mu(p+\eta)] b_i}{(i+p+\eta)[1 + \mu(i + \gamma|\beta| - 2)] \left(\frac{c+p+\eta}{c+i+p+\eta}\right)^\delta} z^i. \quad (16)$$

If the subordination (16) holds true , then there exist an analytic function w with $w(0) = 0$ and $|w(z)| < 1$ such that

$$1 + \sum_{k=2}^{\infty} a_k b_k z^k = 1 + \frac{[\gamma|\beta|(1-\mu) + (\gamma|\beta| - 1)\mu(p+\eta)] b_i}{(i+p+\eta)[1 + \mu(i + \gamma|\beta| - 2)] \left(\frac{c+p+\eta}{c+i+p+\eta}\right)^\delta} \{w(z)\}^i.$$

From the hypothesis of the theorem , there exists an analytic function w given by

$$\{w(z)\}^i = \frac{(i+p+\eta)[1 + \mu(i + \gamma|\beta| - 2)] \left(\frac{c+p+\eta}{c+i+p+\eta}\right)^\delta}{[\gamma|\beta|(1-\mu) + (\gamma|\beta| - 1)\mu(p+\eta)] b_i} \times \sum_{k=2}^{\infty} a_k b_k z^k ,$$

which readily yields $w(0) = 0$. Thus for such function w , using the hypothesis in the coefficient inequality for the class $L(p, \eta, \delta)$, we get

$$|w(z)|^i \leq \frac{(i+p+\eta)[1 + \mu(i + \gamma|\beta| - 2)] \left(\frac{c+p+\eta}{c+i+p+\eta}\right)^\delta}{[\gamma|\beta|(1-\mu) + (\gamma|\beta| - 1)\mu(p+\eta)] b_i} \sum_{k=2}^{\infty} a_k b_k |z|^k$$

$$\leq |z|^2 \frac{(i+p+\eta)[1 + \mu(i + \gamma|\beta| - 2)] \left(\frac{c+p+\eta}{c+i+p+\eta}\right)^\delta}{[\gamma|\beta|(1-\mu) + (\gamma|\beta| - 1)\mu(p+\eta)] b_i} \sum_{k=2}^{\infty} a_k b_k$$

$$\leq |z| < 1 .$$

Therefore ,the subordination (16) holds true , thus the theorem is proved.

Theorem 4.

Let $f \in L(p, \eta, \delta)$, $p(z)$ be given by

$$p(z) = z^{p+\eta} + \sum_{s=1}^m b_{sj-(s-1)(p+\eta)} z^{sj-(s-2)(p+\eta)} , \quad (17)$$

$(p + \eta) > \lambda$, $j \geq 2$, $m \geq 2$, and suppose that

$$\sum_{k=2}^{\infty} (k+p+\eta-\lambda)_{\lambda+1} a_k \leq \sum_{s=1}^m \frac{\Gamma(sj-(s-2)(p+\eta)+1)\Gamma(p+\eta-\lambda+1-v)\Gamma(3-\lambda-n+p+\eta)}{\Gamma(sj-(s-2)(p+\eta)-\lambda+1-v)\Gamma(2+p+\eta-\lambda)\Gamma(p+\eta-\lambda+1-n)} \times b_{sj-(s-1)(p+\eta)} \quad (18)$$

for $\lambda = 0$ or 1

$(0 \leq n, v < 1)$ and $2 \leq \lambda \leq k$ ($0 < n, v < 1$) ,

where $(k+p+\eta-\lambda)_{\lambda+1}$ denotes the pochhammer symbol defined by

$$(k+p+\eta-\lambda)_{\lambda+1} = (k+p+\eta-\lambda)(k+p+\eta-\lambda+1) \dots (k+p+\eta).$$

Then for $z = re^{i\theta}$ ($0 < r < 1$),

$$\int_0^{2\pi} |D_z^{\lambda+n} f(z)|^\tau d\theta$$

$$\leq \int_0^{2\pi} \left| \frac{\Gamma(p + \eta - \lambda + 1 - v)}{\Gamma(p + \eta - \lambda + 1 - n)} z^{v-n} D_z^{\lambda+v} p(z) \right|^\tau d\theta, \quad (\tau > 0). \quad (19)$$

Proof. By means of the fractional derivative formula (13) and Definition 2 , we find from (1) that

$$D_z^{\lambda+n} f(z) = \frac{\Gamma(p + \eta + 1)}{\Gamma(p + \eta - \lambda + 1 - n)} z^{p+\eta-\lambda-n} \times \left[1 + \sum_{n=2}^{\infty} \frac{\Gamma(k + p + \eta + 1)\Gamma(p + \eta - \lambda + 1 - n)}{\Gamma(p + \eta + 1)\Gamma(k + p + \eta - \lambda + 1 - n)} a_k z^k \right] = \frac{\Gamma(p + \eta + 1)}{\Gamma(p + \eta - \lambda + 1 - n)} z^{p+\eta-\lambda-n} \times \left[1 + \sum_{k=2}^{\infty} (k + p + \eta - \lambda)_{\lambda+1} \frac{\Gamma(p + \eta - \lambda + 1 - n)}{\Gamma(p + \eta + 1)} \varphi(k) a_k z^k \right]$$

where $\varphi(k) =$

$$\frac{\Gamma(k + p + \eta - \lambda)}{\Gamma(k + p + \eta + 1 - \lambda - n)} \begin{cases} \lambda = 0 \text{ or } 1 & (0 \leq n < 1) \\ 2 \leq \lambda \leq k & (0 < n < 1) \end{cases}$$

$k \geq 2, k \in \mathbb{N}$.

Since $\varphi(k)$ is a decreasing function of k , we

have

$$0 \leq \varphi(k) \leq \varphi(2) =$$

$$\frac{\Gamma(2 + p + \eta - \lambda)}{\Gamma(3 + p + \eta - \lambda - n)} \begin{cases} \lambda = 0 \text{ or } 1 & (0 \leq n < 1) \\ 2 \leq \lambda \leq k & (0 < n < 1) \end{cases}$$

$k \geq 2, k \in \mathbb{N}$.

Similarly , by using (17) ,(13) and Definition 2 ,

we obtain

$$D_z^{\lambda+v} p(z) = \frac{\Gamma(p + \eta + 1)}{\Gamma(p + \eta - \lambda + 1 - v)} z^{p+\eta-\lambda-v} \left[1 + \sum_{s=1}^m \frac{\Gamma(sj - (s - 2)(p + \eta) + 1)\Gamma(p + \eta - \lambda + 1 - v)}{\Gamma(p + \eta + 1)\Gamma(sj - (s - 2)(p + \eta) - \lambda + 1 - v)} \times b_{sj-(s-1)(p+\eta)} z^{sj-(s-1)(p+\eta)} \right]$$

Thus , we have

$$\frac{\Gamma(p + \eta - \lambda + 1 - v)}{\Gamma(p + \eta - \lambda + 1 - n)} z^{v-n} D_z^{\lambda+v} p(z) = \frac{\Gamma(p + \eta + 1)}{\Gamma(p + \eta - \lambda + 1 - n)} z^{p+\eta-\lambda-n} \times \left[1 + \sum_{s=1}^m \frac{\Gamma(sj - (s - 2)(p + \eta) + 1)\Gamma(p + \eta - \lambda + 1 - v)}{\Gamma(p + \eta + 1)\Gamma(sj - (s - 2)(p + \eta) - \lambda + 1 - v)} \times b_{sj-(s-1)(p+\eta)} z^{sj-(s-1)(p+\eta)} \right]$$

For $z = re^{i\theta} (0 < r < 1)$, we must show that

$$\int_0^{2\pi} \left| 1 + \sum_{k=2}^{\infty} (k + p + \eta - \lambda)_{\lambda+1} \frac{\Gamma(p + \eta - \lambda + 1 - n)}{\Gamma(p + \eta + 1)} \varphi(k) a_k z^k \right|^\tau d\theta \leq \int_0^{2\pi} \left| 1 + \sum_{s=1}^m \frac{\Gamma(sj - (s - 2)(p + \eta) + 1)\Gamma(p + \eta - \lambda + 1 - v)}{\Gamma(p + \eta + 1)\Gamma(sj - (s - 2)(p + \eta) - \lambda + 1 - v)} \times b_{sj-(s-1)(p+\eta)} z^{sj-(s-1)(p+\eta)} \right|^\tau d\theta, (\tau > 0).$$

By applying Theorem 1. It suffices to show that

$$1 + \sum_{k=2}^{\infty} (k + p + \eta - \lambda)_{\lambda+1} \frac{\Gamma(p + \eta - \lambda + 1 - n)}{\Gamma(p + \eta + 1)} \varphi(k) a_k z^k < 1 + \sum_{s=1}^m \frac{\Gamma(sj - (s - 2)(p + \eta) + 1)\Gamma(p + \eta - \lambda + 1 - v)}{\Gamma(p + \eta + 1)\Gamma(sj - (s - 2)(p + \eta) - \lambda + 1 - v)} \times b_{sj-(s-1)(p+\eta)} z^{sj-(s-1)(p+\eta)}. \quad (20)$$

By setting

$$1 + \sum_{k=2}^{\infty} (k + p + \eta - \lambda)_{\lambda+1} \frac{\Gamma(p + \eta - \lambda + 1 - n)}{\Gamma(p + \eta + 1)} \varphi(k) a_k z^k = 1 + \sum_{s=1}^m \frac{\Gamma(sj - (s - 2)(p + \eta) + 1)\Gamma(p + \eta - \lambda + 1 - v)}{\Gamma(p + \eta + 1)\Gamma(sj - (s - 2)(p + \eta) - \lambda + 1 - v)} \times b_{sj-(s-1)(p+\eta)} \{w(z)\}^{sj-(s-1)(p+\eta)}$$

we find that

$$\{w(z)\}^{sj-(s-1)(p+\eta)}$$

$$= \sum_{k=2}^{\infty} (k+p+\eta-\lambda)_{\lambda+1} \frac{\Gamma(p+\eta-\lambda+1-n)}{\Gamma(p+\eta+1)} \varphi(k) a_k z^k$$

$$\times \frac{1}{\sum_{s=1}^m \frac{\Gamma(sj-(s-2)(p+\eta)+1)\Gamma(p+\eta-\lambda+1-v)}{\Gamma(p+\eta+1)\Gamma(sj-(s-2)(p+\eta)-\lambda+1-v)} b_{sj-(s-1)(p+\eta)}}$$

which readily yields $w(0) = 0$. Therefore ,

we have

$$|w(z)|^{sj-(s-1)(p+\eta)} \leq$$

$$\sum_{k=2}^{\infty} (k+p+\eta-\lambda)_{\lambda+1} \frac{\Gamma(p+\eta-\lambda+1-n)}{\Gamma(p+\eta+1)} \varphi(k) a_k |z|^k$$

$$\times \frac{1}{\sum_{s=1}^m \frac{\Gamma(sj-(s-2)(p+\eta)+1)\Gamma(p+\eta-\lambda+1-v)}{\Gamma(p+\eta+1)\Gamma(sj-(s-2)(p+\eta)-\lambda+1-v)} b_{sj-(s-1)(p+\eta)}}$$

$$\leq |z|^2$$

$$\times \frac{\varphi(2) \frac{\Gamma(p+\eta-\lambda+1-n)}{\Gamma(p+\eta+1)}}{\sum_{s=1}^m \frac{\Gamma(sj-(s-2)(p+\eta)+1)\Gamma(p+\eta-\lambda+1-v)}{\Gamma(p+\eta+1)\Gamma(sj-(s-2)(p+\eta)-\lambda+1-v)} b_{sj-(s-1)(p+\eta)}}$$

$$\times \sum_{k=2}^{\infty} (k+p+\eta-\lambda)_{\lambda+1} a_k$$

$$= |z|^2$$

$$\times \frac{\frac{\Gamma(2+p+\eta-\lambda)}{\Gamma(3+p+\eta-\lambda-n)} \frac{\Gamma(p+\eta-\lambda+1-n)}{\Gamma(p+\eta+1)}}{\sum_{s=1}^m \frac{\Gamma(sj-(s-2)(p+\eta)+1)\Gamma(p+\eta-\lambda+1-v)}{\Gamma(p+\eta+1)\Gamma(sj-(s-2)(p+\eta)-\lambda+1-v)} b_{sj-(s-1)(p+\eta)}}$$

$$\times \sum_{k=2}^{\infty} (k+p+\eta-\lambda)_{\lambda+1} a_k \leq |z|^2 < 1,$$

by means of the hypothesis (18) of Theorem 4.

Now , we prove the following property by using the definition of f_i which given by

$$f_i(z) = z^{p+\eta} + \sum_{k=2}^{\infty} a_{k,i} z^{k+p+\eta}. \quad (21)$$

Theorem 5.

Let f of the form (1) be in the class $L(p, \eta, \delta)$ such that $f(U)$ is convex. Assume that f satisfies the inequality (5). Then , for the Cesàro operator [3-6] of f defined by the relation

$$\sigma_k(z) = \sum_{k=2}^{\infty} \frac{1}{k+1} \left(\sum_{i=0}^k a_{k,i} \right) z^{k+p+\eta},$$

$$(p = 1, 2, \dots, 0 \leq \eta < 1, z \in U)$$

with $\sigma_0(z) = 0, \sigma_1(z) = z^{p+\eta}$, we have $\sigma_k(z) \in L(p, \eta, \delta)$.

Proof. Since f_i which is defined by (21) satisfies (5) then in view of Theorem 1 , we have

$$\sum_{k=2}^{\infty} a_{k,i} (k+p+\eta) [1 + \mu(k + \gamma|\beta| - 2)]$$

$$\times \left(\frac{c+p+\eta}{c+k+p+\eta} \right)^\delta$$

$$\leq \gamma|\beta|(1-\mu) + (\gamma|\beta| - 1)\mu(p+\eta).$$

For all $k \in \mathbb{N}_0$, we have

$$\sigma_k(z) = 0 + z^{p+\eta}$$

$$+ \sum_{k=2}^{\infty} \frac{1}{k+1} \left(\sum_{i=0}^k a_{k,i} \right) z^{k+p+\eta}, \quad (z \in U).$$

Hence

$$\sum_{k=2}^{\infty} (k+p+\eta) [1 + \mu(k + \gamma|\beta| - 2)]$$

$$\times \left(\frac{c+p+\eta}{c+k+p+\eta} \right)^\delta \frac{1}{k+1} \left(\sum_{i=0}^k a_{k,i} \right)$$

$$= \frac{1}{k+1} \sum_{i=0}^n \left(\sum_{k=2}^{\infty} (k+p+\eta)[1+\mu(k+\gamma|\beta|-2)] \right. \\ \left. \times \left(\frac{c+p+\eta}{c+k+p+\eta} \right)^{\delta} a_{k,i} \right)$$

$$< \frac{1}{k+1} \sum_{i=0}^k (\gamma|\beta|(1-\mu) + (\gamma|\beta|-1)\mu(p+\eta))$$

$$= \gamma|\beta|(1-\mu) + (\gamma|\beta|-1)\mu(p+\eta).$$

This implies that $\sigma_k(z) \in L(p, \eta, \delta)$.

References

- [1] M. Darus and R. W. Ibrahim, *Coefficient inequalities for a new class of univalent functions*. Lobachevskii J. Math., 29(2008),221-229.
- [2] J.E. Littlewood, *On inequalities in the theory of functions*,Proc. LondonMath. Soc., 23(1925), 482-519.
- [3] J. Miao, *The Ces`aro operator is bounded on H^p for $0 < p < 1$* , Proc. Amer. Math. Soc., 116(4)(1992),1077-1079.
- [4] J.-h. Shi and G.-b. Ren, *Boundedness of the Ces`aro operator on mixed norm spaces*, Proc. Amer. Math. Soc., 126(12)(1998) , 3553-3560.
- [5] A. G. Siskakis, *Composition semigroups and the Ces`aro operator on H^p* , J. London Math. Soc.(2),36(1)(1987),153-164.
- [6] A. G. Siskakis, *The Ces`aro operator is bounded on H^1* ,Proc. Amer. Math. Soc., 110(2)(1990),461-462.
- [7] Tariq O. Salim, *A class of multivalent functions involving a generalized linear operator and subordination*, Int. J. Open Problems Complex Analysis, 2(2)(2010).82-94.