On Data Fitting Using B-Spline Functions

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ABSTRACT:

In this research a new formula of B-spline functions have been used to approximate a set of given data points (x_i, y_i) $i=1,2,\ldots,n$ & $0 \le x_i \le 1$ for both uniform and non uniform spaced, that is when $h_i = x_{i+1} - x_i$ $i=1,2,\ldots,n-1$ satisfy either $h_i = h_{i+1}$ $, i=1,2,\ldots,n-1$ or $h_i \ne h_{i+1}$ $, i=1,2,\ldots,n-1$.

The first order B-spline $B_i^{\,1}(x)$, the second order B-spline $B_i^{\,2}(x)$ and the third order B-spline $B_i^3(x)$ have been used. In the second & third order B-spline we need the first &second derivatives at x=0, in this research we use Lagrange's interpolation polynomial & Taylor's series to find y' & y'' at x=0 using three point formula for both cases, i.e., uniform & non uniform spaced.

The research contains the flowchart and the algorithm that described our work with its implementation, also the program written using visual basic language.

INTRODUCTION:

Splines are piecewise polynomials of degree n joined together at the break points with n-1 continuous derivatives. The break points are called "knots" [8].

There are many types of spline [11]: classical spline, B-spline, Bezier and non uniform rational B-spline (NURBS) .Even thought they all are based on different mathematical concepts, they have one thing in common, and the "control points" are editable.

In this research we deal with B-Splines [9], which are standard representation of smooth non-linear geometry in numerical calculations. B-Splines were introduced around 1940's in the context of approximation theory by Schoenberg. B-spline mean basis and the letter B in B-spline stands for basis.

Scaling the knots uniformly will not change the definition of the basis function. But changing the relative positioning of the knots will change the shape and therefore change the definition of the basis functions [10]. If the knots are evenly spaced the bases are called "uniform B-Splines", otherwise, they are "non uniform B-spline".

In this work a new formula of B-spline functions introduced by Mustafa in [10] have been used to approximate a given set of data points using:

(1) First order B-spline $B_i^1(x)$.

(2) Second order B-spline $B_i^2(x)$.

(3) Third order B-spline $B_i^3(x)$.

The rule for (2) and (3) contains first & second derivatives so we use some special types of numerical differentiation to approximate it.

1- ***Definitions & Properties Of B-Spline:***

Definition (1-1) [2]: ***Bezier curves*** are a class of approximation Splines. They are defined using control points, but don't necessarily pass through all the control points. The general form of Bezier curve is:

$$y_a(x) = \sum_k p_k B_{k,n}(x) \qquad (1)$$

Where p_k the k^{th} control points, and $B_{k,n}$ is a Bernstein polynomial:

$$B_{k,n}(x) = c(n,k)\, x^k (1-x)^{n-k} \qquad (2)$$

, where $c(n,k)$ is a binomial coefficient.

Definition (1-2) [3]: A ***B-spline*** is a generalization of the Bezier curve. Let a vector known as the knot vector be defined

$$T = \{t_0, t_1, \ldots, t_m\},$$

Where T is a non decreasing sequence with $t_i \in [0,1]$, and defines control points $p_0, \ldots, p_n$ .Define the degree as

$$p = m - n - 1.$$

Define the basis function as

$$B_{i,0}(x) = \begin{cases} 1 & if \quad x_i \le x < x_{i+1} \;\&\; x_i < x_{i+1} \\ 0 & otherwise \end{cases}$$

$$B_{i,p}(x) = \frac{x - x_i}{x_{i+p} - x_i} B_{i,p-1}(x) + \frac{x_{i+p+1} - x}{x_{i+p+1} - x_{i+1}} B_{i+1,p-1}(x) \qquad (3)$$

The above relation is called ***the recurrence relation***.

<u>Definition (1-3) [2]:</u> the ***B-Splines of order zero*** are defined by

$$B_i^{\,0}(x)=\begin{cases}1 & \text{if} \quad x_i \le x < x_{i+1} \\ 0 & \text{otherwise}\end{cases}, \qquad (5)$$

The ***B-spline basis functions of order one*** are defined by

$$B_i^{\,1}(x)=\frac{x-x_i}{x_{i+1}-x_i}B_i^{\,0}(x)+\frac{x_{i+2}-x}{x_{i+2}-x_{i+1}}B_{i+1}^{\,0}(x)$$

$$=\begin{cases}\dfrac{x-x_i}{x_{i+1}-x_i} & \text{if} \quad x_i \le x < x_{i+1} \\ \dfrac{x_{i+2}-x_i}{x_{i+2}-x_{i+1}} & \text{if} \quad x_{i+1} \le x < x_{i+2} \\ 0 & \text{otherwise}\end{cases}, \qquad (6)$$

The ***B-spline basis functions of order two*** are defined by

$$B_i^{\,2}(x)=\frac{x-x_i}{x_{i+2}-x_i}B_i^{\,1}(x)+\frac{x_{i+3}-x}{x_{i+3}-x_{i+1}}B_{i+1}^{\,1}(x)$$

$$=\begin{cases}\dfrac{(x-x_i)^2}{(x_{i+2}-x_i)(x_{i+1}-x_i)} & \text{if} \quad x_i \le x < x_{i+1} \\ \dfrac{(x-x_i)(x_{i+2}-x)}{(x_{i+2}-x_i)(x_{i+2}-x_{i+1})}+\dfrac{(x_{i+3}-x)(x-x_{i+1})}{(x_{i+3}-x_{i+1})(x_{i+2}-x_{i+1})} & \text{if} \quad x_{i+1} \le x < x_{i+2} \\ \dfrac{(x_{i+3}-x)^2}{(x_{i+3}-x_{i+1})(x_{i+3}-x_{i+2})} & \text{if} \quad x_{i+2} \le x < x_{i+3} \\ 0 & \text{otherwise}\end{cases}, \qquad (7)$$

And finally ***the B-spline basis functions of order three*** are defined by

$$B_i^{3}(x)=\frac{x-x_i}{x_{i+3}-x_i}B_i^{2}(x)+\frac{x_{i+4}-x}{x_{i+4}-x_{i+1}}B_{i+1}^{2}(x) \qquad (8)$$

=

$$\begin{cases}\frac{(x-x_i)^3}{(x_{i+3}-x_i)(x_{i+2}-x_i)(x_{i+1}-x_i)} & if \quad x_i \le x < x_{i+1}\\ \frac{(x-x_i)^2(x_{i+2}-x)}{(x_{i+3}-x_i)(x_{i+2}-x_i)(x_{i+2}-x_{i+1})}+\frac{(x-x_i)(x_{i+3}-x)(x-x_{i+1})}{(x_{i+3}-x_i)(x_{i+3}-x_{i+1})(x_{i+2}-x_{i+1})} & \\ \quad +\frac{(x_{i+4}-x)(x-x_{i+1})^2}{(x_{i+4}-x_{i+1})(x_{i+3}-x_{i+1})(x_{i+2}-x_{i+1})} & if \quad x_{i+1} \le x < x_{i+2}\\ \frac{(x-x_i)(x_{i+3}-x)^2}{(x_{i+3}-x_i)(x_{i+3}-x_{i+1})(x_{i+3}-x_{i+2})}+\frac{(x_{i+4}-x)(x-x_{i+1})(x_{i+3}-x)}{(x_{i+4}-x_{i+1})(x_{i+3}-x_{i+1})(x_{i+3}-x_{i+2})} & \\ \quad +\frac{(x_{i+4}-x)^2(x-x_{i+2})}{(x_{i+4}-x_{i+1})(x_{i+4}-x_{i+2})(x_{i+3}-x_{i+2})} & if \quad x_{i+2} \le x < x_{i+3}\\ \frac{(x_{i+4}-x)^3}{(x_{i+4}-x_{i+1})(x_{i+4}-x_{i+2})(x_{i+4}-x_{i+3})} & if \quad x_{i+3} \le x < x_{i+4}\\ 0 & otherwise\end{cases}$$

<u>Definition (1-4) [2]:</u> in the above definitions x_i are called parametric knot values, for an open curve, they are given by:

$$x_j=\begin{cases}0 & , j<k+1\\ j-k & , k+1\le j\le n\\ n-k+1 & , j>n\end{cases} \qquad (9)$$

Where $0\le j\le n+k+1$, and the range of x is $0\le x\le n-k+1$.

<u>Theorem 1:</u> suppose that $k\ge 0$, and suppose that $x_i < x_{i+k+1}$, then for all $x\in\Re$,

$1-B_i^{k}(x)>0\ , for\ all\ x\in[x_i,x_{i+k+1})$

$2-B_i^{k}(x)=0\ , for\ all\ x\notin[x_i,x_{i+k+1})$

$3-\sum_{i=-\infty}^{\infty}B_i^{k}(x)=1,\ for\ any\ integer\ k>0$

<u>Proof:</u> [5]

2- Numerical Differentiation Using Lagrange's Interpolation Polynomial & Taylor series[4]:

Suppose that $\{x_0, x_1, \ldots, x_n\}$ are (n+1) distinct numbers in some interval I and that

$f \in c^{n+1}(I)$, We have

$$f(x) = \sum_{k=0}^{n} f(x_k)\, L_k(x) + \frac{(x-x_0)\ldots(x-x_n)}{(n+1)!} f^{(n+1)}(\xi(x)) \qquad (10)$$

For some $\xi(x)$ lies in I, where $L_k(x)$ denotes the $k\,th$ Lagrange coefficient polynomial for f at $x_0, x_1, \ldots, x_n$.Differentiating (10) we obtain:

$$\begin{aligned} f'(x) &= \sum_{k=0}^{n} f(x_k) L_k'(x) + D_x\left[\frac{(x-x_0)\ldots(x-x_n)}{(n+1)!}\right] f^{(n+1)}(\xi(x)) \\ &+ \left[\frac{(x-x_0)\ldots(x-x_n)}{(n+1)!}\right] D_x\left[f^{(n+1)}(\xi(x))\right] \end{aligned} \qquad (11)$$

Put $x = x_j$ in (11), we get

$$f'(x_j) = \sum_{k=0}^{n} f(x_k) L_k'(x_j) + D_x\left[\frac{(x_j-x_0)\ldots(x_j-x_n)}{(n+1)!}\right] f^{(n+1)}(\xi(x_j)) + 0 \qquad (12)$$

So,

$$f'(x_j) = \sum_{k=0}^{n} f(x_k) L_k'(x_j) + \frac{f^{(n+1)}(\xi(x_j))}{(n+1)!} \prod_{k=0,k\neq j}^{n} (x_j - x_k) \qquad (13)$$

Equation (13) is called an (n+1)-point formula to approximate $f'(x_j)$ since a linear combination of the (n+1) values $f(x_k)$ is used for $k = 0,1,\ldots,n$.

In this work we use three- point formula:

$$L_0(x)=\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} \quad \Rightarrow \quad L_0'(x)=\frac{2x-x_1-x_2}{(x_0-x_1)(x_0-x_2)}$$
$$L_1(x)=\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \quad \Rightarrow \quad L_1'(x)=\frac{2x-x_0-x_2}{(x_1-x_0)(x_1-x_2)}$$
$$L_2(x)=\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \quad \Rightarrow \quad L_2'(x)=\frac{2x-x_0-x_1}{(x_2-x_0)(x_2-x_1)}$$

Substitute the above equations in (13) we get

$$f'(x_j)= f(x_0)\frac{(2x-x_1-x_2)}{(x_0-x_1)(x_0-x_2)}+ f(x_1)\frac{(2x-x_0-x_2)}{(x_1-x_0)(x_1-x_2)} + f(x_2)\frac{(2x-x_0-x_1)}{(x_2-x_0)(x_2-x_1)} , \qquad (14)$$

&

$$f''(x_j)= f(x_0)\frac{2}{(x_0-x_1)(x_0-x_2)}+ f(x_1)\frac{2}{(x_1-x_0)(x_1-x_2)} + f(x_2)\frac{2}{(x_2-x_0)(x_2-x_1)} \qquad (15)$$

For j=0, 1, 2, equations (14) & (15) used for unequal spaced.

The three point formula can be converting to find $f'(x)$ for equally spaced, that is, when $x_1 = x_0 + h$, $x_2 = x_1 + h = x_0 + 2h$ *for some* $h \neq 0$.

Using equation (14) with $x_j = x_0, x_1 = x_0 + h, x_2 = x_0 + 2h$ gives:

$$f'(x_j)= f(x_0)\frac{(2x-x_1-x_2)}{(x_0-x_1)(x_0-x_2)}+ f(x_1)\frac{(2x-x_0-x_2)}{(x_1-x_0)(x_1-x_2)} + f(x_2)\frac{(2x-x_0-x_1)}{(x_2-x_0)(x_2-x_1)} \qquad (16)$$

To find $f''(x_0)$ we use Taylor series formula, as follows:

$$f(x_0+h)=f(x_0)+hf'(x_0)+\frac{h^2}{2!}f''(x_0) \qquad (17)$$

Substitute (16) in (17) to get $f''(x_0)$;

$$f(x_0+h)=f(x_0)+\frac{h}{2h}\{-3f(x_0)+4f(x_0+h)-f(x_0+2h)\}+\frac{h^2}{2!}f''(x_0)$$

$$\frac{h^2}{2!}f''(x_0)=f(x_0+h)-f(x_0)+\frac{3}{2}f(x_0)-2f(x_0+h)+\frac{1}{2}f(x_0+2h)$$
$$\frac{h^2}{2!}f''(x_0)=\frac{1}{2}f(x_0)-f(x_0+h)+\frac{1}{2}f(x_0+2h)$$
$$\frac{h^2}{2!}f''(x_0)=\frac{1}{2}\{f(x_0)-2f(x_0+h)+f(x_0+2h)\}$$

$$f''(x_0)=\frac{1}{h^2}\{f(x_0)-2f(x_0+h)+f(x_0+2h)\} \qquad (18)$$

Equations (16) & (18) used for equally spaced.

3- ***B-spline curves:***

B-spline curves [7] are the proper and powerful generalization of Bezier curves. They provide local control on the curve shape as opposed to global control by using a special set of blending functions that provide local influence. They also provide the ability to add control points without increasing the degree of the curve.

The theory of B-spline curves separates the degree of resulting curve from the number of the given points. The B-spline curve defined by the (n+1) control points p_i given by:

$$y_a(x)=\sum_{k=0}^{n}p_k B_i^k(x) \quad , 0\le x\le 1. \qquad (19)$$

Now a new formula of: the first order B-spline $B_i^1(x)$, second order B-spline $B_i^2(x)$ and third order B-spline $B_i^3(x)$ have been explained and discussed in more details.

(3-1) The First Order B-Spline $B_i^1(x)$ *[10]:*

This kind of B-spline is called linear spline; it is defined from the recurrence relation defined by equation (3), then we

$$B_i^1(x) = \begin{cases} \dfrac{x - x_i}{x_{i+1} - x_i} & if \quad x_i \le x < x_{i+1} \\ \dfrac{x_{i+2} - x_i}{x_{i+2} - x_{i+1}} & if \quad x_{i+1} \le x < x_{i+2} \\ 0 & otherwise \end{cases}$$

Now, if we take k=1& n=1, we obtain by using equation (9): $\lfloor x_0 \ \ x_1 \ \ x_2 \ \ x_3 \rfloor = [0 \ \ 0 \ 1 \ 1] \quad \& \, 0 \le x \le 1$

Substitute this in equation (19), we get

$$y_a(x) = p_0 B_0^1(x) + p_1 B_1^1(x)$$
$$= p_0 \begin{cases} \dfrac{x - x_0}{x_1 - x_0} & x_0 \le x < x_1 \\ \dfrac{x_2 - x}{x_2 - x_1} & x_1 \le x < x_2 \\ 0 & otherwise \end{cases} + p_1 \begin{cases} \dfrac{x - x_1}{x_2 - x_1} & x_1 \le x < x_2 \\ \dfrac{x_3 - x}{x_3 - x_2} & x_2 \le x < x_3 \\ 0 & otherwise \end{cases}$$

Therefore, we have:

$$y_a(x) = p_0(1 - x) + p_1 x \qquad 0 \le x \le 1 \qquad (20)$$

The control points p_0 & p_1 obtained by substituting x=0 & x=1 in $B_i^1(x)$ that is $p_0 = y_0$, $p_1 = y_n$.

(3-2) The Second Order B-Spline $B_i^2(x)$[10]:

This kind of B-spline is called is called quadratic spline, by equation (3), we get:

$$B_i^2(x) = \begin{cases} \dfrac{(x-x_i)^2}{(x_{i+2}-x_i)(x_{i+1}-x_i)} & if \quad x_i \le x < x_{i+1} \\ \dfrac{(x-x_i)(x_{i+2}-x)}{(x_{i+2}-x_i)(x_{i+2}-x_{i+1})} + \dfrac{(x_{i+3}-x)(x-x_{i+1})}{(x_{i+3}-x_{i+1})(x_{i+2}-x_{i+1})} & if \quad x_{i+1} \le x < x_{i+2} \\ \dfrac{(x_{i+3}-x)^2}{(x_{i+3}-x_{i+1})(x_{i+3}-x_{i+2})} & if \quad x_{i+2} \le x < x_{i+3} \\ 0 & otherwise \end{cases}$$

Now, if we take k=2 & n=2, we get by equation (9) that:
$[x_0\ x_1\ x_2\ x_3\ x_4\ x_5] = [0\,0\,0\,1\,1\,1]$ & $0 \le x \le 1$.

Substitute in equation (19) we have:

$$y_a(x) = p_0 B_0^2(x) + p_1 B_1^2(x) + p_2 B_2^2(x)$$

$$= p_0 \begin{cases} \dfrac{(x-x_0)^2}{(x_2-x_0)(x_1-x_0)} & if \quad x_0 \le x < x_1 \\ \dfrac{(x-x_0)(x_2-x)}{(x_2-x_0)(x_2-x_1)} + \dfrac{(x_3-x)(x-x_1)}{(x_3-x_1)(x_2-x_1)} & if \quad x_1 \le x < x_2 \\ \dfrac{(x_3-x)^2}{(x_3-x_1)(x_3-x_2)} & if \quad x_2 \le x < x_3 \\ 0 & otherwise \end{cases}$$ 1

$$+\, p_1 \begin{cases} \dfrac{(x-x_1)^2}{(x_3-x_1)(x_2-x_1)} & if \quad x_1 \le x < x_2 \\ \dfrac{(x-x_1)(x_3-x)}{(x_3-x_1)(x_3-x_2)} + \dfrac{(x_4-x)(x-x_2)}{(x_4-x_2)(x_3-x_2)} & if \quad x_2 \le x < x_3 \\ \dfrac{(x_4-x)^2}{(x_4-x_2)(x_4-x_3)} & if \quad x_3 \le x < x_4 \\ 0 & otherwise \end{cases}$$

$$+\, p_2 \begin{cases} \dfrac{(x-x_2)^2}{(x_4-x_2)(x_3-x_2)} & if \quad x_2 \le x < x_3 \\ \dfrac{(x-x_2)(x_4-x)}{(x_4-x_2)(x_4-x_3)} + \dfrac{(x_5-x)(x-x_3)}{(x_5-x_3)(x_4-x_3)} & if \quad x_3 \le x < x_4 \\ \dfrac{(x_5-x)^2}{(x_5-x_3)(x_5-x_4)} & if \quad x_4 \le x < x_5 \\ 0 & otherwise \end{cases}$$

Therefore, we have:

$$y_a(x) = p_0(1-x)^2 + 2p_1x(1-x) + p_2x^2 \qquad (21)$$

p_0 & p_2 Can be found by substitute x=0 & x=1 in $B_i^{\,2}(x)$ that is $p_0 = y_0$, $p_2 = y_n$.
p_1 Can be found by taking the first derivative of (21) w.r.t x when x=0, we get:

$$\left.\frac{dy_a(x)}{dx}\right|_{x=0} = -2p_0(1-x) + 2p_1[x*-1+(1-x)] + 2p_2x$$

$$\left.\frac{dy_a(x)}{dx}\right|_{x=0} = -2p_0 + 2p_1 \qquad \Rightarrow \qquad p_1 = \frac{1}{2}\left.\frac{dy_a(x)}{dx}\right|_{x=0} + p_0$$

Here $\left.\dfrac{dy_a(x)}{dx}\right|_{x=0}$ find by using either equation (14) or (16).

(3-3) The Third Order B-Spline $B_i^3(x)$*[10]:*

The third order B-spline is called also cubic spline, from equation (3) we know that:

$$B_i^3(x)=\begin{cases}\dfrac{(x-x_i)^3}{(x_{i+3}-x_i)(x_{i+2}-x_i)(x_{i+1}-x_i)} & if \quad x_i \le x < x_{i+1}\\[2ex] \dfrac{(x-x_i)^2(x_{i+2}-x)}{(x_{i+3}-x_i)(x_{i+2}-x_i)(x_{i+2}-x_{i+1})}+\dfrac{(x-x_i)(x_{i+3}-x)(x-x_{i+1})}{(x_{i+3}-x_i)(x_{i+3}-x_{i+1})(x_{i+2}-x_{i+1})} & \\ \qquad +\dfrac{(x_{i+4}-x)(x-x_{i+1})^2}{(x_{i+4}-x_{i+1})(x_{i+3}-x_{i+1})(x_{i+2}-x_{i+1})} & if \quad x_{i+1} \le x < x_{i+2}\\[2ex] \dfrac{(x-x_i)(x_{i+3}-x)^2}{(x_{i+3}-x_i)(x_{i+3}-x_{i+1})(x_{i+3}-x_{i+2})}+\dfrac{(x_{i+4}-x)(x-x_{i+1})(x_{i+3}-x)}{(x_{i+4}-x_{i+1})(x_{i+3}-x_{i+1})(x_{i+3}-x_{i+2})} & \\ \qquad +\dfrac{(x_{i+4}-x)^2(x-x_{i+2})}{(x_{i+4}-x_{i+1})(x_{i+4}-x_{i+2})(x_{i+3}-x_{i+2})} & if \quad x_{i+2} \le x < x_{i+3}\\[2ex] \dfrac{(x_{i+4}-x)^3}{(x_{i+4}-x_{i+1})(x_{i+4}-x_{i+2})(x_{i+4}-x_{i+3})} & if \quad x_{i+3} \le x < x_{i+4}\\ 0 & otherwise\end{cases}$$

Now if we take k=3 & n=3 we obtain by using equation (9):

$$[x_0\ x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7]=[0\,0\,0\,0\,1\,1\,1\,1] \quad \& \quad 0 \le x \le 1.$$

By do the same thing described in (3-1) & (3-2), we get:

$$y_a(x)=p_0(1-x)^3+3p_1(1-x)^2x+3p_2(1-x)x^2+p_3x^3 \qquad (22)$$

p_0 & p_3 Can be found by substitute x=0 & x=1 in $B_i^3(x)$ that is $p_0=y_0$, $p_3=y_n$.

To obtain p_1 & p_2 differentiate (22) twice when x=0 to get p_1 & p_2 as follows:

$$\left.\frac{dy_a(x)}{dx}\right|_{x=0}=-3p_0(1-x)^2+3p_1\left[(1-x)^2-2x(1-x)\right]+3p_2\left[(1-x)*2x-x^2\right]+3p_3x^2$$

$$\left.\frac{dy_a(x)}{dx}\right|_{x=0} = -3p_0 + 3p_1 \quad \Rightarrow \quad p_1 = \frac{1}{3}\left.\frac{dy_a(x)}{dx}\right|_{x=0} + p_0$$

Here $\left.\frac{dy_a(x)}{dx}\right|_{x=0}$ find by using either equation (14) or (16).

Now differentiate (22) again w.r.t x to get:

$$\left.\frac{d^2y_a(x)}{dx^2}\right|_{x=0} = 6p_0(1-x) + 3p_1[-2(1-x) + 2x - 2(1-x)] + 3p_2[2(1-x) - 2x - 2x] + 6p_3x$$

$$\left.\frac{d^2y_a(x)}{dx^2}\right|_{x=0} = 6p_0 - 12p_1 + 6p_2 \quad \Rightarrow \quad p_2 = \frac{1}{6}\left.\frac{d^2y_a(x)}{dx^2}\right|_{x=0} - p_0 + 2p_1$$

Here $\left.\frac{d^2y_a(x)}{dx^2}\right|_{x=0}$ find by using either equation (15) or (18).

For more details about the properties of these new formulas of B-Spline functions see the work done by Mustafa in [10].