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# Solvability of the Quaternary Continuous Classical Boundary Optimal Control Dominated by Quaternary Parabolic System

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## Abstract

The purpose of this paper is to study the solvability of the quaternary continuous classical boundary optimal control vector problem dominated by quaternary nonlinear parabolic boundary value problem with state constraints. The existence theorem for a quaternary continuous classical boundary optimal control vector with equality and inequality state constraints is stated and demonstrated under suitable assumptions. The mathematical formulation of the quaternary adjoint eqs. associated with the quaternary nonlinear parabolic boundary value problem with state constraints is discovered. The Fréchet derivative of the cost function and the constraints functions are derived. The necessary and sufficient theorems (conditions) for optimality of the quaternary continuous classical boundary optimal control vector problem are stated and demonstrated under suitable assumptions.

**Keywords:** Adjoint Eqs. , Constraint Continue Classical Optimal Control Vector, Necessary Conditions, Quaternary Nonlinear Parabolic System, Sufficient Condition, The Fréchet Derivative.

## Introduction

Optimal control problem (OCP) means endogenously controlling a parameter in a mathematical model to produce an optimal (cost) output. The problem comprises an objective (or cost) function, which is a function of the state and control variables, the constraints on the control, and the dominating system. The problem seeks to optimize the objective function subject to the constraints construed by the model describing the evolution of the underlying system. Optimal control problems play an important role in many practical applications, such as in medicine[1], aircraft [2], economics[3], robotics [4], weather conditions [5]and many other scientific fields. There are two types of optimal control problems; the classical and the relax type, each one is either continuous or discrete, also each one of them is dominated either by ODEqs [6-8] or by PDEqs[9].

The study of the continuous classical type began with the continuous classical boundary optimal control problems dominated by nonlinear parabolic or elliptic or hyperbolic PDEs. Then, these studies were generalized to deal with systems dominated by coupling nonlinear PDEs of these three types[10,11], and then were generalized also to deal systems dominated by triple nonlinear PDEs of these three types [12] . In each type of these classical continuous boundary optimal control problems, the problem consists of; an initial or a boundary value problem (the dominating eqs.), the objective (cost) function of the classical continuous control vector, and the constraints on the state vector (equality and inequality state constraints). The study in each one of these problems included; the state and proof for the existence theorem of a continuous classical boundary optimal control that satisfies the state constraints under suitable conditions, the derivation of the mathematical formulation of the adjoint eqs. associated with the state eqs. , the derivation of the Freichet derivative of the cost function and the state constraints functions, and the state and proof of the necessary (conditions) theorem and the sufficient (conditions) theorem for optimality. All these studies encouraged us to study the generalization of these problems to include the quaternary continuous classical boundary optimal control vector problem dominated by quaternary nonlinear parabolic boundary value problem. According to this idea of generalization, it is necessary to find a mathematical model for the dominating eq. , as well as the cost function and the constraints on the state function, the spaces of definition for the control, and the state vectors, all of which need to be generalized. Hence all the theorems and the results included in the above studies for the continuous classical boundary optimal control dominated by the couple and the triple nonlinear PDEs of the parabolic type must be stated and proved for the "new" proposed quaternary continuous classical boundary optimal control problem.

In this paper; the quaternary continuous classical boundary optimal control vector problem dominated by quaternary nonlinear parabolic boundary value problem is proposed, it starts at first with some given principles and mathematical concepts, and then the description for the problem is given, the main results of this work consist of; the existence theorem of a quaternary continuous classical boundary optimal control vector (minimizing the cost function and satisfying the equality and inequality state constraints) is stated and demonstrated under suitable assumptions. The mathematical formulation for the adjoint quaternary eqs. associated with the quaternary nonlinear parabolic boundary value problem is discovered and then the Fréchet derivative for the cost function and the state constraints functions (equality and inequality state constraints) are obtained. Finally, the necessary (conditions) theorem for optimality and the sufficient (conditions) theorem for the proposed problem are stated and demonstrated under suitable assumptions.

## Materials and Methods

The following mathematical concepts are important and are used next in this paper.

**Definition 1** [13]**:** Let $\Omega \subset \mathbb{R}^n$ , a function $g: \Omega \times \mathbb{R}^4 \to \mathbb{R}$ is called of the *Carathéodory Type*, if it is continuous w.r.t. $y$ and $u$ for fixed $x \in \Omega$ and measurable w.r.t. $x$ if it is fixed $(y, u)$.

**Definition 2** [13]**:** Let $\psi: \mathrm{A} \to \mathrm{B}$ be a function from an open set A in a normed space E into a normed space B, and $r \in \mathrm{A}$. If there is a bounded linear operator $L: \mathrm{E} \to \mathrm{B}$ such that: $\lim_{h \to 0} \frac{\|\psi(r+h)-\psi(r)-Lh\|_{\mathrm{B}}}{\|h\|_{\mathrm{A}}} = 0$.

Then, $\psi$ is called Fréchet differentiable at $r$. Moreover, $L$ is called the *Fréchet derivative* of $\psi$ at $r$.

**Definition 3** [14]**:** Let $(\mathrm{E}, \|\cdot\|)$ be a real normed space, $\mathrm{B}(\neq \emptyset) \subset \mathrm{E}$, and let $\psi: \mathrm{B} \to \mathbb{R}$ be a given functional. The functional $\psi$ is called *weakly lower semi continuous* if every sequence $(\mathrm{K}_n)_{n \in \mathbb{N}}$ in a set B converges weakly to some $\overline{\mathrm{K}} \in \mathrm{B}$ , i.e.: $\lim_{n\to\infty} \inf \psi(\mathrm{K}_n) \geq \psi(\overline{\mathrm{K}})$.

**Definition 4** [15]**:** Let $L$ be a bounded linear operator on a Hilbert space $H$. The operator $L^a: H \to H$ **is** defined by: $(Lr, z) = (r, L^a z)$ for any $r, z \in H$ is said to be *the adjoint operator* of $L$.

**Theorem 1** [13]**:** Let $D$ be a measurable subset of $\mathbb{R}^d$ , and $\psi: D \to \mathbb{R}$ with $\psi \in L^1(D, \mathbb{R})$, if $\int_S \psi(v)dv \geq 0$ (or $\leq 0, = 0$), for each measurable set $S \subset D$. Then, $\psi(v) \geq 0$ (or $\leq 0, = 0$), a.e. in $D$.

**Theorem 2**[13]**:** Let $U$ be a convex subset of a vector space $X$, $K$ be a positive cone in a normed space $Z$ with $K^{\circ} \neq \emptyset$ be a convex, $G_0: U \to \mathbb{R}$ , $G_1: U \to \mathbb{R}^m$ $m \geq 0$ , $G_2: U \to Z$, $W = \{u \in U | G_1(u) = 0, G_2(u) \in -K\}$.

If $G_l$, $l = 0,1,2$, are $(m+1)$ −locally continuous at $u \in U$, and have $(m+1)$ −derivatives at $u$, and if $G_0(u)$ has a minimum at $u$ in $W$, then $u$ satisfies the *"Kuhn-Tucker-Lagrange's Multipliers"* , i.e.

There exist $0 \leq \lambda_0 \in \mathbb{R}$, $\lambda_1 \in \mathbb{R}^m$, $0 \leq \lambda_2 \in \mathbb{Z}^*$, with $\sum_{l=0}^{2} |\lambda_l| = 1$ s.t., $\forall w \in W$

$$\lambda_0 DG_0(u, w-u) + \lambda_1^T DG_1(u, w-u) + \langle \lambda_2, DG_2(u, w-u) \rangle \geq 0, \text{ and } \langle \lambda_2, G_2(u) \rangle = 0.$$

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**Proposition 1** [16]**:** Let $f: D \subset \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^m$ be of the *Carathèdory* type, $F(y) = \int_D f(\chi, y(\chi))\, d\chi$, and $\|f(\chi, y)\| \leq \zeta(\chi) + \eta(\chi)\|y\|^{\alpha}$,$\forall(x, y) \in D \times \mathbb{R}^n$,$y \in L^p(D \times \mathbb{R}^n)$, where$\zeta \in L^1(D \times \mathbb{R})$,$\eta \in L^{\frac{p}{p-\alpha}}(D \times \mathbb{R})$, and $\alpha \in N$, if $p \in [1, \infty)$. Then $F$ is continuous on $L^p(D \times \mathbb{R}^n)$.

**Proposition 2**[16]**:** Let $f$ and $f_y: D \subset \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^m$ be of the *Carathèdory* type, let $F: L^p(\mathrm{D}) \to \mathbb{R}$ s.t. $F(y) = \int_D f(\chi, y(\chi))\, d\chi$, and $\|f(\chi, y)\| \leq \zeta(\chi) + \eta(\chi)\|y\|^{\frac{\beta}{q}}$, $\forall(\chi, y) \in D \times \mathbb{R}^n$,$p, q \neq 0$,$y \in L^p(D \times \mathbb{R}^n)$.

where $\zeta \in L^q(D \times \mathbb{R})$, $\eta \in L^{\frac{pq}{p-\beta}}(D \times \mathbb{R})$, $\beta \in [0, p]$ and $\frac{1}{p} + \frac{1}{q} = 1$ , if $p \neq \infty$, and $\eta = 0$, if $p = \infty$.

Then the Fréchet derivative of $F$ exists and is given by $G'(y)k = \int_D f_y(\chi, y(\chi))k(\chi)\, d\chi$, $\forall y \in L^p(D \times \mathbb{R}^n)$.

In the following section, the description of the proposed problem is given; it includes the mathematical formulation for the model which is represented by the dominating boundary value problem, the cost function, the state constraints functions, the constraints on the control, and the weak form for the dominating eqs.

**Problem Description:**
Let $\Gamma$ be a boundary of the open bounded region$\Omega \subset \mathbb{R}^2$, $Q = I \times \Omega$ , $\Sigma = \Gamma \times I$ , and $x = (x_1, x_2)$. The quaternary continuous classical boundary optimal control vector problem consists of the state quaternary eqs. , which are considered as (in Q):

$$y_{1t} - \sum_{i,j=1}^{2} \frac{\partial}{\partial \chi_i}\left(a_{1ij} \frac{\partial y_1}{\partial x_j}\right) + b_1 y_1 - b_5 y_2 + b_6 y_3 + b_7 y_4 = f_1(\chi, s, y_1), \quad 1$$

$$y_{2t} - \sum_{i,j=1}^{2} \frac{\partial}{\partial \chi_i}\left(a_{2ij} \frac{\partial y_2}{\partial x_j}\right) + b_2 y_2 + b_5 y_1 - b_9 y_3 - b_{11} y_4 = f_2(\chi, s, y_2), \quad 2$$

$$y_{3t} - \sum_{i,j=1}^{2} \frac{\partial}{\partial \chi_i}\left(a_{3ij} \frac{\partial y_3}{\partial x_j}\right) + b_3 y_3 + b_9 y_2 - b_6 y_1 + b_{15} y_4 = f_3(\chi, s, y_3), \quad 3$$

$$y_{4t} - \sum_{i,j=1}^{2} \frac{\partial}{\partial \chi_i}\left(a_{4ij} \frac{\partial y_1}{\partial x_j}\right) + b_4 y_4 - b_7 y_1 + b_{11} y_2 - b_{15} y_3 = f_4(\chi, s, y_4), \quad 4$$

With the following conditions:

$$\frac{\partial y_r}{\partial n_r} = \sum_{l,j=1}^{2} a_{rij} \frac{\partial y_r}{\partial \chi_j} \cos(n_r, \chi_i) = v_r(\chi, s), \text{ on } \Sigma, \quad 5$$

$$y_r(x, 0) = y_r^0(x), \text{ on } \Omega, \quad 6$$

where $\vec{f} \in (L^2(Q))^4 = \boldsymbol{L^2(Q)}$, is a vector of function for each $x = (x_1, x_2) \in \Omega$, $\vec{y} \in (H^2(\Omega))^4 = \boldsymbol{H^2(\Omega)}$ is the quaternary vector state solution corresponding to quaternary continuous classical boundary control vector $\vec{v} \in (L^2(\boldsymbol{\Sigma}))^4 = \boldsymbol{L^2(\Sigma)}$. $a_{ij}(\chi, s)$, $b_{ij}(\chi, s)$, $c_{ij}(\chi, s)$, $d_{ij}(\chi, s)$, $b_{r+3}(\chi, s)$, $b_{2r+3}(\chi, s)$, $b_{3r+3}(\chi, s)$ and $b_{4r+3}(\chi, s) \in C^{\infty}(Q)$, and $n_r$ (for $r = 1,2,3,4$) is an outer normal vector on the boundary $\Sigma$ and $(n_r, \chi_j)$ is the angle between $n_r$ and $\chi_j - axis$.

**The objective function and the state constraints are considered as:**

$$G_0(\vec{v}) = \int_Q [g_{01}(\chi, s, y_1) + g_{02}(\chi, s, y_2) + g_{03}(\chi, s, y_3) + g_{04}(\chi, s, y_4)]d\chi ds + \int_{\Sigma} [k_{01}(\chi, s, w_1) + k_{02}(\chi, s, w_2) + k_{03}(\chi, s, w_3) + k_{04}(\chi, s, w_4)]d\sigma\,, \quad 7$$

$$G_1(\vec{v}) = \int_Q [g_{11}(\chi, s, y_1) + g_{12}(\chi, s, y_2) + g_{13}(\chi, s, y_3) + g_{14}(\chi, s, y_4)]d\chi ds + \int_{\Sigma} [k_{11}(\chi, s, w_1) + k_{12}(\chi, s, w_2) + k_{13}(\chi, s, w_3) + k_{14}(\chi, s, w_4)]d\sigma = 0\,, \quad 8$$

$$G_2(\vec{v}) = \int_Q [g_{21}(\chi, s, y_1) + g_{22}(\chi, s, y_2) + g_{23}(\chi, s, y_3) + g_{24}(\chi, s, y_4)]\, d\chi ds + \int_{\Sigma} [k_{21}(\chi, s, w_1) + k_{22}(\chi, s, w_2) + k_{23}(\chi, s, w_3) + k_{24}(\chi, s, w_4)]d\sigma \leq 0\,, \quad 9$$

**The set of admissible** quaternary continuous classical boundary control vector **is considered as:**

$\overrightarrow{U_A} = \{\vec{v} = (v_1, v_2, v_3, v_4) \in \boldsymbol{L^2(\Sigma)} \,|\, \vec{v} \in \vec{U} \subset \mathbb{R}^4\ a.e. in\ \Sigma, G_1(\vec{v}) = 0, G_2(\vec{v}) \leq 0\}$,$\vec{U}$ is a convex set.

Let $\vec{W} = W_1 \times W_2 \times W_3 \times W_4$

$$= \{\vec{w} : \vec{w} = (w_1, w_2, w_3, w_4) \in \boldsymbol{H^1(\Omega)}\}.$$

**The weak form of the** state quaternary eqs. (when $\vec{y} \in \boldsymbol{H^2(\Omega)}$) is:

$$(y_{1s}, w_1) + a_1(s, y_1, w_1) + (b_1(s)y_1, w_1)_{\Omega} - (b_5(s)y_2, w_1)_{\Omega} + (b_6(s)y_3, w_1)_{\Omega} + (b_7(s)y_4, w_1)_{\Omega} = (f_1, w_1)_{\Omega} + (v_1, w_1)_{\Gamma}\,, \quad 10$$

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$(y_1(0), w_1)_\Omega = (y_1^0, w_1)_\Omega$ ,   11

$(y_{2s}, w_2) + a_2(s, y_2, w_2) + (b_2(s)y_2, w_2)_\Omega + (b_5(s)y_1, w_2)_\Omega - (b_9(s)y_3, w_2)_\Omega - (b_{11}(s)y_4, w_2)_\Omega = (f_2, w_2)_\Omega + (v_2, w_2)_\Gamma$ ,   12

$(y_2(0), w_2)_\Omega = (y_2^0, w_2)_\Omega$ ,   13

$(y_{3s}, w_3) + a_3(s, y_3, w_3) + (b_3(s)y_3, w_3)_\Omega + (b_9(s)y_2, w_3)_\Omega - (b_6(s)y_1, w_3)_\Omega + (b_{15}(s)y_4, w_3)_\Omega = (f_3, w_3)_\Omega + (v_3, w_3)_\Gamma$ ,   14

$(y_3(0), w_3)_\Omega = (y_3^0, w_3)_\Omega$ ,   15

$(y_{4s}, w_4) + a_4(s, y_4, w_4) + (b_4(s)y_4, w_4)_\Omega - (b_7(s)y_1, w_4)_\Omega + (b_{11}(s)y_2, w_4)_\Omega - (b_{15}(s)y_3, w_4)_\Omega = (f_4, w_4)_\Omega + (v_4, w_4)_\Gamma$ ,   16

$(y_4(0), w_4)_\Omega = (y_4^0, w_4)_\Omega$ ,   17

where $\forall r = 1,2,3,4$,

$$a_r(s, y_r, w_r) = \int_\Omega \sum_{i,j=1}^n a_{rij}(\chi, s) \frac{\partial y_r}{\partial \chi_j} \frac{\partial v_r}{\partial \chi_i} d\chi.$$

**Assumptions (A):**

(i) $f_r$, $\forall r = 1,2,3,4$ is of the *Carathèdory* type, and satisfies

$| f_r(\chi, s, y_r) | \leq \eta_r(\chi, s) + c_r|y_r|$, where $(\chi, s) \in Q, u_r, y_r \in \mathbb{R}, c_r > 0$ and $\eta_r \in L^2(Q, \mathbb{R})$.

(ii) $f_r$, $\forall r = 1,2,3,4$ satisfies the Lipchitz conditions w.r.t. $y$, i.e.

$| f_r(\chi, s, y_r) - f_r(\chi, s, \hat{y}_r)| \leq L_r| y_r - \hat{y}_r|$, where $\hat{y}_r, y_r \in \mathbb{R}$ and $L_r > 0$,

(iii) $| a_r(s, y_r, w_r) | \leq \alpha_r\| y_r\|_1\| w_r\|_1$,

$|(b_r(s)\, y_r, w_r)_\Omega | \leq \beta_r\| y_r\|_0\| w_r\|_0$,

$a_r(s, y_r, y_r) \geq \overline{\alpha}_r\| y_r\|_1^2$,

$(b_r(s)\, y_r, y_r)_\Omega \geq \overline{\beta}_r\| y_r\|_0^2, \forall r = 1,2,3,4$,

$|(b_{r+3}(s)\, y_r, w_1)_\Omega | \leq \epsilon_r\| y_r\|_0\| w_1\|_0$,

$\forall r = 2,3,4$,

$|(b_{2r+3}(s)\, y_r, w_2)_\Omega | \leq \bar{\epsilon}_r\| y_r\|_0\| w_2\|_0$ ,

$\forall r = 1,3,4$,

$|(b_{3r+3}(s)\, y_r, w_3)_\Omega | \leq \hat{\epsilon}_r\| y_r\|_0\| w_3\|_0$ ,

$\forall r = 1,2,4$,

$|(b_{4r+3}(s)\, y_r, w_4)_\Omega | \leq \tilde{\epsilon}_r\| y_r\|_0\| w_4\|_0$ ,

$\forall r = 1,2,3$,

$$c(s, \vec{y}, \vec{y}) = a_1(s, y_1, y_1) + (b_1(s)\, y_1, y_1)_\Omega + a_2(s, y_2, y_2) + (b_2(s)\, y_2, y_2)_\Omega + a_3(s, y_3, y_3) + (b_3(s)\, y_3, y_3)_\Omega + a_4(s, y_4, y_4) + (b_4(s)\, y_4, y_4)_\Omega$$

$c(t, \vec{y}, \vec{y}) \geq \bar{\alpha}\|\vec{y}\|_1^2$ , $\|\vec{y}\|_1^2 = \sum_{r=1}^4 \| y_r\|_1^2$, With $\alpha_r, \beta_r, \epsilon_r$ $(r = 1,2,3,4)$and $\bar{\alpha}$ are real positive constants.

**Theorem 3**[12]**:** With Assumptions (A), for each fixed quaternary continuous classical boundary control vector $\vec{v} \in \boldsymbol{L^2(\Omega)}$, Eq.10 – Eq.17 has a unique quaternary vector state solution

$\vec{y} \in \left(L^2(\mathrm{I}, \mathrm{W})\right)^4 = \boldsymbol{L^2}(\mathbf{I}, \mathbf{W})$, $\overrightarrow{y_s} \in \left(L^2(\mathrm{I}, W^*)\right)^4 = \boldsymbol{L^2(I, W^*)}$.

**Theorem 4**[12]**:** With Assumptions (A), the following two cases are held:

**a-** Let $\vec{y}$, $\vec{y} + \Delta\vec{y}$ be the quaternary vector states solution corresponding to the bounded quaternary continuous classical boundary control vectors $\vec{v}$ ,$\vec{v} + \Delta\vec{v} \in \boldsymbol{L^2(\Sigma)}$ resp., then ( with $\boldsymbol{\kappa}$ denote for various cosntants):

$$\|\Delta\vec{\boldsymbol{y}}\|_{\boldsymbol{L^\infty(I, L^2(\Omega))}} \leq \boldsymbol{\kappa}\|\Delta\vec{\boldsymbol{v}}\|_{\boldsymbol{\Sigma}} ,$$

$$\|\Delta\vec{\boldsymbol{y}}\|_{\boldsymbol{L^2(Q)}} \leq \boldsymbol{\kappa}\|\Delta\vec{\boldsymbol{v}}\|_{\boldsymbol{\Sigma}},$$

$$\|\Delta\vec{\boldsymbol{y}}\|_{\boldsymbol{L^2(I,V)}} \leq \boldsymbol{\kappa}\|\Delta\vec{\boldsymbol{v}}\|_{\boldsymbol{\Sigma}} .$$

**b-** The operator $\vec{v} \to \vec{y}_{\vec{v}}$ from $\boldsymbol{L^2(\Sigma)}$ into $\boldsymbol{L^\infty}\left(\boldsymbol{I}, \boldsymbol{L^2(\Omega)}\right)$or into $\boldsymbol{L^2(Q)}$ is continuous.

**Assumptions (B):**

The functions $g_{lr}$ , $k_{lr}(\forall l = 0,1,2$, and $r = 1,2,3,4)$ are of the *Carathèdory* type on $(Q \times \mathbb{R})$ and on $(\Sigma \times \mathbb{R})$ resp., and satisfy:

$| g_{lr}(\chi, s, y_r)| \leq \gamma_{lr}(\chi, s) + c_{lr}(y_r)^2$,
$| k_{lr}(\chi, s, v_r)| \leq \delta_{lr}(\chi, s) + d_{lr}(v_r)^2$,

where $y_r, v_r \in \mathbb{R}$ with $\gamma_{lr} \in L^1(Q)$, $\delta_{lr} \in L^1(\Sigma)$ .

**Lemma 1:** With Assumptions (B), the functional $G_l(\vec{u})$ is continuous on $\boldsymbol{L^2(\Sigma)}$ for each $l = 0,1,2$.

**Proof:** The continuity of the integrals $\int_Q g_{lr}(\chi, s, y_r)\, d\chi ds$, $\int_\Sigma k_{lr}(\chi, s, v_r)\, d\sigma$ on $L^2(Q)$,$L^2(\Sigma)$ respe, $\forall r = 1,2,3,4$ & $l = 0,1,2$ are obtained from Assumptions (B), with using Proposition 1, and hence $G_l(\vec{v})$ for each $l = 0,1,2$ is continuous on $\boldsymbol{L^2(\Sigma)}$ for each $l = 0,1,2$.

**Theorem 5**[12]**:** If $\vec{V}$ is compact in $\vec{U}_{\vec{A}} \neq \emptyset$. If $G_0(\vec{v})$ is convex w.r.t. $\vec{v}$ for fixed $(\chi, s, \vec{y})$. Then there exists a quaternary continuous classical boundary optimal control vector.

## Results and Discussion

**Main Results**

In this part, the existence theorem for a quaternary continuous classical boundary optimal control vector satisfying the equality and inequality constraints is stated and demonstrated. The mathematical formulation for the quaternary adjoint eqs. associated with the quaternary nonlinear state eqs. is discovered. Then the Fréchet derivative for the cost functional and the state constraints functions are obtained. The necessary and sufficient theorems for optimality are stated and demonstrated under suitable assumptions.

**Theorem 6:** If $\vec{U}_{\vec{A}} \neq \emptyset$, $G_l(\vec{v})$ (for each $l = 0,2$) is convex w.r.t. $\vec{v}$ and for fixed $(\chi, s, \vec{y})$, $G_1(\vec{v})$ is independent of $\vec{v}$. Then there exists a constraints quaternary continuous classical boundary optimal control vector.

**Proof:** From the hypotheses on $\vec{V}$, $\vec{U}_A$ is weakly compact. Since $\vec{U}_{\vec{A}} \neq \emptyset$, then there exists a minimizing sequence$\{\vec{v}_k\} \in \vec{U}_A$ ,$\forall K$, s.t $\lim_{n\to\infty} G_0(\vec{v}_k) = inf_{\vec{\bar{v}} \in \vec{W}_A} G_0(\vec{\bar{v}})$. Then there exists a subsequence of $\{\vec{v}_k\}$ say again $\{\vec{v}_k\}$ s.t. $\vec{v}_k \xrightarrow{w} \vec{v}$ in $\boldsymbol{L^2(\Sigma)}$ and $\|\vec{v}_k\|_\Sigma \leq c, \forall k$ .Hence by Theorem 3, there exists a sequence for the quaternary state solutions$\{\vec{y}_k\}$ corresponding to the sequence of the quaternary continuous classical boundary control vector $\{\vec{v}_k\}$.

In Theorem 3, it is proved that the $\vec{y}_k \xrightarrow{S} \vec{y}$ in $\boldsymbol{L^2(Q)}$, and $\vec{y}$ is a quaternary state solution of the Eq. 10- Eq. 17, corresponding to the quaternary continuous classical boundary control vector $\vec{v}$. It remains to show that the limit $\vec{v}$ satisfies Eq.7- Eq.9.

Since $G_1(\vec{v}_k) = \int_Q [g_{11}(\chi, s, y_{1k}) + g_{12}(\chi, s, y_{2k}) + g_{13}(\chi, s, y_{3k}) + g_{14}(\chi, s, y_{4k})] d\chi ds.$

From Lemma 1, with $\vec{y}_n \xrightarrow{S} \vec{y}$ in $\boldsymbol{L^2(Q)}$, and by Proposition 1, one gets:

$$\int_Q g_{1r}(\chi, s, y_{rk}) d\chi ds \to \int_Q g_{1r}(\chi, s, y_r) d\chi ds.$$

Which gives, $G_1(\vec{v}) = \lim_{k\to\infty} G_1(\vec{v}_k) = 0$.

From Lemma 1, the function $g_{lr}(\chi, s, y_r)$ is continuous w.r.t. $y_r$ ($\forall r = 1,2,3,4$ & $l = 0,2$), and one obtains

$$\int_Q g_{lr}(\chi, s, y_{rk}) d\chi ds \to \int_Q g_{lr}(\chi, s, y_r) d\chi ds. \quad 18$$

From the hypotheses on $k_{lr}$, the function $k_{lr}(\chi, s, v_r)$ is weakly lower semi continuous w.r.t. $v_r$, then by using (18), the following inequality is obtained:

$$\int_Q g_{lr}(\chi, s, y_r) d\chi ds + \int_\Sigma k_{lr}(\chi, s, w_r) d\sigma \leq$$
$$\lim_{k\to\infty} inf \int_\Sigma k_{lr}(\chi, s, w_{rk}) d\sigma + \int_Q g_{lr}(\chi, s, y_r) d\chi ds$$
$$= \lim_{k\to\infty} inf \int_\Sigma k_{lr}(\chi, s, w_{rk}) d\sigma +$$
$$\lim_{k\to\infty} inf \int_Q g_{lr}(\chi, s, y_{rk}) d\chi ds.$$

i.e. $G_l(\vec{v}) \leq \lim_{k\to\infty} inf\, G_l(\vec{v}_k)$ , for $l = 0,2$, but $G_2(\vec{v}_k) \leq 0$,then $G_2(\vec{v}) \leq 0$ and one gets that $\vec{v} \in \vec{W}_A$.

And

$$G_0(\vec{v}) \leq \lim_{k\to\infty} inf\, G_0(\vec{v}_k) = \lim_{k\to\infty} G_0(\vec{v}_k) = inf_{\vec{\bar{v}} \in \vec{W}_A} G_0(\vec{\bar{v}}) \Longrightarrow G_0(\vec{v}) = min_{\vec{\bar{v}} \in \vec{W}_A} G_0(\vec{\bar{v}}).$$

Thus $\vec{v}$ is a constraints quaternary continuous classical boundary optimal control vector.

**The Necessary and the sufficient theorems for Optimality:**

The following assumptions are needed to discover the Fréchet derivative for the cost function, which is used next in the proof of the necessary and the sufficient conditions for optimality.

**Assumptions (C):**

If, $f_{r_{y_r}}$, $g_{lr_{y_r}}$ and $h_{lr_{y_r}}$ (for $l = 0,1,2$ & $r = 1,2,3,4$) are of the *Carathèdory* type on $(Q \times \mathbb{R})$, and on $(\Sigma \times \mathbb{R})$ respe, s.t. $\left|g_{lr_{y_r}}(\chi, s, y_r)\right| \leq \bar{\gamma}_{lr}(\chi, s) + c_{lr}|y_r|$, $\left|k_{lr_{y_r}}(\chi, s, w_r)\right| \leq \bar{\delta}_{lr}(\chi, s) + d_{lr}|w_r|$ and $\left|f_{r_{y_r}}(\chi, s, y_r)\right| \leq \bar{L}_r(\chi, s)$,

where $(\chi, s) \in Q$, $y_r, v_r \in \mathbb{R}$, $\bar{\gamma}_{lr}$, $\bar{L}_r \in L^2(Q)$ and $\bar{\delta}_{lr} \in L^2(\Sigma)$.

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Baghdad Science Journal

**Theorem 7:** Dropping the index $l$ in $g_l$, $h_l$, for each $l = 0,1,2$, consider Eq. 7, and:

$$H(\chi, s, y_r, z_r, v_r) = \sum_{r=1}^{4}( z_r f_r(\chi, s, y_r) + g_r(\chi, s, y_r) + k_r(\chi, s, v_r)) .$$

The adjoin $z_r = z_{rv_r}$ or $z_{v_r}$ associated with state eqs. are formulated in Q as:

$$-z_{1s} - \sum_{i,j=1}^{n} \frac{\partial}{\partial \chi_i}\left(a_{ij}\frac{\partial z_1}{\partial \chi_j}\right) + b_1 z_1 - b_5 z_2 + b_6 z_3 + b_7 z_4 = z_1 f_{1_{y_1}}(y_1) + g_{y_1}(y_1), \quad 19$$

$$-z_{2s} - \sum_{i,j=1}^{n} \frac{\partial}{\partial \chi_i}\left(b_{ij}\frac{\partial z_2}{\partial \chi_j}\right) + b_2 z_2 + b_5 z_1 - b_9 z_3 - b_{11} z_4 = z_2 f_{2_{y_2}}(y_2) + g_{y_2}(y_2), \quad 20$$

$$-z_{3s} - \sum_{i,j=1}^{n} \frac{\partial}{\partial \chi_i}\left(c_{ij}\frac{\partial z_3}{\partial \chi_j}\right) + b_3 z_3 + b_9 z_2 - b_6 z_1 + b_{15} z_4 = z_3 f_{3_{y_3}}(y_3) + g_{y_3}(y_3), \quad 21$$

$$-z_{4s} - \sum_{i,j=1}^{n} \frac{\partial}{\partial \chi_i}\left(d_{ij}\frac{\partial z_1}{\partial \chi_j}\right) + b_4 z_4 - b_7 z_1 + b_{11} z_2 - b_{15} z_3 = z_4 f_{4_{y_4}}(y_4) + g_{y_4}(y_4) . \quad 22$$

With the following conditions:

$$\frac{\partial z_r}{\partial n_r} = 0, \text{ on } \Sigma, \quad 23$$

$$z_r(T) = 0, \text{ on } \Omega. \quad 24$$

Then the Fréchet derivative of $G$ is:

$$G'(\vec{v})\overrightarrow{\Delta v} = \int_{\Sigma} \begin{pmatrix} z_1 + k_{1_{v_1}} \\ z_2 + k_{2_{v_2}} \\ z_3 + k_{3_{v_3}} \\ z_4 + k_{4_{v_4}} \end{pmatrix} \cdot \begin{pmatrix} \Delta v_1 \\ \Delta v_2 \\ \Delta v_3 \\ \Delta v_4 \end{pmatrix} d\sigma.$$

**Proof:** The weak from of the quaternary adjoint eqs. for each $v_r \in V$, $\forall r = 1,2,3,4$, is:

$$-(z_{1s}, w_1) + a_1(s, z_1, w_1) + (b_1(s)z_1, w_1)_\Omega - (b_5(s)z_2, w_1)_\Omega + (b_6(s)z_3, w_1)_\Omega + (b_7(s)z_4, w_1)_\Omega = (z_1 f_{1_{y_1}}, w_1)_\Omega + (g_{y_1}, w_1)_\Omega, \quad 25$$

$$-(z_{2s}, w_2) + a_2(s, z_2, w_2) + (b_2(s)z_2, w_2)_\Omega + (b_5(s)z_1, w_2)_\Omega - (b_9(s)z_3, w_2)_\Omega - (b_{11}(s)z_4, w_2)_\Omega = (z_2 f_{2_{y_2}}, w_2)_\Omega + (g_{y_2}, w_2)_\Omega \quad 26$$

$$-(z_{3s}, w_3) + a_3(s, z_3, w_3) + (b_3(s)z_3, w_3)_\Omega + (b_9(s)z_2, w_3)_\Omega - (b_6(s)z_1, w_3)_\Omega + (b_{15}(s)z_4, w_3)_\Omega = (z_3 f_{3_{y_3}}, w_3)_\Omega + (g_{y_3}, w_3)_\Omega , \quad 27$$

$$-(z_{4s}, w_4) + a_4(s, z_4, w_4) + (b_4(s)z_4, w_4)_\Omega - (b_7(s)z_1, w_4)_\Omega + (b_{11}(s)z_2, w_4)_\Omega - (b_{15}(s)z_3, w)_\Omega = (z_4 f_{4_{y_4}}, w_4)_\Omega + (g_{y_4}, w_4)_\Omega , \quad 28$$

Now, utilizing $y_r = \Delta y_r$ , $w_r = z_r, \forall r = 1,2,3,4$, in Eq.10, Eq.12, Eq.14& Eq.16 resp., integrating both sides w.r.t. t from 0 to T to get:

$$\int_0^T (\Delta y_{1s}, z_1) dt + \int_0^T [ a_1(s, \Delta y_1, z_1) + (b_1(s)\Delta y_1, z_1)_\Omega - (b_5(s)\Delta y_2, z_1)_\Omega + (b_6(s)\Delta y_3, z_1)_\Omega + (b_7(s)\Delta y_4, z_1)_\Omega] ds = \int_0^T (f_1(y_1 + \Delta y_1), z_1)_\Omega ds - \int_0^T (f_1(y_1), z_1)_\Omega ds + \int_0^T (\Delta v_1, z_1)_\Gamma ds , \quad 29$$

$$\int_0^T (\Delta y_{2s}, z_2) ds + \int_o^T [ a_2(s, \Delta y_2, z_2) + (b_2(s)\Delta y_2, z_2)_\Omega + (b_5(s)\Delta y_1, z_2)_\Omega - (b_9(s)\Delta y_3, z_2)_\Omega - (b_{11}(s)\Delta y_4, z_2)_\Omega] ds = \int_0^T (f_2(y_2 + \Delta y_2), z_2)_\Omega ds - \int_0^T (f_2(y_2), z_2)_\Omega ds + \int_0^T (\Delta v_2, z_2)_\Gamma ds, \quad 30$$

$$\int_0^T (\Delta y_{3s}, z_3) dt + \int_0^T [ a_3(s, \Delta y_3, z_3) + (b_3(s)\Delta y_3, z_3)_\Omega + (b_9(s)\Delta y_2, z_3)_\Omega - (b_6(s)\Delta y_1, z_3)_\Omega + (b_{15}(s)\Delta y_4, z_3)_\Omega] ds = \int_0^T (f_3(y_3 + \Delta y_3), z_3)_\Omega ds - \int_0^T (f_3(y_3), z_3)_\Omega ds + \int_0^T (\Delta v_3, z_3)_\Gamma ds , \quad 31$$

$$\int_0^T (\Delta y_{4s}, z_4) dt + \int_0^T [ a_4(s, \Delta y_4, z_4) + (b_4(s)\Delta y_4, z_4)_\Omega - (b_7(s)\Delta y_1, z_4)_\Omega + (b_{11}(s)\Delta y_2, z_4)_\Omega - (b_{15}(s)\Delta y_3, z_4)_\Omega] ds = \int_0^T (f_4(y_4 + \Delta y_4), z_4)_\Omega ds - \int_0^T (f_4(y_4), z_4)_\Omega ds + \int_0^T (\Delta v_4, z_4)_\Gamma ds , \quad 32$$

Adding Eq.29-Eq.32 together, to get:

$$\int_0^T (\overrightarrow{\Delta y_s}, \vec{z}) ds + \int_0^T c(t, \vec{y}, \vec{y}) ds = \int_0^T (f_1(y_1 + \Delta y_1), z_1)_\Omega ds - \int_0^T (f_1(y_1), z_1)_\Omega ds + \int_0^T (\Delta v_1, z_1)_\Gamma ds + \int_0^T (f_2(y_2 + \Delta y_2), z_2)_\Omega ds - \int_0^T (f_2(y_2), z_2)_\Omega ds + \int_0^T (\Delta v_2, z_2)_\Gamma ds + \int_0^T (f_3(y_3 + \Delta y_3), z_3)_\Omega ds -$$

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$\int_0^T (f_3(y_3), z_3)_\Omega ds + \int_0^T (\Delta v_3, z_3)_\Gamma ds + \int_0^T (f_4(y_4 + \Delta y_4), z_4)_\Omega ds - \int_0^T (f_4(y_4), z_4)_\Omega ds + \int_0^T (\Delta v_4, z_4)_\Gamma ds$ . 33

Now, from Proposition 2:

$\int_0^T (f_r(y_r + \Delta y_r), z_r)_\Omega ds - \int_0^T (f_r(y_r), z_r)_\Omega ds = \int_0^T ( f_{r_{y_r}} \Delta y_r, z_r) ds + \varepsilon_{r1}(\Delta y_r)\|\Delta y_r\|_Q$ ,

where $\varepsilon_{r1}(\Delta y_r) \to 0$ & $\|\Delta y_r\|_Q \to 0, \forall r = 1,2,3.$

Hence

$\sum_{r=1}^4 [\int_0^T (f_r(y_r + \Delta y_r), z_r)_\Omega ds - \int_0^T (f_r(y_r), z_r)_\Omega ds] = \sum_{r=1}^4 (\int_0^T (f_{r_{y_r}} \Delta y_r, z_r)\, ds) + \varepsilon_1(\overrightarrow{\Delta v})\|\overrightarrow{\Delta v}\|_\Sigma$, 34

where $\sum_{r=1}^4 \varepsilon_{r1}(\Delta y_r)\|\Delta y_r\|_Q = \varepsilon_1(\overrightarrow{\Delta y})\|\overrightarrow{\Delta y}\|_Q$.

Now, substituting Eq. 33 in Eq. 34, to get:

$\int_0^T (\overrightarrow{\Delta y_s}, \vec{z})\, ds + \int_0^T c(t, \vec{y}, \vec{y}) ds = \sum_{r=1}^4 (\int_0^T \left( f_{r_{y_r}} \Delta y_r, z_r \right) ds) + \varepsilon_1(\overrightarrow{\Delta v})\|\overrightarrow{\Delta v}\|_\Sigma + \int_0^T (\Delta v_1, z_1)_\Gamma\, ds + \int_0^T (\Delta v_2, z_2)_\Gamma\, ds + \int_0^T (\Delta v_3, z_3)_\Gamma\, ds + \int_0^T (\Delta v_4, z_4)_\Gamma\, ds,$ 35

where, $\|\overrightarrow{\Delta v}\|_\Sigma \to 0, \varepsilon_1(\overrightarrow{\Delta v}) \to 0$ .

Substituting $w_r = \Delta y_r$ in the Eq.25- Eq.28 resp., integration of both sides w.r.t. t from 0 to T, integration by parts the first term in the L.H.S. of each one of them, and finally adding all the resulting eqs. together, to obtain:

$\int_0^T (\overrightarrow{\Delta y_s}, \vec{z})\, ds + \int_0^T c(t, \vec{y}, \vec{y}) ds = \int_0^T \left( z_1 f_{1_{y_1}}, \Delta y_1 \right)_\Omega ds + \int_0^T \left( g_{1_{y_1}}, \Delta y_1 \right)_\Omega ds + \int_0^T \left( z_2 f_{2_{y_2}}, \Delta y_2 \right)_\Omega ds + \int_0^T \left( g_{2_{y_2}}, \Delta y_2 \right)_\Omega ds + \int_0^T \left( z_3 f_{3_{y_3}}, \Delta y_3 \right)_\Omega ds + \int_0^T (g_{3_{y_3}}, \Delta y_3)_\Omega\, ds + \int_0^T (z_4 f_{4_{y_4}}, \Delta y_4)_\Omega ds + \int_0^T (g_{4_{y_4}}, \Delta y_4)_\Omega ds$ 36

Subtracting Eq. 36 from Eq. 35, to get:

$\int_0^T \left( g_{1_{y_1}}, \Delta y_1 \right)_\Omega ds + \int_0^T \left( g_{2_{y_2}}, \Delta y_2 \right)_\Omega ds + \int_0^T (g_{3_{y_3}}, \Delta y_3)_\Omega\, ds + \int_0^T (g_{4_{y_4}}, \Delta y_4)_\Omega\, ds = \varepsilon_1(\overrightarrow{\Delta v})\|\overrightarrow{\Delta v}\|_\Sigma + \int_0^T (\Delta v_1, z_1)_\Gamma\, ds + \int_0^T (\Delta v_2, z_2)_\Gamma\, ds + \int_0^T (\Delta v_3, z_3)_\Gamma\, ds + \int_0^T (\Delta v_4, z_4)_\Gamma ds.$ 37

By the definition of the Fréchet derivative and Theorem 4, one has:

$G(\vec{v} + \overrightarrow{\Delta v}) - G(\vec{v}) = \sum_{r=1}^4 \int_Q g_{r_{y_r}} \Delta y_r\, d\chi ds + \sum_{r=1}^4 \int_\Sigma k_{r_{v_r}} \Delta v_r\, d\sigma + \varepsilon_4(\overrightarrow{\Delta v})\|\overrightarrow{\Delta v}\|_\Sigma,$ 38

where $\varepsilon_4(\overrightarrow{\Delta v}) = (\varepsilon_2 + \varepsilon_3)(\overrightarrow{\Delta v}) \to 0$ and $\|\overrightarrow{\Delta v}\|_\Sigma \to 0$ .

Now, substituting Eq.37 in Eq.38, to get:

$G(\vec{v} + \overrightarrow{\Delta v}) - G(\vec{v}) = \sum_{r=1}^4 \int_\Sigma (\Delta v_r, z_r)\, d\sigma + \sum_{r=1}^4 \int_\Sigma k_{r_{v_r}} \Delta v_r\, d\sigma + \varepsilon_4(\overrightarrow{\Delta v})\|\overrightarrow{\Delta v}\|_\Sigma$ .

Since the Fréchet derivative of G is $G(\vec{v} + \overrightarrow{\Delta v}) - G(\vec{v}) = (G'(\vec{v}), \overrightarrow{\Delta v}) + \varepsilon_4(\overrightarrow{\Delta v})\|\overrightarrow{\Delta v}\|_\Sigma$, then

$$G'(\vec{v})\overrightarrow{\Delta v} = \int_\Sigma \begin{pmatrix} z_1 + k_{1_{v_1}} \\ z_2 + k_{2_{v_2}} \\ z_3 + k_{3_{v_3}} \\ z_4 + k_{4_{v_4}} \end{pmatrix} \cdot \begin{pmatrix} \Delta v_1 \\ \Delta v_2 \\ \Delta v_3 \\ \Delta v_4 \end{pmatrix} d\sigma.$$

**Theorem 8:** If $\vec{v} \in \overrightarrow{W}_A$ is a quaternary continuous classical boundary optimal control vector, then there exist $\lambda_l \in \mathbb{R}$, $l = 0,1,2$, with $\lambda_0, \lambda_2 \geq 0$, $\sum_{l=0}^2 |\lambda_l| = 1$ , s.t. the following conditions are satisfied

$\sum_{l=0}^2 \lambda_l G'_l(\vec{v})(\overrightarrow{v'} - \vec{v}) \geq 0, \overrightarrow{v'} \in \overrightarrow{W}$ , 39

$\lambda_2 G_2(\vec{v}) = 0.$ 40

Moreover (35) is equivalent to

$H_{\vec{v}}(\chi, s, \vec{y}, \vec{z}, \vec{v})\vec{v} = \min_{\overrightarrow{v'} \in V} H_{\vec{v}}(\chi, s, \vec{y}, \vec{z}, \vec{v})\overrightarrow{v'}$, on $\Sigma$. 41

**Proof:** Since $G_l(\vec{v})$ and $G'_l(\vec{v})$ are continuous and linear w.r.t.$(\overrightarrow{v'} - \vec{v})$, then $G_l(\vec{v})$ is $\rho$- differentiable at $\vec{v} \in \overrightarrow{U}, \forall \rho$ and then by theorem 2, there exist $\lambda_l \in \mathbb{R}$, for $l = 0,1,2$ with $\lambda_0, \lambda_2 \geq 0$ and $\sum_{l=0}^2 |\lambda_l| = 1$, s.t Eq.39 and Eq.41 hold, i.e.

$\sum_{l=0}^2 \lambda_l G'_l(\vec{v})(\overrightarrow{v'} - \vec{v}) \geq 0, \overrightarrow{v'} \in \overrightarrow{U}$ . 42

Applying Theorem 7, setting $\overrightarrow{\Delta v} = (\overrightarrow{v'} - \vec{v})$, then from Eq.42, one has

$\sum_{l=0}^2 \sum_{r=1}^4 \int_\Sigma \lambda_l \left( z_{lr} + k_{lr_{v_r}} \right) (v'_r - v_r)\, d\sigma \geq 0$ .

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Let $z_r = \sum_{l=0}^{2} \lambda_l z_{lr}$ and $k_{r_{u_r}} = \sum_{l=0}^{2} \lambda_l k_{lr_{v_r}}$, $\forall r = 1,2,3,4$, then

$$\int_\Sigma H_{\vec{v}}(\chi, s, \vec{y}, \vec{z}, \vec{v}) \cdot \overrightarrow{\Delta v}\, d\sigma \geq 0 \,. \qquad 43$$

To prove Eq.43 is equivalent to Eq.41, let $\{\overrightarrow{v'_k}\}$ be a dense sequence in $\vec{U}_A$, $\mu$ be the Lebesgue measure on $\Sigma$ and let $S \subset \Sigma$ be a subset, s.t

$$\overrightarrow{v'}(\chi, s) = \begin{cases} \vec{v}_k(\chi, s) & , if\, (\chi, s) \in S \\ \vec{v}(\chi, s) & , if\, (\chi, s) \notin S \end{cases}.$$

Therefore (43) becomes:

$$\int_S H_{\vec{v}}(\chi, s, \vec{y}, \vec{z}, \vec{v}) \cdot (\vec{v}_k - \vec{v})\, d\sigma \geq 0 \,.$$

From Theorem 1, one obtains that:

$H_{\vec{v}}(\chi, s, \vec{y}, \vec{z}, \vec{v}) \cdot (\vec{v}_k - \vec{v}) \geq 0$ , in $\Sigma$ and then in $P = \cap_k P_k$, with $P_k = \Sigma - \Sigma_k$ and $\mu(\Sigma_k) = 0, \forall k$, but $P$ is independent on $k$, hence $\mu(\Sigma - \mathrm{P}) = \mu(\cup_k \Sigma_k) = 0$, but $\{\overrightarrow{v'_k}\}$ is dense in $\vec{U}_A$, so the above inequality becomes
$H_{\vec{v}}(\chi, s, \vec{y}, \vec{z}, \vec{v}) \cdot \left(\overrightarrow{v'} - \vec{v}\right) \geq 0, \forall \overrightarrow{v'} \in \vec{U}_A$ , a.e. in $\Sigma$,

$$\Rightarrow H_{\vec{v}}(\chi, s, \vec{y}, \vec{z}, \vec{v})\vec{v} = \min_{\overrightarrow{v'} \in V} H_{\vec{v}}(\chi, s, \vec{y}, \vec{z}, \vec{v})\overrightarrow{v'}, \text{on } \Sigma.$$

The converse is obtained directly.

**Theorem 9:** Suppose for each $r = 1,2,3,4$ that $f_r$, $g_{1r}$ are affine w.r.t. $y_r$ ,$\forall(\chi, s) \in Q$, $k_{1r}$ is affine w.r.t. $v_r$, $\forall(\chi, s) \in \Sigma$ and the functions $g_{0r}$, $g_{2r}$ are convex w.r.t. $y_r$, $\forall(\chi, s) \in Q$, the functions $k_{0r}$, $k_{2r}$ are continuous w.r.t. $v_r$, $\forall(\chi, s) \in \Sigma$. Then the necessary conditions in Theorem 6 with $\lambda_0 > 0$ are sufficient.

**Proof:** Suppose $\vec{v} \in \vec{U}_A$ and satisfies the following conditions:

$$\int_\Sigma H_{\vec{v}}(\chi, s, \vec{y}, \vec{z}, \vec{v}) \cdot \overrightarrow{\Delta v}\, d\sigma \geq 0\,, \forall \overrightarrow{v'} \in \vec{U}_A,$$

$$\lambda_2 G_2(\vec{v}) = 0 \,.$$

Let $G(\vec{v}) = \sum_{l=0}^{2} \lambda_l G_l(\vec{v})$, then from Theorem 7, one has

$$G'(\vec{v}) \cdot \overrightarrow{\Delta v} = \sum_{l=0}^{2} \lambda_l G'_l(\vec{v}) \cdot \overrightarrow{\Delta v} = \sum_{l=0}^{2} \sum_{r=1}^{4} \int_\Sigma \lambda_l \left(z_{lr} + k_{lr_{v_r}}\right) \Delta v_r\, d\sigma.$$

Since the functions $f_r$ (in the R.H.S. of the Eq.1-Eq.4) are affine, then

$$f_r(\chi, s, y_r) = f_{r1}(\chi, s) y_r + f_{r2}(\chi, s), \forall r = 1,2,3,4.$$

Let $\vec{v}$ and $\vec{\bar{v}}$ be two quaternary continuous classical boundary control vectors, then $\vec{y} = \vec{y}_{\vec{v}}$ and $\vec{\bar{y}} = \vec{\bar{y}}_{\vec{\bar{v}}}$ are their corresponding quaternary vector state solutions (by Theorem3), i.e., from the first eq. for $y_1$, one has:

$$y_{1s} - \sum_{i,j=1}^{2} \frac{\partial}{\partial \chi_i}\left(a_{1ij} \frac{\partial y_1}{\partial \chi_j}\right) + b_1 y_1 - b_5 y_2 + b_6 y_3 + b_7 y_4 = f_{11} y_1 + f_{12}, \text{ in } Q\,, \qquad 44$$

$$y_1(\chi, 0) = y_1^0(\chi), \text{ on } \Omega\,, \qquad 45$$

$$\sum_{i,j=1}^{2} a_{1ij} \frac{\partial y_1}{\partial \chi_j} \cos(n_1, \chi_i) = v_1(\chi, s), \text{ on } \Sigma\,, \qquad 46$$

and

$$\bar{y}_{1s} - \sum_{i,j=1}^{2} \frac{\partial}{\partial \chi_i}\left(a_{1ij} \frac{\partial \bar{y}_1}{\partial \chi_j}\right) + b_1 \bar{y}_1 - b_5 \bar{y}_2 + b_6 \bar{y}_3 + b_7 \bar{y}_4 = f_{11} \bar{y}_1 + f_{12}, \text{ in } Q\,, \qquad 47$$

$$\bar{y}_1(\chi, 0) = y_1^0(\chi), \text{ on } \Omega, \qquad 48$$

$$\sum_{i,j=1}^{2} a_{1ij} \frac{\partial \bar{y}_1}{\partial \chi_j} \cos(n_1, \chi_i) = \bar{v}_1(\chi, s), \text{ on } \Sigma\,. \qquad 49$$

By multiplying Eq.44- Eq.46 by $\theta \in [0,1]$ and Eq.47- Eq.49 by $(1 - \theta)$, resp. and then adding each pair of the obtained eqs. together, to get:

$$(\theta y_1 + (1-\theta)\bar{y}_1)_s - \sum_{i,j=1}^{2} \frac{\partial}{\partial \chi_i}\left(a_{1ij} \frac{\partial(\theta y_1 + (1-\theta)\bar{y}_1)}{\partial \chi_j}\right) + b_1(\theta y_1 + (1-\theta)\bar{y}_1) - b_5(\theta y_2 + (1-\theta)\bar{y}_2) + b_6(\theta y_3 + (1-\theta)\bar{y}_3) + b_7(\theta y_4 + (1-\theta)\bar{y}_4) = f_{11}(\theta y_1 + (1-\theta)\bar{y}_1) + f_{12}, \qquad 50$$

$$\theta y_1(\chi, 0) + (1-\theta)\bar{y}_1(\chi, 0) = y_1^0, \qquad 51$$

$$\sum_{i,j=1}^{2} a_{1ij} \frac{\partial(\theta y_1 + (1-\theta)\bar{y}_1)}{\partial \chi_j} \cos(n_1, \chi_i) = \theta v_1 + (1-\theta)\bar{v}_1\,, \qquad 52$$

For the remainder eqs. , the above same way can be used to get that:

$$(\theta y_2 + (1-\theta)\bar{y}_2)_s - \sum_{i,j=1}^{2} \frac{\partial}{\partial \chi_i}\left(a_{2ij} \frac{\partial(\theta y_2 + (1-\theta)\bar{y}_2)}{\partial \chi_j}\right) + b_2(\theta y_2 + (1-\theta)\bar{y}_2) + b_5(\theta y_1 + (1-\theta)\bar{y}_1) - b_9(\theta y_3 + (1-\theta)\bar{y}_3) - b_{11}(\theta y_4 + (1-\theta)\bar{y}_4) = f_{21}(\theta y_1 + (1-\theta)\bar{y}_1) + f_{22}\,, \qquad 53$$

$$(\theta y_3 + (1-\theta)\bar{y}_3)_s - \sum_{i,j=1}^{2} \frac{\partial}{\partial \chi_i}\left(a_{3ij} \frac{\partial(\theta y_3 + (1-\theta)\bar{y}_3)}{\partial \chi_j}\right) + b_3(\theta y_3 + (1-\theta)\bar{y}_3) + b_9(\theta y_2 + (1-\theta)\bar{y}_2) - b_6(\theta y_1 + (1-\theta)\bar{y}_1) + b_{15}(\theta y_4 + (1-\theta)\bar{y}_4) = f_{31}(\theta y_1 + (1-\theta)\bar{y}_1) + f_{32}, \qquad 54$$

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Baghdad Science Journal

$(\theta y_4 + (1-\theta)\bar{y}_4)_s - \sum_{i,j=1}^{2} \frac{\partial}{\partial \chi_i}\left(a_{4ij}\frac{\partial(\theta y_3+(1-\theta)\bar{y}_3)}{\partial \chi_j}\right) + b_4(\theta y_4 + (1-\theta)\bar{y}_4) - b_7(\theta y_1 + (1-\theta)\bar{y}_1) + b_{11}(\theta y_2 + (1-\theta)\bar{y}_2) - b_{15}(\theta y_3 + (1-\theta)\bar{y}_3) = f_{41}(\theta y_1 + (1-\theta)\bar{y}_1) + f_{42}$, 55

$\theta y_r(\chi,0) + (1-\theta)\bar{y}_r(\chi,0) = y_r^0(\chi)$, for $r = 2,3,4,$ 56

$\sum_{i,j=1}^{2} a_{rij}\frac{\partial(\theta y_r+(1-\theta)\bar{y}_r)}{\partial \chi_j}\cos(n_r,\chi_i) = \theta v_r + (1-\theta)\bar{v}_r$ , for $r = 2,3,4,$ 57

From Eq.50- Eq.57, one gets that the quaternary continuous classical boundary control vector $\vec{\bar{v}} = \theta\vec{v} + (1-\theta)\vec{\bar{v}}$ has the corresponding quaternary state vector $\overline{\vec{y}} = \theta\vec{y} + (1-\theta)\vec{\bar{y}}$.

Thus $\vec{v} \to \vec{y}_{\vec{v}}$ is convex- linear w.r.t. $(\vec{y},\vec{v})$ for each $(\chi,s) \in Q$.

Also, since

$g_{1r}(\chi,s,y_r) = I_{1r}(\chi,s)y_r + I_{2r}(\chi,s)$, $k_{1r}(\chi,s,y_r) = I_{1r}(\chi,s)v_r + I_{2r}(\chi,s)$.

Then for the two quaternary continuous classical boundary controls vectors $\vec{v}$ and $\vec{\bar{v}}$ and their corresponding quaternary vector state $\vec{y} = \vec{y}_{\vec{v}}$ and $\vec{\bar{y}} = \vec{\bar{y}}_{\vec{\bar{v}}}$, one obtains

$G_1(\theta\vec{v} + (1-\theta)\vec{\bar{v}}) = \sum_{r=1}^{4}\int_Q g_{1r}(\chi,s,\theta y_r + (1-\theta)\bar{y}_r)d\chi ds + \sum_{r=1}^{4}\int_\Sigma k_{1r}(\chi,s,\theta v_r + (1-\theta)\bar{v}_r)\, d\sigma$. Since the operator $\vec{v} \to \vec{y}_{\vec{u}}$ is convex linear, then:

$G_1(\theta\vec{v} + (1-\theta)\vec{\bar{v}}) = \sum_{r=1}^{4}\int_Q I_{1r}(\chi,s)(\theta y_r + (1-\theta)\bar{y}_r) + I_{2r}(\chi,s)d\chi ds +$

$\sum_{r=1}^{4}\int_\Sigma I_{1r}(\chi,s)(\theta v_r + (1-\theta)\bar{v}_r) + I_{2r}(\chi,s)\, d\sigma$ ,
$= \theta G_1(\vec{v}) + (1-\theta)G_1(\vec{\bar{v}})$,

i.e., $G_1(\vec{u})$ is convex- linear w.r.t. $(\vec{y},\vec{v})$.

From the hypotheses on the functions $g_{0r}$ ,$g_{2r}$ ($h_{0r}$,$h_{2r}$) $\forall r = 1,2,3,4$, one gets that the integrals $\sum_{r=1}^{4}\int_Q g_{0r}\,dxdt$,$\sum_{r=1}^{4}\int_Q g_{2r}\,dxdt$ ($\sum_{r=1}^{4}\int_\Sigma k_{0r}\,d\sigma$ and $\sum_{r=1}^{4}\int_\Sigma h_{2r}\,d\sigma$) are convex w.r.t. $\vec{y}$(w.r.t. $\vec{v}$) , which gives $G_0(\vec{v})$ and $G_2(\vec{v})$ are convex w.r.t. $(\vec{y},\vec{v})$, and hence $G(\vec{v})$ is convex w.r.t. $(\vec{y},\vec{v})$.

On the other hand, from Theorem 8 and Proposition 2, the following is satisfied

$G'(\vec{v})\cdot\overrightarrow{\Delta v} \geq 0$, which means $G(\vec{v})$ has a minimum at $\vec{v}$, i.e.

$\sum_{l=0}^{2}\lambda_l G_l(\vec{v}) \leq \sum_{l=0}^{2}\lambda_l G_l(\vec{\bar{v}})$. 58

Let $\vec{\bar{v}} \in \vec{U}_A$ with $\lambda_2 \geq 0$, then from Eq.40 and Eq.58:

$\lambda_0 G_0(\vec{v}) \leq \lambda_0 G_0(\vec{\bar{v}}) \Rightarrow G_0(\vec{v}) \leq G_0(\vec{\bar{v}}), \forall\vec{\bar{v}} \in \vec{U}_A$ & $(\lambda_0 > 0)$.

Hence $\vec{v}$ is a quaternary continuous classical boundary optimal control vector.

## Conclusion

In this paper, the quaternary continuous classical boundary optimal control vector dominated by the quaternary nonlinear parabolic boundary value problem with state constraints is proposed. The existence theorem for a quaternary continuous classical boundary optimal control vector with equality and inequality constraints is stated and demonstrated under suitable hypotheses. The mathematical formulation of the adjoint quaternary eqs. associated with the quaternary nonlinear parabolic state eqs. is discovered. The Fréchet derivative for the objective function and the state constraints functions are obtained. Finally, under suitable hypotheses both the necessary (conditions) theorem and the sufficient (conditions) theorem for optimality of the quaternary continuous classical boundary optimal control vector problem are stated and demonstrated.

The future scope of this paper is very interesting in the field of applied mathematics since the proposed model represents a generalization for the heat eq. Furthermore, it represents multi objectives problems which have many applications. On the other hand the results are very important because they give the green light for the ability to solve such problems numerically.

## Author's Declaration

- Conflicts of Interest: None.
- Ethical Clearance: The project was approved by the local ethical committee at University of Mustansiriyah

## Author's Contribution Statement

J. A. A. and F. J. N contributed to the design and implementation of the research, the proof of the theorems, and the writing of the manuscript.

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# قابلية الحل للسيطرة الامثلية الحدودية الرباعية المستمرة التقليدية المسيطر بالنظام الرباعي المكافيء

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**الخلاصة**

الهدف من هذا البحث هو دراسة قابلية الحل لمتجه السيطرة الامثلية الحدودية الرباعية التقليدية المستمرة المسيطرة بمسالة القيم الحدودية الغير خطية المكافئة بوجود قيود الحالة. تم ذكر وإثبات مبرهنة الوجود لمتجه السيطرة الامثلية الحدودية التقليدية المستمرة الرباعية مع قيدي التساوي والتباين للحالة وبوجود شروط ملائمة. تم ايجاد الصيغة الرياضية لمسالة القيم الحدودية الرباعية المرافقة والمصاحبة لمسالة القيم الحدودية الرباعية الغير خطية المكافئة . تم ايجاد مشتقة فريشيه لدالة الكلفة ولدوال القيود. بوجود شروط مناسبة تم ذكر نص واثبات مبرهنتي الشروط الضرورية والشروط الكافية لوجود متجهة السيطرة الامثلية الحدودية الرباعية المستمرة التقليدية.

**الكلمات المفتاحية:** المعادلات المرافقة، متجه السيطرة الامثلية الحدودية الرباعية المقيدة، الشروط الضرورية المثلى، نظام القطع المكافئ الرباعي غير الخطي، الشروط المثلى الكافية، مشتقة فريشيه.