Weighted (k, n)–arcs of Type $(n - 4, n)$ in $PG(2, 8)$

Fuad. K. Hameed Habeeb M. Abood Zainab A. Abudljebar

Mathematics Department- College of Education for pure Sciences
University of Basrah

Abstract. The main purpose of this paper is to construct a $(k, n; f)$ −arcs of type $(n - 4, n)$ in projective plane of order 8. We proved that there exist $(66, 16; f)$ −arc of type (12, 16) when the points of weight 0 form a $(7, 3)$ −arc of type (7, 0, 42, 24) and $(66, 15; f) -$ arc of type (11, 15) when the points of weight 0 form (7, 3) − arc of types (3, 12, 30, 28) and (2,15, 27, 29).

Key words. Projective Plane**,** weighted (k, n) −arcs.

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1. Introduction

The notion of weighted (k, n) −arcs was proposed by Scafati [13] in 1971. In 1977, Barlotti [2] suggested the study of sets with weighted points. He presented the definition of $(k, n; \{\omega_i\})$ −sets of kind s in $PG(r, q)$, for which two-dimensional yields the definition of $(k, n; \{\omega_i\})$ −arcs. Barnabei [3] in 1979 studied these types of arcs and obtained some particular results about the existence or non-existence of these arcs. She gave some examples of these arcs in $PG(2, q)$ by using a computer programme. D'Agostini [4] in 1979 studied caps with weighted points in $PG(n, q), n \geq 2$. Some relations between parameters of $(k, n; f)$ −cap and its characters were found. She studied a weighted (k, n)-arcs of type $(n - 2,\ n)$in $PG(2, q)$.Wilson [14] in 1986 proved that there is $(88, 14; f) -$ arc of type (11, 14) in the Galois plane of order 9. Also he proved that there is $(10, 7; f)$ −arc of type (4,7) in the $PG(2, 3)$. In 1989, Hummed [9] studied the existence and non-existence weighted of (k, n)-arcs in $PG(2, 9)$ and he proved that there exist (81, 12; f) –arc of type (9, 12) and (85, 13; f) –arc of type (10, 13) .

Mahmood [12] in 1990 discussed a $(k, n; f)$ −arcs in $PG(2, 5)$. She proved there is $(21, 11; f)$ −arc of type (6, 11), $(20, 10, f)$ −arc of type (5, 10) and $(15, 8, f)$ −arc of type (3, 8). The extensions work of the weighted arcs in $PG(2, 9)$ was investigated by Abbas [1] in 2011. He proved there exists $(81, 12; f) -$ arc of type (9, 12) and $(76, 11; f)$ −arc of type (8, 11) .

2. Preliminaries

Definition 2.1 [11]. Let $GF(p) = Z/pZ,\ p$ prime and let $f(x)$ be irreducible polynomial of degree h over GF(p), then
$GF(p^h)$ = GF(p)[x]/$(f(x))$ = { $a_0 + a_1 t + \cdots + a_{h-1} t^{h-1}$: a_i in GF(p), $f(t) = 0$}

Definition 2.2[11]. A projective plane over $GF(q)$ is 2-dimentional projective space and denoted by $PG(2, q)$ or π which contains $q^2 + q + 1$ points and $q^2 + q + 1$ lines, every line contains $q + 1$ points and through every point there pass $q + 1$ lines and satisfy the following axioms:
(i) Any two distinct points determine a unique line ;
(ii) Any two distinct lines intersect in exactly one point ;
(iii) There exist four distinct points such that no three of them are on a line.

Definition 2.3 [11]. A (k, n)-arc $\mathcal{H}$ in a finite projective plane is a set of k points such that no $(n + 1)$ of them are collinear. A $(k$, 2)−arc denoted by k −arc which is a set of k points such that no three points are collinear. Let ℓ be any line in $PG(2, q)$, if ℓ intersect $\mathcal{H}$ in i- points, then ℓ is called i-secant to $\mathcal{H}$, and let τ_i , ρ_i and $\sigma_i$are respectively denoted the total number of i −secant to $\mathcal{H}$, the number of *i*-secant through a point p of $\mathcal{H}$ and the number of *i*-secant through a point Q in $\pi\backslash\mathcal{H}$.

Lemma 2. 4 [11]. For the (k,n) −arc $\mathcal{H}$, the following equations are hold:

$$\sum_{i=0}^{n} \tau_{i=} q^2+q+1 \; ; \qquad (2.1)$$

$$\sum_{i=1}^{n} i\tau_i = k(q+1)\; ; \qquad (2.2)$$

$$\sum_{i=2}^{n} (i(i-1)/2)\,\tau_i = k(k-1)/2\; ; \qquad (2.3)$$

$$\sum_{i=1}^{n} \rho_i = q+1 \; ; \qquad (2.4)$$

$$\sum_{i=2}^{n} (i-1)\rho_i = k-1 \qquad (2.5)$$

$$\sum_{i=1}^{n} \sigma_i = q+1 \qquad (2.6)$$

$$\sum_{i=1}^{n} i\,\sigma_i = k \qquad (2.7)$$

$$i\tau_i = \sum_{p\in\mathcal{H}} \rho_i \qquad (2.8)$$

$$(q+1-i)\tau_i = \sum_{Q\in\pi\backslash\mathcal{H}} \sigma_i \qquad (2.9)$$

Definition 2.5 [4]. Let π be a projective plane of order q and denoted by P and a R are respectively the sets of points and lines of π. Let f be a function from P into the set N of non-negative integers and call the weight of $p\in P$ the value $f(p)$ and the support of f the set of points of the plane have non-zero weight. By using f we can define the function $F: R\to Z^+$ such that for any $r\in R$, $F(r)=\sum_{p\in r} f(P)$. We call $F(r)$ the weight of the line r.

Definition 2.6 [4]. A $(k,n;f)$-arc of the plane π is a subset K of the points of the plane such that:

(i) K is the support of f;
(ii) $k=|K|$;
(iii) $n=max\{F(r): r\in R\}$.

Denote $l_j=|f^{-1}(j)|$ to the number of points having weight j for $j=0,1,\dots,\omega$, where $\omega=max_{p\in P} f(p)$, V_i^j to the number of lines of weight i through a point of weight j and $W=\sum_{j=1}^{\omega} Jl_j=\sum_{p\in P} f(p)$. For a $(k,n;f)$ −arc, t_i the number of lines having weight i for $i=0,1,\dots,n$. We have the following important lemma:

Lemma 2. 7 [9].
(i) $\omega\le n-m$;
(ii) If p is any point of the plane, then $\sum_{r\in[p]} F(r)=W+qf(p)$, where $[p]$ denote the set of lines through p;
(iii) The weight W of a $(k,n;f)$ −arc satisfies $m(q+1)\le W\le (n-\omega)q+n$;
(iv) Let K be a $(k,n;f)$ −arc of type $(m,n), m>0$ and let p be a point having weight s,
then V_m^s and V_n^s are determined of p and are given by:
$$V_m^s=(q(n-s)-W+n)/(n-m)$$
and
$$V_n^s=(q(s-m)+W-m)/(n-m);$$
(v) $q\equiv 0\ mod(n-m)$;
(vi) the characters of $(k,n;f)$ −arc K of type (m,n) are given by
$$t_m=(q+1/n-m)[n\,(q^2+q+1/q+1)-W]$$
and
$$t_n=(q+1/n-m)[W-m(q^2+q+1/q+1)]$$
.

3. $(\boldsymbol{k},\boldsymbol{n};\boldsymbol{f})$ −arcs of type $(\boldsymbol{n}-\boldsymbol{4},\boldsymbol{n})$

For $(k,n;f)$ −arc of type $(n-4,n)$, it is necessary that $n\ge 4$. If $n=4$ we have to consider a $(k,4)$-arc having only 0-secants and 4-secants .

Lemma 3.1. The existence of $(k,n;f)$ −arc of type $(n-4,n)$ with $n\ge 5$ requires

$$q\equiv 0\ (mod\ 4)$$

Proof . Directly, from Lemma 2.7 case (v).

Lemma 3.2 [5]. The existence of a $(k,n;f)$ −arc of type $(n-4,n)$ with $n\ge 5$ requires $l_i=0,\ i\ge 3$.

We used Lemma 2.7 case (iii) to get

$$(n-4)(q+1)\le W\le (n-4)(q+1)+4$$

Lemma 3. 3. For a $(k\,n;f)$ −arc of type $(n-4,n)$ in $PG(2,q)$, $q=2^h$, $h\ge 1$ with W minimal ($W=(n-4)(q+1)$) we have :

$$V_{n-4}^0 = q+1, \quad V_{n-4}^1 = \frac{3q+4}{4},$$

$$V_{n-4}^2 = \frac{q+2}{2}, \; V_n^0 = 0,$$

$$V_n^1 = \frac{q}{4}, \quad V_n^2 = \frac{q}{2}$$

Proof. From Lemma 2.7 case (iv), by substituting $m = n-4$ for $\operatorname{Im} f = \{0,1,2\}$.

Corollary 3. 4. There is no point of weight 0 on n −secants of the $(k,n;f)$ −arc of type $(n-4,n)$.

For the case $l_0 > 0,\ l_1 > 0,\ l_2 > 0,\ l_3 = 0$, we have the maximum weight of the points of the $(k,n;f)$ −arc is $\omega = 2$, and we use the minimal case ($W = (n-4)(q+1)$) and Lemma 2.7 to find the following:

$$t_n + t_{n-4} = q^2 + q + 1$$

By counting the number of n −secants (t_n) and $n-4$ −secants (t_{n-4}). And counting the total incidence we get

$$nt_n + (n-4)t_{n-4} = W(q+1) = (n-4)(q+1)^2$$

Consequently, we get

$$t_n = \frac{1}{4}(n-4) \qquad (3.1)$$

$$t_{n-4} = \frac{1}{4}(4q^2 + 8q - nq + 4) \qquad (3.2)$$

Now, from Corollary 3. 4 there is no points of weight 0 on n-secants. Suppose that on n-secants there are β points of weight 1 and δ points of weight 2. Then counting the points of n −secants lines, it follows that: $\beta + \delta = q + 1$

And counting the weights of points on n −secants , we have

$$\beta + 2\delta = n$$

Solving these two equations, we obtain

$$\beta = 2(q+1) - n \qquad (3.\ 3)$$

$$\delta = n - (q+1) \qquad (3.\ 4)$$

Counting incidences between the points of weight 2 and n −secants, we get

$$l_2 V_n^2 = t_n \delta$$

Making use of the Lemma 3. 3, equation (3. 1) and equation (3. 4) we obtain

$$l_2 = (n-4)(n-q-1)/2 \qquad (3.\ 5)$$

Similarly, counting incidences between the points of weight 1 and n −secants we have

$$l_1 V_n^1 = t_n \beta$$

Hence , by using Lemma 2.4, equation (3. 2) and equation (3. 3) we get

$$l_1 = (n-4)(2q+2-n) \qquad (3.6)$$

From equations (3. 5) and (3. 6), counting the points in the plane

$$l_0 + l_1 + l_2 = q^2 + q + 1$$

$$l_0 = q^2 + q + 1 - (n-4)(2q+2-n) - \frac{(n-4)(n-q-1)}{2}$$

Hence

$$2q^2 + (14-3n)q + n^2 - 7n$$

$$+14 - 2l_0 = 0 \qquad (3.\ 7)$$

The solution of equation (3. 7) exists with respect to q if $(17-3n)^2 - 8(n^2 - 7n + 14 - 2l_0)$ is square, then

$$(n-14)^2 - (112 - 16l_0) = square \qquad (3.8)$$

We discuss the $(k,n;f)$ −arc of type $(n-4,n)$ in $PG(2,8)$ where the points of weight 0 is 7. For the value of $l_0 = 7$, the equation (3.7) becomes

$$2q^2 + (14-3n)q + n^2 - 7n = 0 \qquad (3.\ 9)$$

From the equation (3. 8) , we get

$$(n-14)^2=\mu^2$$

The solution of the equation (3. 9) is either $q=n-7$ or $2q=n$. In $PG(2,8)$we have two solutions for n being non-negative integers with $q\equiv 0\ mod\ (n-m)$ which are $n=16$ or $n=15$.

4. $(k,16;f)$ −arcs of Type (12, 16) in $PG(2,8)$

By Lemma 2.4 cases (i) and (iii) we have:
(i) $0\le\omega\le 4$;
(ii) $108\le W\le 128$.

Lemma 4. 1. For a $(66,16;f)$ −arc of type (12, 16) in $PG(2,8)$ with $W=108$ we have

$$V_{16}^0=0,\quad V_{16}^1=2,\quad V_{16}^2=4V_{12}^0=9,$$

$$V_{12}^1=7,\quad V_{12}^2=5$$

Proof. From Lemma 3. 3 directly, we get the requirements by putting $n=16,\ q=8$.

Corollary 4. 2. There are no points of weight 0 lie on any 16 −secants.

Lemma 4. 3. For the existence of $(66,16;f)$ −arc of type (12, 16) with 7 points of weight zero in $PG(2,8)$ we have :
(i) The number of 16-secants t_{16} is 24;
(ii) The number of 12-secants t_{12} is 49;
(iii) The number of points of weight 2 (l_2) is 42;
(iv) The number of points of weight 1 (l_1) is 24.

Proof. From equations (3. 1), (3. 2), (3. 5) and (3. 6) we obtain (i), (ii), (iii) and (iv) respectively.

Let T_{12} be 12 −secant of $(66,16;f)$ −arc have on it α points of weight 0, β points of weight 1 and δ points of weight 2 then

$$\alpha+\beta+\delta=9$$
$$\beta+2\delta=12$$

So the possibilities of non-negative integers solutions α,β and δ are listed in the table

Type of16-secant	δ	β	α
T_{12}^3	6	0	3
T_{12}^2	5	2	2
T_{12}^1	4	4	1
T_{12}^0	3	6	0

Table (4. 1)

Now, let A_{16} be $16-$ secant of $(66,16;f)-$ arc. Since there are no points 0f weight 0 on a 16 - secants, then we suppose β points of

of weight 1 and δ points of weight 2, then

$$\beta+\delta=9\ ,\ \beta+2\delta=16$$

Thus, $\delta=7$ and $\beta=2$. Hence, we proved the following lemma:

Lemma 4. 4. The lines of $PG(2,8)$ are partitioned into five classes with respect to a minimal $(66.16;f)$ −arc of type (12, 16) as follows:

(i) A_{16}^0 which contains no points of weight 0, 2 points of weight 1 and 7 points of weight 2;
(ii) T_{12}^3 which contains 3points of weight 0, no points of weight 1 and 6 points of weight 2;
(iii) T_{12}^2 which contains 2 points of weight 0, 2 points of weight 1 and 5 points of weight 2;
(iv) T_{12}^1 which contains 1 point of weight 0, 4 points of weight 1 and 4 points of weight 2;
(v) T_{12}^0 which contains no points of weight 0,6 points of weight 1 and 3 points of weight 2.

Corollary 4. 5. There is no point of weight 1 on the 3-secant of (7.3) −arc formed by the points of weight 0.

From the equations (1. 1), (1. 2) and (1. 3), the following equations are obtained:

$$\tau_0+\tau_1+\tau_2+\tau_3=73$$
$$\tau_1+2\tau_2+3\tau_3=63$$
$$\tau_2+3\tau_3=21,$$

where τ_i is the number of i −secants of $(k,3)$ −arc. The possible solutions of these equations are listed in the following table:

Type of	τ_3	τ_2	τ_1	τ_0

$\mathcal{R}_i$				
$\mathcal{R}_7$	7	0	42	24
$\mathcal{R}_5$	5	6	36	26
$\mathcal{R}_4$	4	9	33	27
$\mathcal{R}_3$	3	12	30	28
$\mathcal{R}_2$	2	15	27	29
$\mathcal{R}_1$	1	18	24	30

Table (4. 2)

Where $\mathcal{R}_i$ represent $(7,3)$ −arc having i 3-secasnts.

Let $\mathcal{H}$ be a $(7,3)$ −arc, K be a $(66,16;f)$ −arc of type (12, 16) and $Q \in PG(2,8)\backslash\mathcal{H}$. Suppose that through Q pass S_1 represent the number of 3-secants of $\mathcal{H}$ which are 12-secants of K, S_2 represent the number of 2-secant of $\mathcal{H}$ which are 12-secants of K, S_3 represent the number of 1-secants of $\mathcal{H}$ which are 12- secants of K,S_4 represent the number of 0-secants of $\mathcal{H}$which are 12-secants of $K$and S_5 represent the number of 0-secants of $\mathcal{H}$ which are 16-secants of K. By counting the number of the lines which are pass through a point we get

$$S_1 + S_2 + S_3 + S_4 + S_5 = q + 1 = 9$$

From Lemma 4. 4 we get the following equations:

$$6S_1 + 5S_2 + 4S_3 + 3S_4 + 7S_5 = l_2 \quad (4.\ 2)$$
$$2S_2 + 4S_3 + 6S_4 + 2S_5 = l_1 \quad (4.\ 3)$$
$$3S_1 + 2S_2 + S_3 = l_0 \quad (4.\ 4)$$

Let Q_2 be a points of weight 2 in $PG(2,8)\backslash \mathcal{R}_i$. From Lemma 4.1, the number of the lines of weight 16 through any point of weight 2 is 4 so $S_5 = 4$. Thus, the equations (4. 1), (4. 2), (4. 3) and (4. 4) becomes:

$$S_1 + S_2 + S_3 + S_4 = 5$$
$$5S_1 + 4S_2 + 3S_3 + 2S_4 = 17$$
$$2S_2 + 4S_3 + 6S_4 = 16$$
$$3S_1 + 2S_2 + S_3 = 7$$

The solutions of the above system are given in the following table:

Type of points	S_1	S_2	S_3	S_4	S_5
A_1	2	0	1	2	4
A_2	1	2	0	2	4
A_3	1	1	2	1	4
A_4	0	3	1	1	4
A_5	1	0	4	0	4
A_6	0	2	3	0	4

Table (4. 3)

Let Q_1 be a points of weight 1 in $PG(2,8)\backslash \mathcal{R}_i$. From Lemma 4.1, the number of the lines of weight 16 through any point of weight 1 is 2, so $S_5 = 2$ and from Corollary 4.5 we deduce $S_1 = 0$. Put $S_5 = 2$ and $S_1 = 0$, the equations (4. 1), (4. 2), (4. 3) and (4. 4) becomes:

$$S_2 + S_3 + S_4 = 7$$
$$5S_2 + 4S_3 + 3S_4 = 28$$
$$S_2 + 3S_3 + 5S_4 = 21$$
$$2S_2 + S_3 = 7$$

The solutions of the above system are given in the following table:

Type of point	S_1	S_2	S_3	S_4	S_5
B_1	0	3	1	3	2
B_2	0	2	3	2	2
B_3	0	1	5	1	2
B_4	0	0	7	0	2

Table (4. 4**)**

Let P be a points of weight 0 in $\mathcal{H}$. From Lemma 4.1, The number of the lines of weight 16 through any point of weight 0 is 0, so $S_5 = 0$. Since the 0-secant of $\mathcal{H}$ having no points of the $(k,3)$ −arc, then $S_4 = 0$. Putting $S_5 = 0$ and $S_4 = 0$, the equations (4. 1), (4. 2), (4. 3) and (4. 4) becomes:

$$S_1 + S_2 + S_3 = 9$$
$$6S_1 + 5S_2 + 4S_3 = 42$$
$$2S_2 + 4S_3 = 24$$
$$2S_1 + S_2 = 6$$

The possible solutions of these equations are listed in the following table :

Type of points	S_1	S_2	S_3	S_4	S_5
C_1	0	6	3	0	0
C_2	1	4	4	0	0
C_3	2	2	5	0	0
C_4	3	0	6	0	0

Table (4. 5)

From [15], there are six projectively distinct $(7,3)$ −arc in $PG(2,8)$ which are listed in the following table:

$\mathcal{R}_i$	Distinct $(7,3)$ −arc							τ_3	τ_2	τ_1	τ_0
$\mathcal{R}_7$	P_1	P_2	P_3	P_{53}	P_{37}	P_{38}	P_{73}	7	0	42	24
$\mathcal{R}_5$	P_1	P_2	P_3	P_{53}	P_{37}	P_{28}	P_{73}	5	6	36	26
$\mathcal{R}_4$	P_1	P_2	P_3	P_{53}	P_{37}	P_{24}	P_{73}	4	9	33	27
$\mathcal{R}_3$	P_1	P_2	P_3	P_{53}	P_{37}	P_{24}	P_{73}	3	12	30	28
$\mathcal{R}_2$	P_1	P_2	P_3	P_{13}	P_{37}	P_{24}	P_{73}	2	15	27	29
$\mathcal{R}_1$	P_1	P_2	P_3	P_{53}	P_{13}	P_4	P_{11}	1	18	24	30

Table (4. 6)

Let $\mathcal{R}_7$ be a $(7,3)$ −arc of type (7,0,42, 24) represent a points of weight zero.

$$\mathcal{R}_7 = \{P_1, P_2, P_3, P_{53}, P_{37}, P_{38}, P_{73}\}$$

Lemma 4.6: The points of weight zero of $(66, 16;\ f)$ −arc K are the points of type C_4 with respect to $\mathcal{R}_7$.

Proof. Since the number of 2-secants of $\mathcal{R}_7$ is 0 and 2-secants of $\mathcal{R}_7$ are 12-secants of K which is nominating S_2, so S_2 is 0. Therefore, the points of weight 0 are only points of type C_4 [Table (4. 5)].

Lemma 4.7. The points of weight 2 of $(66,16; f)$ −arc K of type (12, 16) are points of type A_5 when the points of weight zero formed $\mathcal{R}_7$.

Proof. From the Table (4. 2), the number of 2-secants of $\mathcal{R}_7$ is 0 and every 2-secants of $\mathcal{R}_7$ are 12-secants of K. So $S_2 = 0$. Since the number of 0-secants of $\mathcal{R}_7$ from Table (4. 2) is 24 and from Lemma (4. 3) the number of 16-secants is 24 and every lines of weight 16 of K is 0-secants of $\mathcal{R}_7$. Hence there is no line of weight 12 of K is 0-secant of $\mathcal{R}_7$, this implies S_4 equal zero. From the Table (4. 3), the only type of points in which $S_2 = S_4 = 0$ is A_5.

Lemma 4.8. The points of type B_4 represent the points of weight 1 of K where the points of weight 0 $\mathcal{R}_7$.

Proof. From Lemma 4.7 we have $S_2 = S_4 = 0$, so B_4 is the only type of points which represents point of weight 1 [Table (4. 4)].

From Lemmas 4. 3, 4. 4 and Table (4. 2) we deduce the following lemmas:

Lemma 4.9. The points of weight two form $(42, 7)$ −arc of type (24,7, 0, 42, 0, 0, 0, 0).

Lemma 4.10. The points of weight one form $(24, 4)$ −arc of type (42, 0, 24, 0, 7).

Hence, we deduce the following theorem:

Theorem 4.11. There is $(66, 16; f)$ −arc K of type (12, 16) in $PG(2,8)$ when the points of weight 0 form $(7,3)$ −arc $\mathcal{H}$ of type (7, 0, 42, 24) .

Theorem 4.12. There is no $(66, 16; f)$ −arc of type (12, 16) in $PG(2,8)$ when the points of weight zero form $(7,3)$ −arc having five, four , three, two and one 3-secant.

Proof. Suppose the points of weight zero form $\mathcal{R}_5$and $\mathcal{R}_4$. So the number of points of weight 1 equal 32 and 31respectively which contradict Lemma 4.3.
If the points of weight 0 form $\mathcal{R}_3$ and $\mathcal{R}_1$, then there exist 12-secant of K which is 1-secant of $\mathcal{R}_i$ contains 5 points of weight 1. This contradict Lemma 4.4 .
When the points of weight 0 form $\mathcal{R}_2$, there exist 0-secant of $\mathcal{R}_2$ having 4 points of weight 2 and 3 points of weight 1. This contradict Lemma 4. 4.

5.$(k, 15; f)$ −arcs of type(11, 15) in $PG(2,8)$

By Lemma 2.7(i) and (iii) we get the following

(i) $0 \leq \omega \leq 4$

(ii) $99 \leq W \leq 119$

Lemma 5. 1. For a $(66, 15; f)$ −arc of type (11, 15) in $PG(2, 8)$ with $W = 99$ we have

$$V_{15}^{0} = 0, \quad V_{15}^{1} = 2, \quad V_{15}^{2} = 4V_{11}^{0} = 9,$$

$$V_{11}^{1} = 7, \quad V_{11}^{2} = 5$$

Proof: Put $n = 15$ and $q = 8$ in Lemma (3. 3), we obtain solutions of V_{11}^{s} and V_{15}^{s} for $s = \{0, 1, 2\}$.

Corollary 5. 2. There are no points of weight zero lie on any 15-secants of a $(66, 15; f)-$ arc .

Now, we classify the lines of the plane with respect to the $(66.15; f)$ −arc of type (11, 15). Let U_{11} be 11-secant having on it ε points of weight 0, μ points of weight 1 and γ points of weight 2, then $\varepsilon + \mu + \gamma = 9$

And counting the weights of the points on U_{11} , it follows that $\mu + 2\gamma = 11$

Let U_{15} be 15-secant having on it μ points of weight 2 and γ points of weight 2, then

$$\mu + \gamma = 9$$

$$\mu + 2\gamma = 15$$

We summaries the solutions of these equations in the following table :

Type of the lines	Point of weight 0	Point of weight 1	Point of weight 2
U_{11}^{3}	3	1	5
U_{11}^{2}	2	3	4
U_{11}^{1}	1	5	3
U_{11}^{0}	0	7	2
U_{15}	0	3	6

Table(5. 1)

By substituting $q = 8$ and $n = 15$ in equations (3. 1), (3. 2), (3. 5) and (3. 6) we get

$$t_{15} = 22, \quad t_{11} = 51, \quad l_2 = 33 \text{ and } l_1 = 33.$$

Lemma 5. 3. The points of weight zero form $(7, 3)$ −arc of type $(\tau_3, \tau_2, \tau_1, \tau_0)$.

Remark 5.4. Let $P \in \mathcal{H}$ and suppose that through there pass ℓ_3 3-secants , ℓ_2 2-secants, ℓ_1 1-secants, then by using equations (2. 4) and (2. 5) , the following obtained:

$$\ell_3 + \ell_2 + \ell_1 = 9$$

$$\ell_2 + 2\ell_3 = 6$$

The possible solutions of these equations are listed in the following table:

Type of the point	ℓ_3	ℓ_2	ℓ_1
Type 1	3	0	6
Type 2	2	2	5
Type 3	1	4	4
Type 4	0	6	3

Table (5. 2)

Suppose there are A points of type 1, B points of type 2, C points of type 3 and D points of type 4, then by using equation (2. 8) , the following equations are obtained

$$A + B + C + D = k = 7 \quad (5.\ 1)$$
$$3A + 2B + C = 3\tau_3 \quad (5.\ 2)$$
$$2B + 4C + 6D = 2\tau_2 \quad (5.\ 3)$$
$$6A + 5B + 4C + 3D = \tau_1 \quad (5.\ 4)$$

Let $Q \notin \mathcal{H}$, and suppose that through Q there pass ℓ_3 3-secants, ℓ_2 2-secants, ℓ_1 1-secants and ℓ_0 0-secants. Then , by the equations (2. 6) and (2. 7), it follows that:

$$\ell_3 + \ell_2 + \ell_1 + \ell_0 = 9$$
$$\ell_1 + 2\ell_2 + 3\ell_3 = 7$$

From above equations we have eight non-negative integral solutions as follows:

Type of the point	ℓ_3	ℓ_2	ℓ_1	ℓ_0
E_1	2	0	1	6
E_2	1	2	0	6
E_3	1	1	2	5
E_4	1	0	4	4
E_5	0	3	1	5
E_6	0	2	3	4
E_7	0	1	5	3
E_8	0	0	7	2

Table (5. 3)

Lemma 5.5. The points of weight 2 of the $(66, i6; f)$ −arc K are points of type E_i, $i = 1, \dots, 6$ with respect to the (7, 3)-arc $\mathcal{H}$.

Proof. By Lemma 5.1, through a point of weight 2 there pass four 15-secants of K which represent 0-secants of $\mathcal{H}$ and five 11-secants of K which are i-secants of $\mathcal{H}, i = 0, 1, 2, 3$. Hence the number of 0-secants which pass through a point of weight 0 must be at least 4. Then, the type of points of K, which not satisfied the condition above are the points of type E_7, E_8 [Table (5. 3)].

Remark 5.6. Suppose that φ_i be the number of the points of type E_i, $i = 1, \dots, 8$ which represent the points of the plane of order 8 excluding the points of $\mathcal{H}$, then we obtained the following:

$$\sum_{i=1}^{8} \varphi_i = 73 - 7 = 66$$

Making use of equation (2. 9), we obtain

$$2\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 = 6\tau_3 \quad (5.5)$$
$$2\varphi_2 + \varphi_3 + 3\varphi_5 + 2\varphi_6 + \varphi_7 = 7\tau_2 \quad (5.6)$$
$$\varphi_1 + 2\varphi_3 + 4\varphi_4 + \varphi_5 + 3\varphi_6 + 5\varphi_7 + 7\varphi_8 = = 8 \quad (5.7)$$
$$6\varphi_1 + 6\varphi_2 + 5\varphi_3 + 4\varphi_4 + 5\varphi_5 + 4\varphi_6 + 3\varphi_7 + 2\varphi_8 = 9\tau_0 \quad (5.8)$$

From [8], there are six projectively distinct $(7, 3)$ −arcs in $PG(2, 8)$ which are listed in the following table:

$\mathcal{M}_i$	Distinct (7, 3)-arc							τ_3	τ_2	τ_1	τ_0
$\mathcal{M}_7$	P_1	P_2	P_3	P_{53}	P_{37}	P_{38}	P_{73}	7	0	42	24
$\mathcal{M}_5$	P_1	P_2	P_3	P_{53}	P_{37}	P_5	P_{28}	5	6	36	26
$\mathcal{M}_4$	P_1	P_2	P_3	P_{53}	P_{37}	P_5	P_6	4	9	33	27
$\mathcal{M}_3$	P_1	P_2	P_3	P_{53}	P_{37}	P_5	P_{11}	3	12	30	28
$\mathcal{M}_2$	P_1	P_2	P_3	P_{53}	P_{10}	P_4	P_{11}	2	15	27	29
$\mathcal{M}_1$	P_1	P_2	P_3	P_{53}	P_{13}	P_4	P_{11}	1	18	24	30

Table (5. 4)

Where $\mathcal{M}_i$ represent $(7, 3)$ −arc with i 3-secants.

Theorem 5.7. There is no $(66, 15; f)$ −arc $K$of type (11, 15) in $PG(2, 8)$ having Imf ={0, 1, 2}, where the seven points of weight 0 form (7, 3)-arc having 7, 5, and 4 3-secants.

Proof. Suppose the points of weight 0 form $\mathcal{M}_7$, So the number of points of weight 2 equal 35, this contradiction because $l_2 = 33$. And if the seven points of weight 0 form $\mathcal{M}_5$ and $\mathcal{M}_4$, then there exist 11-secant of K which 1-secant of $\mathcal{M}_i$ that contains 6 points of weight1, this lead to contradict the Table (5.1).

From Table (5. 4) the points of $(7, 3)$ −arc of type (3, 12, 30, 28) is the set

$$\mathcal{M}_3 = \{P_1, P_2, P_3, P_5, P_{11}, P_{37}, P_{53}\}$$

Let $P \in \mathcal{M}_3$, since there are 3 2-secants which meet at most in two points of $\mathcal{M}_3$, then $A = 0$. By putting $\tau_3 = 3, \tau_2 = 12, \tau_1 = 30$ and $A = 0$ in the equations(5.1), (5. 2), (5. 3) and (5. 4) then we have the only non-negative integer solution is $B = 3, C = 3$ and $D = 1$ **.** Suppose $Q \notin \mathcal{M}_3$, the three 3-secants are meet in a point of $\mathcal{M}_3$, and every two 2-secants of $\mathcal{M}_3$ are intersects at point $Q \notin \mathcal{M}_3$and not lies on any 3-secant of $\mathcal{M}_3$. Therefore, $|E_1| = |E_2| = 0$.

Putting $\tau_3 = 3,\ \tau_2 = 12,\ \tau_1 = 30,\ \tau_0 = 28$ and $|E_1| = |E_2| = 0$, the equations (5. 5), (5. 6), (5. 7) and (5. 8) becomes

$\varphi_3 + \varphi_4 = 18$

$3\varphi_5 + 2\varphi_6 + \varphi_7 = 84$

$2\varphi_3 + 4\varphi_4 + \varphi_5 + 3\varphi_6 + 5\varphi_7 + 7\varphi_8 = 240$

$5\varphi_3 + 4\varphi_4 + 5\varphi_5 + 4\varphi_6 + 3\varphi_7 + 2\varphi_8 = 252$

By classification the points in the plane of order 8 with respect to $\mathcal{M}_3$, we get the only non- negative integer r solution of the above equations which is:
$\varphi_1 = 0,\ \varphi_2 = 0,\ \varphi_3 = 12, \varphi_4 = 6,\ \varphi_5 = 3,\ \varphi_6 = 21,\ \varphi_7 = 20,\ \varphi_8 = 4.$

Hence we deduce the following theorem:

Theorem 5.8. There is $(66, 15; f)$ −arc K of type (11, 15) in $PG(2, 8)$ having Imf ={0, 1, 2}for which the seven points of weight 0 form $\mathcal{M}_3$.

Now, we discuss the case when points of weight 0 form $\mathcal{M}_2$. From Table (5. 4) the points of $\mathcal{M}_2$ of type (2, 15, 27, 30) is the set

$$\mathcal{M}_2 = \{P_1, P_2, P_3, P_4, P_{10}, P_{11}, P_{53}\}$$

Since there are only two 2-secant which meet in one point of $\mathcal{M}_2$, therefore $A = 0$ and there is only one point of type B [Table (5. 2)]. This mean$B = 1$.
By substituting $\tau_3 = 2, \tau_2 = 15, \tau_1 = 27,\ A = 0$ and $B = 1$in the equations (5.1), (5. 2), (5.3) and (5. 4) we have the only non-negative integral solution is $A = 0,\ B = 1,\ C = 4$ and $D = 2$.

Suppose $Q \notin \mathcal{M}_2$, hence every two 3-secants of $\mathcal{M}_2$ are not intersect at a point of $Q \notin \mathcal{M}_2$, therefore, $\varphi_1 = 0$. But every two 2-secants of $\mathcal{M}_2$ are intersects at a point $Q \notin \mathcal{M}_2$ and not lies on any 3-secants of $\mathcal{M}_2$.Therefore,$\varphi_2 = 0$.
Substituting $\tau_3 = 2,\ \tau_2 = 15,\ \tau_1 = 27,\ \tau_0 = 29$, and $\varphi_1 = \varphi_2 = 0$, the equations (5. 5), (5. 6), (5. 7) and (5. 8) becomes
$\varphi_3 + \varphi_4 = 12$
$\varphi_3 + 3\varphi_5 + 2\varphi_6 + \varphi_7 = 105$
$2\varphi_3 + 4\varphi_4 + \varphi_5 + 3\varphi_6 + 5\varphi_7 + 7\varphi_8 = 216$
$5\varphi_3 + 4\varphi_4 + 5\varphi_5 + 4\varphi_6 + 3\varphi_7 + 2\varphi_8 = 261$

By classification the points in the plane of order 8 with respect to $\mathcal{M}_2$, we get: $\varphi_1 = 0$, $\varphi_2 = 0,\ \varphi_3 = 10, \varphi_4 = 2,\ \varphi_5 = 7,\ \varphi_6 = 29,\ \varphi_7 = 16,\ \varphi_8 = 2.$

Hence we deduce the following theorem:

Theorem 5.9. There is $(66, 15; f)$ −arc K of type (11,15) in $PG(2, 8)$ having Imf $= \{0, 1, 2\}$ for which the seven points of weight 0 form $\mathcal{M}_2$.

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