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Similarity Analysis of Differential Equations By Lie Group Technique

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Abstract

The goal of similarity transformation method is to provide the solution for differential equations by using Lie group technique. However higher order differential equations are going to be reduced to a lower order. For partial differential equation at least a reduction in the number of independent variables is sought and in favorable cases it reduces to ordinary differential equation with special solution [4]. Also we investigated completely the Similarity analysis of 2nd order differential equations.

Key words: Lie group, Similarity transformation method.

I . Introduction

The similarity method (Blumen and Cole 1974) for solving second order ordinary differential equations, based on invariance under continuous Lie group of transformation.[2][3] .The goal of similarity method is the expression of a solution in terms of quadratic in the case of ordinary differential equation of first order and reduction in order for higher order equation. The differential equations describing the realistic situations normally are of non Linear type and corresponding solution are not obtainable easily even by computer oriented methods. Many times, exact solutions of such equations are preferred over numerical one. Quite commonly, exact analytical solution of system of partial differential equations in applied fields like fluid dynamics and general relativity are extremely valuable as they provide more insight into extreme cases which is not possible through numerical treatment [6].

One of the most attractive aspects of Lie's method of symmetries is generality, on another words; all solving method for differential equations can be correlated to particular forms of the symmetry generator. However, for first order ordinary differential equation, Lie's method seems to be, in principle, not as useful as in higher order case [5].

So, first of all, let us introduce the operator U of the infinitesimal transformation as the symbol for a directional derivative in the plane, consider the parameter group of transformation ε.

II <u>Group of Transformations</u>

Consider the set of transformations

$$\left.\begin{aligned} X^* &= X(x,y;\alpha) \\ Y^* &= Y(x,y;\alpha) \end{aligned}\right\} \qquad (1)$$

that depends on a parameter α .

Any particular value of α determines one transformation of the set and each transformation of the set may be looked upon as mapping any point *(x,y)* in the *x y* plane into an image point *(x*, y*).*

For example of such a set of transformations is

$$\left.\begin{aligned} x^* &= x\cos\alpha - y\sin\alpha \\ y^* &= x\sin\alpha + y\cos\alpha \end{aligned}\right\} \qquad (2)$$

These transformations are called rotation because of the position image point *(x*, y*)* can be determined by rotating the radius vector to the source point *(x, y)* it will be counterclockwise through an angle α .

III <u>Infinitesimal Transformation</u>

Let α_0 be value of α , which corresponds to the identical transformation

$$\left.\begin{aligned} X^* &= X(x,y,\alpha_0) = x \\ Y^* &= Y(x,y,\alpha_0) = y \end{aligned}\right\} \qquad (3)$$

The expanding of (1) in a Taylor's series round the point $\alpha = \alpha_0$, found that:

$$x^* = x + \left(\frac{\partial X}{\partial \alpha}\right)_{\alpha=\alpha_0} (\alpha - \alpha_0) + \left(\frac{\partial^2 X}{\partial \alpha^2}\right)_{\alpha=\alpha_0} \frac{(\alpha-\alpha_0)^2}{2} + \cdots$$

$$y^* = y + \left(\frac{\partial Y}{\partial \alpha}\right)_{\alpha=\alpha_0} (\alpha - \alpha_0) + \left(\frac{\partial^2 Y}{\partial \alpha^2}\right)_{\alpha=\alpha_0} \frac{(\alpha-\alpha_0)^2}{2} + \cdots \qquad (4)$$

The partial derivative $\left(\frac{\partial X}{\partial \alpha}\right)_{\alpha=\alpha_0}$ and $\left(\frac{\partial Y}{\partial \alpha}\right)_{\alpha=\alpha_0}$ are continuous function of *x and y* and denote, respectively, by $\zeta(x,y)$ and $\eta(x,y)$, which are called the coefficients of infinitesimal transformations.

For sufficiently small $(\alpha - \alpha_0)$ is which denoted by ε , we can then rewrite the coordinates of the image point *(x*, y*)* as

$$\left.\begin{aligned} X^* &= x + \varsigma(x,y)\varepsilon + O(\varepsilon^2) \\ Y^* &= y + \eta(x,y)\varepsilon + O(\varepsilon^2) \end{aligned}\right\} \qquad (5)$$

This is called infinitesimal transformation, while the infinitesimal generator transformation on (1) is defined by

$$U = \sum_{i=1}^{2} \zeta(x_i)\frac{\partial}{\partial x_i} = \zeta_1 \frac{\partial}{\partial x_1} + \zeta_2 \frac{\partial}{\partial x_2} \qquad (6)$$

IV <u>Invariant ODE</u>

Definition [8]

The one- parameter Lie group of transformations

$x^* = X(x, y, \varepsilon),$

$y^* = Y(x, y, \varepsilon),$

Since the general formula of ordinary differential equation can expressed as

$y^{(n)} = f\left(x, y, y^{(1)}...., y^{(n-1)}\right)$

while

$$y^{(k)} = \frac{d^k y}{dx^k}, k = 1,2,......, n \quad \text{...} (7)$$

Which it is invariant ordinary differential equation if and only if its n^{th} extension transformation infinitesimal is defined by

$x^* = X(x, y; \varepsilon),$

$y^* = Y(x, y; \varepsilon),$

$y^{*(1)} = Y^{(1)}(x, y, y^1; \varepsilon),$

$$y^{*(k)} = Y^{(k)}\left(x, y, y',...y^{(k)}, \varepsilon\right) = \frac{\frac{\partial Y^{(k-1)}}{\partial x} + y^{(1)}\frac{\partial Y^{(K-1)}}{\partial y} + \cdots + y^{(k)}\frac{\partial Y^{(k-1)}}{\partial y^{(k)}}}{\frac{\partial X}{\partial x}(x; \varepsilon) + y^{(1)}\frac{\partial X}{\partial y}(x; \varepsilon)} \quad .... (8)$$

Then equation (7) invariant at $k=n$.

V *Ordinary Differential Equations and Lie Symmetries [1],[7]*

The basic idea for the existence Lie symmetries for a system of ordinary differential equations is

$$\dot{x} = g(x) \qquad x \in C^n \qquad (9)$$

Which it is analytic in the neighborhood of the origin at x=0. The n-derivatives with .respect .to time for equation (9) can be written as

$$\dot{x}^i = g^i(x), \quad i = 1.......n \quad \text{..}(10)$$

using the i^{th} component of x and g.

Since our interest is Lie symmetries around equilibrium point, we set $G(0)=0$.

Without loss of generality, we define Lie symmetries of (9) or (10) as follows:

Let X_g be the vector field accompanying with eq. (9), we get:

$$X_g = \sum_{i=1}^{n} \phi^i(x)\frac{\partial}{\partial x^i} \quad \text{...} \quad (11)$$

Thus the fundamental definition of Lie symmetries that there exist a vector field X_ϕ which can be written in the following form

$$X_\phi = \sum_{i=1}^{n} \phi^i(x)\frac{\partial}{\partial x^i} \quad \text{..}(12)$$

this is called Lie symmetry of the system (9) if the vector field (12) commutes with X_g in the sense of the Lie brackets, this is:

$$\left[X_g, X_\phi\right] = X_g X_\phi - X_\phi X_g = 0 \quad \text{..} (13)$$

In general, one has to set Lie symmetries in the form (12) if

$$X = \varepsilon(x,t)\frac{\partial}{\partial t} + \sum_{i=1}^{n}\phi^i(x,t)\frac{\partial}{\partial x^i} \quad \ldots\ldots (14)$$

VI *Similarity Method for 2nd Order ODE*

We shall provide the necessary details of the similarity method (Blumen and Cole 1974) for solving second order ordinary differential equations, based on invariance under continuous Lie group of transformation. The goal of this method is the expression of a solution in terms of quadratic in the case of ordinary differential equation of first order and reduction in order for higher order equation. For partial differential equation at least a reduction in the number of independent variables is sought and in favorable cases a reduction to ordinary differential equation with special solution. So, first of all, let us introduce the U- symbol of the infinitesimal transformation as the symbol for a directional derivative in the plane, consider the parameter group of transformation ε

$$x_1 = x + \xi(x,y)\ \varepsilon$$
$$y_1 = y + \eta(x,y)\ \varepsilon$$

Where ξ and η are the infinitesimal and the corresponding infinitesimal transformation

$$\left.\begin{aligned} x^* &= x_1 + \xi(x_1,y_1)\ \delta\varepsilon \\ y^* &= y_1 + \eta(x_1,y_1)\ \delta\varepsilon \end{aligned}\right\} \ldots\ldots (15)$$

Now consider a function $f(x^*, y^*)$ defined over the plane varies along the path curve of a given initial point(x, y).

Such that

$$\begin{aligned} \delta f &= f(x^*,y^*) - f(x_1,y_1) \\ &= f(x_1 + \xi(x_1,y_1)\delta\varepsilon, y_1 + \eta(x_1,y_1)\delta\varepsilon) - f(x_1,y_1) \\ &= f(x_1,y_1) + \xi(x_1,y_1)\delta\varepsilon\frac{\partial f}{\partial x_1}(x_1,y_1) + \eta(x_1,y_1)\delta\varepsilon\frac{\partial f}{\partial y_1}(x_1,y_1) - f(x_1,y_1) + o(\varepsilon^2) \\ &\cong \left[\xi(x_1,y_1)\frac{\partial f}{\partial x_1}(x_1,y_1) + \eta(x_1,y_1)\frac{\partial f}{\partial y_1}(x_1,y_1)\right] \ldots\ldots (16) \end{aligned}$$

In particular as $\varepsilon \to 0$; we approach the initial point *(x, y)* so

$$\lim_{\varepsilon\to 0}\frac{\delta f}{\delta\varepsilon} = \frac{\partial f}{\partial\varepsilon} = \xi(x,y)\frac{\partial f}{\partial x} + \eta(x,y)\frac{\partial f}{\partial y} \quad \ldots\ldots (17)$$

We define (17) as an operator *U* act *f* as

$$Uf = \xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} \quad \ldots\ldots (18)$$

where

$$U = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}$$

On expanding at $\varepsilon = 0$ using (15) and (18), we get

$$\begin{aligned} f(x_1,y_1) &= f\left[x + \xi(x,y)\varepsilon, y + \eta(x,y)\varepsilon\right] \\ &= f(x,y) + \xi(x,y)\varepsilon\frac{\partial f}{\partial x} + \eta(x,y)\varepsilon\frac{\partial f}{\partial y} + o(\varepsilon^2) \\ &= f(x,y) + \varepsilon\left[\xi(x,y)\frac{\partial f}{\partial x} + \eta(x,y)\frac{\partial f}{\partial y}\right] + o(\varepsilon^2) \\ &= f(x,y) + \varepsilon Uf(x,y) + o(\varepsilon^2) \ldots\ldots (19) \end{aligned}$$

Thus;

$f(x_1, y_1) + \varepsilon Uf(x, y) + o(\varepsilon^2)$

Evidently, necessary and sufficient condition for invariance is

$UF = 0$ For all (x, y) i.e.

$$\xi(x, y)\frac{\partial f}{\partial x} + \eta(x, y)\frac{\partial f}{\partial y} = 0, \quad \ldots\ldots(20)$$

Now, the corresponding transformation in the derivative is

$$y_1' = \frac{dy_1}{dx_1} = \frac{dy + \varepsilon d\eta}{dx + \varepsilon d\xi} = \frac{\frac{dy}{dx} + \varepsilon\frac{d\eta}{dx}}{1 + \varepsilon\frac{d\xi}{dx}}$$

$$= \left(y' + \varepsilon\frac{d\eta}{dx}\right)\left(1 + \varepsilon\frac{d\xi}{dx}\right)^{-1} \quad where \;\; y' = \frac{dy}{dx}$$

$$= \left(y' + \varepsilon\frac{d\eta}{dx}\right)\left(\left(1 - \varepsilon\frac{d\xi}{dx}\right) + O(\varepsilon)^2\right)$$

$$= y' - \varepsilon y'\frac{d\xi}{dx} + \varepsilon\frac{d\eta}{dx} + 0(\varepsilon)^2$$

Since the coefficient of $O(\varepsilon)^2$ that is contain the term $\left(\varepsilon^2\frac{d\eta}{dx}\frac{d\xi}{dx}\right)$ can be neglect it because it isn't

$$\approx y' + \varepsilon\left[\frac{d\eta}{dx} - y'\frac{d\xi}{dx}\right]$$

$$= y' + \varepsilon\eta(x, y, y') \quad \ldots\ldots \quad (21)$$

Where the changes $(d\eta, d\xi)$ are the changes in (η, ξ) as we travel in the direction y' at the original point (x, y)

Since

$$\frac{d\xi}{dx} = \xi_x + \xi_y y' \quad \text{and}$$

$$\frac{d\eta}{dx} = \eta_x + \eta_y y' \quad \ldots\ldots \; . \; (22)$$

Therefore,

$$\eta = \frac{d\eta}{dx} - y'\frac{d\xi}{dx} = \eta_x + y'[\eta_y - \xi_x] - \xi_y y'^2 \quad \ldots\ldots \quad \ldots (23)$$

Equation (22) with (23) is infinitesimal transformation in (x, y, y'), we can thus write the infinitesimal operator U' which is the extension of U to the (x ,y ,y') spaces as

$$U'F = \xi(x, y)\frac{\partial F}{dx} + \eta(x, y)\frac{\partial F}{\partial y} + \eta(x, y, y')\frac{\partial F}{dy'} \quad \ldots\ldots (24)$$

So, to work out the criterion for

$H = y' - w(x, y) = 0$, where $y' = \frac{dy}{dx}$

By invariance criterion (24)

$$U'H = \xi \frac{\partial H}{dx} + \eta \frac{\partial H}{\partial y} + \eta \frac{\partial H}{dy'} = 0 \quad \text{...} \quad (25)$$

Here

H (x, y, y') =y' –w(x, y)

$$\frac{\partial H}{\partial x} = -w_x$$

$$\frac{\partial H}{\partial y} = -w_y$$

$$\frac{\partial H}{\partial y'} = 1$$

$$\eta = \eta_x + y'(\eta_y - \xi_x) - \xi_y y'^2 \quad \text{...} \quad (26)$$

From the given equation

$y' = w(x, y)$

So $\eta' = \eta_x + w(\eta_y - \xi_x) - \xi_y w'^2$ Using (25) with (26), we get

$$U'H = \xi(-w_x) + \eta(-w_y) + \left[\eta_x + w'(\eta_y - \xi_x) - \xi_w w'^2\right] = 0$$

$$\eta_x + w(\eta_y - \xi_x) - \xi_y w'^2 = \xi w_x + \eta w_y$$

This is required condition for finding the ordinary deformational equation, admits a group. In the same way and in order to find the criterion that a second order deformational equation admits a group: from (26) with (22)

$$\eta = \frac{d}{dx}\eta(x, y) - y'\frac{d}{dx}\xi(x, y)$$

$$= \eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2 \quad \text{...} \quad (27)$$

So, the derivatives being taken along the curve y(x) whose slope y' is

$$U''F = \xi(x, y)\frac{\partial F}{\partial x} + \eta(x, y)\frac{\partial F}{\partial y} + \eta'(x, y, y')\frac{\partial F}{\partial y'} + \eta''(x, y, y', y'')\frac{\partial F}{\partial y''} \text{...} \quad (28)$$

For F(x, y, y', y") where

$$\eta'' = \frac{d\eta'}{dx}(x, y, y') - y''\frac{d\xi}{dx}(x, y) \quad \text{...} \quad (29)$$

Then we can write the derivative of (28) as follows:

$$y_1'' = \frac{dy'}{dx} = \frac{dy' + \varepsilon\, d\,\eta'}{dx + \varepsilon\, d\xi}$$

$$= y'' + \varepsilon\left[\frac{d\eta'}{dx} - y''\frac{d\xi}{dx}\right]$$

$$= y'' + \varepsilon\eta''(x, y, y', y'') \quad \text{...} \quad (30)$$

when η'' is given by (29), equation (30) with help of (28) takes the form:

$$\eta'' = \left[\eta_{xx} + \eta_{xy}y' + (\eta_{yx} - \xi_{xx})y' + (\eta_{yy} - \xi_{yx})y'^2 + (\eta_y - \xi_x)y'' - \xi_{yx}y'^2 - \xi_{yy}y'^3 - 2\xi_y y' y''\right] - y''\left[\xi_x + \xi_y y'\right]$$

$$= \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - \xi_{yy}y'^3 + (\eta_y - 2\xi_x)y'' - 3\xi_y y' y''$$

η'' is linear in y''

Now a second order differential equation

$H(x,y,y',y'')=y''-w(x,y,y')=0$ …………………………………………… (31)

Where $y'=\dfrac{dy}{dx}$ & $y''=\dfrac{d^2y}{dx^2}$

Admits all the transformations of one parameter group

$$Uf=\xi(x,y)\frac{\partial f}{\partial x}+\eta(x,y)\frac{\partial f}{\partial y} \qquad (32)$$

When

$$U''H=\xi\frac{\partial H}{\partial x}+\eta\frac{\partial H}{\partial y}+\eta'\frac{\partial H}{\partial y'}+\eta''\frac{\partial H}{\partial y''}=0, \qquad (33)$$

Differentiating (31) partially, we get

$H(x, y, y', y'') = y''-w(x, y, y') = 0$

$$\frac{\partial H}{\partial x}=-w_x(x,y,y')$$

$$\frac{\partial H}{\partial y}=-w_y(x,y,y')$$

$$\frac{\partial H}{\partial y'}=-w_{y'}(x,y,y')$$

$$\frac{\partial H}{\partial y''}=1 \qquad (34)$$

Now by the use of (33) and $U''H=0$, we get

$$-\xi\frac{\partial w}{\partial x}-\eta\frac{\partial w}{\partial y}-\left[\eta_x+(\eta_y-\xi_x)y'-\xi_y y'^2\right]\frac{\partial w}{\partial y'}+\eta_{xx} \qquad (35)$$

$$+(2\eta_{xy}-\xi_{xx})y'+(\eta_{xx}-2\xi_{xy})y'^2-\xi_{yy}y'^3+\left[\eta_y-2\xi_x-3\xi_y y'\right]y''=0$$

Since $y''=w(x,y,y')=0$ ……………………………………………… (36)

Therefore, we get

$$-\xi w_x-\eta w_y-[\eta_x+(\eta_y-\xi_x)y'-\xi_y y'^2]w_{y'}+\eta_{xx}+(2\eta_{xy}-\xi_{xx})y'$$

$$+(\eta_{xx}-2\xi_{xy})y'^2-\xi_{yy}y'^3+[[\eta_y-2]\xi_x-3\xi_y y']w(x,y,y')=0 \qquad (37)$$

For all x, y, y' which is the required criterion.

So, to work the criterion for $y''=0$

We can write

$$H(x,y,y',y'')=y''=0 \qquad (38)$$

Then by using Lagrange system we can put it as

$$\frac{\partial w}{\partial x}=\frac{\partial w}{\partial y}=\frac{\partial w}{\partial y'}=0$$

Now, by putting $U''H=0$ and using (37), we get

$$\eta_{xx}+(2\eta_{xy}-\xi_{xx})y'+(\eta_{xx}-2\xi_{xy})y'^2-\xi_{yy}y'^3=0 \qquad (39)$$

Therefore, for invariant, we have

$$\eta_{xx}=0 \qquad (40)$$

$$2\eta_{xy}-\xi_{xx}=0 \qquad (41)$$

$$\eta_{xx}-2\xi_{xy}=0 \qquad (42)$$

$$\xi_{yy}=0 \qquad (43)$$

From (40), we get

$\eta(x,y) = b_0(y)x + b_1(y)$ ………………………………………………… (44)

From (41), we get

$\xi(x,y) = a_0(x)y + a_1(x)$ ……………………………………………….. (45)

Now, using (41) in (45), we get

$b_0{}'(y) = a_0{}''(x)y + a_1{}''(x)$ ………………………………………………… (46)

and from (41), we get

$b_0{}''(y)x + b_1{}''(y) = 2a_0{}'(x)$ ……………………………………………….. (47)

From (46) and (47), we have

$b_0{}''(y) = a_0{}''(x) = 0$ ………………………………………………………… (48)

Therefore the solutions are

$a_0(x)=c_1x+c_2$

$a_1(x)=c_3x^2+c_5\ x+c_6$

$b_0\ (y)=c_3y+c_4$

$b_1(y)=c_7+c_8y$ ……………………………………………………….. (49)

Where c_i are constant (i=1,…,8)

Putting in (44) and (45), we get

$\xi(x,y) = c_1xy + c_2y + c_3x^2 + c_5x + c_6$

$\eta(x,y) = c_3xy + c_4x + c_7y + c_8y$

which are required conditions

In order to find out (ξ,η) coefficients of different powers of y' are set to zero, further in the resulting expressions the functions of x which are coefficients of certain power of y are also set to zero. And the equation which is mentioned above can be reduced to a first order equation by finding out two invariant of the group (u, v) which are found from solving the characteristic differential equation[4].

$$\frac{dx}{\xi(x,y)} = \frac{dy}{\eta(x,y)} = \frac{dy'}{\eta'(x,y,y')} \quad \text{……………………………………………} (50)$$

as

u(x,y)= a And

v(x,y,y') = b

Then, the second order differential equation can be expressed as:

$$\frac{dv}{du} = \varphi(u,v)$$

$$= \frac{v_x + v_y y' + v_{y'} y''}{u_x + u_y y'} \quad \text{…………………………………………………} (51)$$

So, the above consideration gives the general form of the second order differential equation which admits a given groups.

It is also shows how the second order differential equation is reduced to a first order equations, this first order differential equation must be integrated to find the solutions.

VII ***Invariance [7]***

The differential equation

$$\frac{dy}{dx} = f(x,y)$$

is said to be invariant under the transformation

$$\left.\begin{aligned} X^* &= \alpha\ x \\ Y^* &= \beta\ y \end{aligned}\right\} \ 0 < \alpha, \beta < \infty$$

Where α , β are the parameters of the transformation above when the differential equation reads the same in the new coordinates.
For example

$$U\Omega = -y\frac{\partial \Omega}{\partial x} + x\frac{\partial \Omega}{\partial y} + m\frac{\partial \Omega}{\partial z} = 0\,,$$

two independent invariants are

$$u = \sqrt{x^2 + y^2} \qquad ,v = \theta - \frac{z}{m} \qquad \text{Where } \theta = \tan^{-1}\frac{y}{x}$$

The general invariant function under this group will be

$$\Omega(x, y, z) = f\left[\sqrt{x^2 + y^2}, \quad \theta - \frac{z}{m}\right]$$

i **<u>Example</u>** Consider the infinitesimal generator $UF(x, y) = y\frac{\partial F(x, y)}{\partial x}$ by using (4) and (5), we can see that

$$\xi = y, \eta = 0$$

$$x^* = x + \varepsilon$$

$$y^* = y$$

$$Ux = y$$

$$U^2 x = U(Ux) = Uy = 0$$

$$Uy = 0$$

$$f(x, y) = y - c = 0 = f(x^*, y^*) = y^* - c = 0$$

Then the curve *f(x,y)* is invariant.
ii **<u>Example</u>**: Rotation Group

in this case $\xi = -y, \eta = x$

$$\eta = 1 + y'^2$$

$$\because U' = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + \left(1 + y'^2\right)\frac{\partial}{\partial y'}$$

Now since we have $H(x, y, y') = 0$ then we can write it as Lagrange system as follows

$$\frac{dx}{-y} = \frac{dy}{x} = \frac{dy'}{1 + y'^2}$$

we can find from 1st and 2nd ratios the 1st solution as

$$\frac{dx}{-y} = \frac{dy}{x} \Rightarrow xdx + ydy = 0$$

$$\Rightarrow x^2 + y^2 = a^2$$

Hence a^2 is arbitrary constant and represents the 1st solution.
Then by taking 2nd and 3rd ratios, we get

$$\frac{dy}{\sqrt{a^2-y^2}}=\frac{dy'}{1+y'^2},$$

Whose integral is

$$\sin^{-1}\frac{y}{a}-\tan^{-1}y'=\alpha \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(52)$$

α represents the 2nd solution

Since the value of $\sin^{-1}(x)$ is equivalent to the value of $\tan^{-1}(x)$ $\quad as\ x\to 0$, then equation (52) becomes;

$$\tan^{-1}\frac{y}{a}-\tan^{-1}y'=\alpha$$

$$\tan[\tan^{-1}\frac{y}{x}-\tan^{-1}y']=\tan\alpha$$

$$\frac{\frac{y}{x}-y'}{1+\frac{y}{x}y'}=\tan\alpha$$

$$\frac{xy'-y}{yy'+x}=\text{constant}$$

s.t $\tan^{-1}\alpha$ =constant

Hence we can conclude the most general differential equation which is invariant under this group as

$$\frac{xy'-y}{yy'+x}=f\left(x^2+y^2\right)$$

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المستخلص
الهدف من طريقة التحويلات المتماثلة هو لأيجاد الحل للمعادلات التفاضلية بأستخدام طريقة زمر لي.
المعادلات ذات الرتب العليا سوف يتم تخفيض رتبتها.بخصوص المعادلات التفاضلية الجزئية على الأقل تقليص عدد المتغيرات المستقلة تم بحثها بحالات ناجحة ثم تحويلها الى معادلات تفاضلية أعتيادية بحلول خاصة.كذلك تم فحص وبصورة كاملة التحليلات المتماثلة للمعادلات التفاضلية.

الكلمات المفتاحية : زمر لي ، طريقة التحويلات المتماثلة