SOLUTION OF SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

Ali Hassan Mohammed **Mohammed Monther Neamah**

Professor Department of Mathematics
Department of Mathematics Faculty of Mathematics and Computer Science
Faculty of Education for Girls, Kufa University Kufa University

Abstract

Our aim in this paper is to solve some special types of second order partial differential equations with variable coefficients and the general form:

$$F_1(x,y)Z_{xx} + F_2(x,y)Z_{xy} + F_3(x,y)Z_{yy} + F_4(x,y)Z_x + F_5(x,y)Z_y + F_6(x,y)Z = 0$$

Such that $F_1(x,y)$,$F_2(x,y)$, $F_3(x,y)$, $F_4(x,y)$, $F_5(x,y)$ and $F_6(x,y)$ are functions of x and y,

by using the assumption

$Z(x,y) = e^{\int xU(x)dx+\int V(y)dy}$. This assumption will transforms the second order partial differential equations to first order linear ordinary differential equations with two independent functions $U(x)$ and $V(y)$.

I.Introduction

The differential equations play an important role in plenty of the fields of the sciences as Physics, Chemistry and other sciences, and therefore the plenty of the scientists were studying this subject and they are trying to find modern methods for getting rid up the difficulties that facing them in the solving of some of these equations.

The researcher Kudaer [4],2006 studied the linear second order ordinary differential equations, which have the form

$y'' + P(x)y' + Q(x)y = 0$

and she used the assumption $y = e^{\int Z(x)dx}$ to find the general solution of it , and the solution depends on the forms of $P(x)$ and $Q(x)$.

The researcher Abd Al-Sada [1], 2006 studied the second order linear (P.D.Es) with constant coefficients and which have the form

$A_1Z_{xx} + A_2Z_{xy} + A_3Z_{yy} + A_4Z_x + A_5Z_y + A_6Z = 0$

, where A_i are constants. (i=1,2,…..,6),and she used the assumption

$Z(x,y) = e^{\int U(x)dx+\int V(y)dy}$ to find the complete solution of it, and the solution depended on the values of A_i .

The researcher Hanon [5] , 2009, studied the second order linear (P.D.Es) , with variable coefficients which have the form

$A(x,y)Z_{xx} + B(x,y)Z_{xy} + C(x,y)Z_{yy} + D(x,y)Z_x + E(x,y)Z_y + F(x,y)Z = 0$

, where some of $A(x,y), B(x,y), C(x,y), D(x,y), E(x,y)$ and $F(x,y)$.

Are functions of x only or y only or both x and y . To solve this kind of equations , she used the assumptions

$Z(x,y) = e^{\int \frac{U(x)}{x}dx + \int V(y)dy}$,

$Z(x,y) = e^{\int U(x)dx + \int \frac{V(y)}{y}dy}$ and

$Z(x,y) = e^{\int \frac{U(x)}{x}dx + \int \frac{V(y)}{y}dy}$

These assumptions represent the complete solution of the above equation and the solution depends on the forms of $A(x,y), B(x,y), C(x,y), D(x,y), E(x,y)$ and $F(x,y)$.

These ideas made us to search functions $U(x)$ and $V(y)$, that give the complete solution of the second order linear partial differential equations with variable coefficients, which have the form

$$F_1(x,y)Z_{xx} + F_2(x,y)Z_{xy} + F_3(x,y)Z_{yy} + F_4(x,y)Z_x + F_5(x,y)Z_y + F_6(x,y)Z = 0$$

and this solution depends on the forms of the functions $F_1(x,y)$,$F_2(x,y)$, $F_3(x,y)$, $F_4(x,y)$, $F_5(x,y)$ and $F_6(x,y)$.

1. Basic definitions

1.1Definition[3]:A partial differential equation(abbreviated P. D. Es) is an equation involving one or more partial derivatives of an unknown function of several variables.

1.2 Definition[3]: A linear partial differential equation of second order in the independent variables x and y is an equation of the form

$$F_1(x,y)Z_{xx} + F_2(x,y)Z_{xy} + F_3(x,y)Z_{yy} + F_4(x,y)Z_x + F_5(x,y)Z_y + F_6(x,y)Z = 0 \quad \dots(1.1)$$

Such that $F_1(x,y)$,$F_2(x,y)$, $F_3(x,y)$, $F_4(x,y)$, $F_5(x,y)$ and $F_6(x,y)$ are functions of x and y.

2. Finding the complete solution

Note: since the equation(1.1) has infinite cases we will choose some types of it.

To solve the equation (1.1), the assumption $Z(x,y) = e^{\int xU(x)dx + \int V(y)dy}$ help us to find the complete solutions of the following kinds:

Kind(1)

1) $A\frac{Z_x}{x} + B\frac{Z_{xy}}{x} + DZ_{yy} = 0 \quad ; \quad x \neq 0$

2) $A\frac{Z_x}{x} + BZ_y + DZ_{yy} + E\frac{Z_{xy}}{x} + FZ = 0 \,; x \neq 0$

3) $Ay^nZ_{xx} + BZ_y + DZ = 0$; n is any real number

Such that A, B, D, E, and F are constants.

Kind(2)

1) $AZ_x + B\,x\sin(x)Z_{yy} = 0$

2) $AZ_{xy} + B\,x\cos(x)Z_{yy} = 0$

3) $A\cosh(y)Z_{xx} + BZ_y = 0$

Such that A and B are constants.

In kind (1), we search functions $U(x)$ and $V(y)$ such that the assumption

$$Z(x,y) = e^{\int xU(x)dx + \int V(y)dy} \quad \dots(1.2)$$

gives the complete solution of the above (P.D.Es.). By finding Z_x,Z_y, Z_{xx}, Z_{yy} and Z_{xy} from (1.2), we get

$Z_x = xU(x)e^{\int xU(x)dx + \int V(y)dy}$

$Z_y = V(y)e^{\int xU(x)dx + \int V(y)dy}$

$Z_{xx} = \left(x^2U^2(x) + xU'(x) + U(x)\right)e^{\int xU(x)dx + \int V(y)dy}$

$Z_{yy} = (V^2(y) + V'(y))e^{\int xU(x)dx + \int V(y)dy}$

$Z_{xy} = xU(x)V(y)e^{\int xU(x)dx + \int V(y)dy}$

By substituting Z_x, Z_y, Z_{xx}, $Z_{yy}$and Z_{xy} in the equations of kind(1) we get

1) $AU(x) + BU(x)V(y) + D(V^2(y) + V'(y) = 0$

2) $AU(x) + BV(y) + DU(x)V(y) + E(V^2(y) + V'(y)) + F = 0$

3) $Ay^n\left(x^2U^2(x)+xU'(x)+U(x)\right)+BV(y)+D=0$; n is any real number.

Respectively the last equations are of the first order(O. D. E.) and contain two independent functions $U(x)$ and $V(y)$.

Solution (1):

Since

$$AU(x)+BU(x)V(y)+D(V^2(y)+V'(y)=0$$

$$\Longrightarrow -U(x)=\frac{D(V^2(y)+V'(y))}{A+BV(y)}=-\lambda$$

$\Longrightarrow U(x)=\lambda$ and

$V'(y)+V^2(y)+A_1V(y)+A_2=0$... (1.3)

Such that $A_1=\frac{\lambda B}{D}$ and $A_2=\frac{\lambda A}{D}$

The equation (1.3) is similar to Riccati's equation [7].

Therefore

i) If $A_2\neq\frac{A_1^2}{4}$, we get $\frac{dV}{(V+\frac{A_1}{2})^2+b^2}+dy=0$;

$b^2=A_2-\frac{A_1^2}{4}$

$\Longrightarrow V(y)=b\tan(f-by)-\frac{A_1}{2}$; f is constant.

Then the complete solution is given by:

$$Z=e^{\frac{\lambda x^2}{2}-\frac{\lambda B}{2D}y}\left[d_1\cos\sqrt{\frac{\lambda A}{D}-\frac{\lambda^2B^2}{4D^2}}\,y+d_2\sin\sqrt{\frac{\lambda A}{D}-\frac{\lambda^2B^2}{4D^2}}\,y\right]$$

where k, $d_1=k\cos f$, $d_2=k\sin f$ and λ are arbitrary constants

ii) If $A_2=\frac{A_1^2}{4}$, we get $\frac{dV}{(V+\frac{A_1}{2})^2}+dy=0$

$\Longrightarrow V(y)=\frac{1}{y-c_1}-\frac{A_1}{2}$, then the complete solution is given by:

$$Z=Ke^{\frac{\lambda x^2}{2}-\frac{\lambda B}{2D}y}(y-c_1)$$

where K, c_1 and λ are arbitrary constants.

Solution (2):

since

$$AU(x)+BV(y)+DU(x)V(y)+E(V^2(y)+V'(y))+F=0$$

$$\Longrightarrow -U(x)=\frac{BV(y)+E\left(V^2(y)+V'(y)\right)+F}{A+DV(y)}=-\lambda$$

$\Longrightarrow U(x)=\lambda$ and

$V'(y)+V^2(y)+A_1V(y)+A_2=0$ ... (1.4)

Such that $A_1=\frac{B+\lambda D}{E}$ and $A_2=\frac{F+\lambda A}{E}$

The equation (1.4) is similar to Riccati's equation [9].

i) If $A_2\neq\frac{A_1^2}{4}$, we get $\frac{dV}{(V+\frac{A_1}{2})^2+b^2}+dy=0$;

$b^2=A_2-\frac{A_1^2}{4}$

$\Longrightarrow V(y)=b\tan(f-by)-\frac{A_1}{2}$; f is constant.

Then the complete solution is given by:

$$Z=e^{\frac{\lambda x^2}{2}-\frac{B+\lambda D}{2E}y}\left[d_1\cos\sqrt{\frac{F+\lambda A}{E}-\left(\frac{B+\lambda D}{2E}\right)^2}\,y+d_2\sin\sqrt{\frac{F+\lambda A}{E}-\left(\frac{B+\lambda D}{2E}\right)^2}\,y\right]$$

,

where k, $d_1=k\cos f$, $d_2=k\sin f$ and λ are arbitrary constants

ii) If $A_2=\frac{A_1^2}{4}$, we get $\frac{dV}{(V+\frac{A_1}{2})^2}+dy=0$

$\Longrightarrow V(y)=\frac{1}{y-c_2}-\frac{A_1}{2}$, then the complete solution is given by:

$$Z=Ke^{\frac{\lambda x^2}{2}-\frac{B+\lambda D}{2E}y}(y-c_2),$$

where K, c_2 and λ are arbitrary constants.

Solution(3): Since

$$Ay^n\left(x^2U^2(x)+xU'+U(x)\right)+BV(y)+D=0$$

$$\Rightarrow -\left(x^2U^2(x)+xU'(x)+U(x)\right)=\frac{BV(y)+D}{Ay^n}=-\lambda^2$$

Then $V(y)=\frac{-(A\lambda^2y^n+D)}{B}$ and

$U'(x)+xU^2(x)+\frac{U(x)}{x}-\frac{\lambda^2}{x}=0$ ...(1.5)

The equation (1.5) is similar to Riccati's equation [7].

Now $U_1 = \frac{\lambda}{x}$ is a particular solution for equation (1.5)

Let $U = U_1 + \frac{1}{t} \Rightarrow U = \frac{\lambda}{x} + \frac{1}{t}$

By substituting U and U' in (1.5), we get

$\frac{dt}{dx} - \left(2\lambda + \frac{1}{x}\right) t = x$...(1.6)

The equation (1.6) is Linear ordinary differential equation [6].

And the general solution for it is given by:

$t = b\, x e^{2\lambda x} - \frac{x}{2\lambda}$; b is constant.

Therefor $U(x) = \frac{\lambda}{x} + \frac{2\lambda}{x(2b\lambda e^{2\lambda x} - 1)}$

If $n \neq -1$, then the complete solution is given by:

$Z = Ke^{\lambda x - \frac{A\lambda^2 y^{n+1}}{n+1} - \frac{D}{B}y}(2b\lambda - e^{-2\lambda x})$,

where K and λ are arbitrary constants.

If $n = 1$, then the complete solution is given by:

$Z = \frac{Ke^{\lambda x - \frac{D}{B}y}}{y^{\frac{A\lambda^2}{B}}}(2b\lambda - e^{-2\lambda x})$,

where K, λ and b are arbitrary constants.

By using assumption (1.2) the equations of kind(2) become

1) $A\frac{U(x)}{\sin(x)} + B\left(V^2(y) + V'(y)\right) = 0$

2) $A\frac{U(x)V(y)}{\cos(x)} + B\left(V^2(y) + V'(y)\right) = 0$

3) $A\cosh(y)(x^2U^2(x) + xU'(x) + U(x)) + BV(y) = 0$

Respectively the last equations are of the first order (O. D. E.) and contain two independent functions $U(x)$ and $V(y)$.

Solution(1): since $A\frac{U(x)}{\sin(x)} + B\left(V^2(y) + V'(y)\right) = 0$

; $x \neq 0, \mp\pi, \mp 2\pi, ..$

$\Rightarrow -\frac{U(x)}{\sin x} = \frac{B}{A}\left(V^2(y) + V'(y)\right) = -\lambda^2$

Therefor $U(x) = \lambda^2 \sin x$ and

$V^2(y) + V'(y) + b^2 = 0$ ...(1.7)

such that $b^2 = \frac{\lambda^2 A}{B}$.

The equation (1.7) is variable separable equation[3].

$\Rightarrow \frac{dV}{V^2+b^2} + dy = 0 \Rightarrow V(y) = b\tan(f - by)$

;f is constant.

Then the complete solution is given by:

$Z = e^{\lambda^2(\sin(x) - x\cos(x))}[d_1\cos\lambda y + d_2 \sin\lambda y]$

where $d_1 = k\cos f$, $d_2 = k\sin f$ and λ are arbitrary constants.

Solution(2): since $A\frac{U(x)V(y)}{\cos(x)} + B\left(V^2(y) + V'(y)\right) = 0$

$\Rightarrow -\frac{U(x)}{\cos(x)} = \frac{B\left(V^2(y) + V'(y)\right)}{AV(y)} = -\lambda^2$;

$x \neq \mp\frac{\pi}{2}, \mp\frac{3\pi}{2}, \mp\frac{5\pi}{2}, \ldots$

Therefor $U(x) = \lambda^2\cos(x)$ and

$V^2(y) + V'(y) + bV(y) = 0$ ...(1.8)

such that $b^2 = \frac{\lambda^2 A}{B}$.

The equation(1.8) is similar to Bernoulli equation[5].

Then $V(y) = \frac{e^{-by}}{\int e^{-by}dy}$, [5]

Hence the complete solution is given by:

$Z = e^{\lambda^2(\cos(x) + x\sin(x))}\left(d_1 e^{-by} + d_2\right)$

where $d_1 = \frac{-k}{b}$, $d_2 = \frac{-km}{b}$, λ, k and m are arbitrary constants.

Solution (3):

since

$A\cosh(y)\,(x^2U^2(x) + xU'(x) + U(x)) + BV(y) = 0$

$\Rightarrow -(x^2U^2(x) + xU'(x) + U(x)) = \frac{BV(y)}{A\cosh(y)} = -\lambda^2$

Then $V(y) = \frac{-A\lambda^2}{B}\cosh(y)$

and $U'(x) + xU^2(x) + \frac{U(x)}{x} - \frac{\lambda^2}{x} = 0$...(1.9)

The equation (1.9) is similar to equation (1.5).

Then the complete solution is given by:

$$Z = Ke^{\lambda x - \frac{A\lambda^2}{B}\sinh(y)}(2b\lambda - e^{-2\lambda x})$$

where K, b and λ are arbitrary constants.

3. conclusion

We found a substitution to solve special kind of partial differential equations as follows :

$Z(x, y) = e^{\int xU(x)dx + \int V(y)dy}$,

this a substitution help us to find solution of this kind of partial differential equations quickly with steps less than known methods.

Reference:

[1] **Abd Al-Sada, N.Z,** "The complete solution of linear second order partial differential equations", M.sc, thesis, University of Kufa, college of Education for Girls, Department of Mathematics, **(2006).**

[2] **A.D.Polyanin, V.F.Zaitsev, and A.Moussiaux,** Handbook of "First Order partial differential equations", Taylor &Francis,London, **(2002).**

[3]**Duchateau,P.,ZACHMANN, D.W.,**"Theory and problems of partial differential equations", Schaum's outline series, Mcgraw-Hill book company, **(1988).**

[4] **Kudaer, R.A.,** "Solving some kinds linear second order Nonhomogeneous differential equations with variable coefficients"
M.sc, thesis, University of Kufa, college of Education for Girls, Department of Mathematics,**(2006).**

[5] **Hanon W.H.,** "On solutions of partial differential equations and their physical applications" M. Sc. thesis, University of AL-Kufa, College of Education for Girls, Department of Mathematics, **(2009).**

[6] **Williams, R.E.** , "Introduction to differential equations and dynamical system", McGraw-Hill Companies, Inc., New York, **(2001).**

[7] **Murphy, G. M.** , "Ordinary differential equations and their solutions", D. Van Nostr and company, Inc. , New York, **(1960).**

"حل المعادلات التفاضلية الجزئية من الرتبة الثانية ذات المعاملات المتغيرة"

أ. علي حسن محمد
جامعة الكوفة
كلية التربية للبنات / قسم الرياضيات
محمد منذر نعمة
جامعة الكوفة
كلية الرياضيات وعلوم الحاسبات/ قسم الرياضيات

المستخلص

الهدف من هذا البحث هو حل المعادلات التفاضلية الجزئية من الرتبة الثانية ذات المعاملات المتغيرة التي صيغتها العامة :

$$F_1(x,y)Z_{xx} + F_2(x,y)Z_{xy} + F_3(x,y)Z_{yy} + F_4(x,y)Z_x + F_5(x,y)Z_y + F_6(x,y)Z = 0$$

حيث ان

$F_5(x,y), F_4(x,y) F_3(x,y), F_2(x,y), F_1(x,y)$ و $F_6(x,y)$ دوال بالمتغيرين x و y.

باستخدام التعويض $Z(x,y) = e^{\int xU(x)dx + \int V(y)dy}$ الذي يحول المعادلات التفاضلية الجزئية من الرتبة الثانية الى معادلات تفاضلية اعتيادية خطية من الرتبة الاولى بالدالتين المستقلتين $U(x)$ و $V(y)$.