(k, n, f) −arcs of Type $(n-11, n)$ and monoidal (k, n, f) −arc in $PG(2, 11)$

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Abstract. The purpose of this work is to construct an examples of (k, n, f) −arcs of type $(n-11, n)$ when the points of weight zero are greater than or equal seventy. We constructed a monoidal(k, n, f) −arc when the points of weight zero form a $(110, 11)$ −arc of type $(10, 121, 2)$, also we constructed a maximum$(100, 10)$ −arc of type $(30, 100, 3)$ in $PG(2, 11)$. Finally, we proved that there are(k, n, f) −arcs of type $(n-11, n)$ when the points of weight one lies on only one n −weighting line of (k, n, f) −arcsand the points of weight two lies on exactly two n −weighting lines of (k, n, f) −arcs.

Key words: Projective plane of order q, (k, n, f) −arcs of two types.

1. Introduction

The notion of(k, n, f)- arcs were proposed by Scafati[13] in 1971. In 1979, Barnabei[4] studied these types of arcs and obtained some particular results about the existence and non-existence of these arcs in $PG(2, q)$ by using a computer program.D'Agostini [5] in 1979

studied caps with weighted points in$PG(2, q)$, some relations between parameters of$(k, n\ f)$-caps and characters were found. In particular,

D'Agostini studied a weighted (k, n) −arcs of type $(\mathrm{n}-2, \mathrm{n})$ in $\mathrm{PG}(2, \mathrm{q})$[6]. Wilson [14] in 1986 proved that there is$(88, 14, f)$-arc of type $(11, 14)$ in the Galois plane of order 9. Also he proved that there is $(10, 7, f)$ −arc of type $(4, 7)$ in $PG(2, 3)$. In 1989, Hameed[8] studied the existence and non-existence weighted (k, n)-arcs in $PG(2, 9)$ and he proved that there exist $(81, 12, f)$-arc of type $(9, 12)$ and $(85, 13, f)$-arc of type $(10, 13)$. In 1990 Mahmood [11] discussed (k, n, f)-arcs in $PG(2, 5)$. She proved there is $(21, 11, f)$ −arc of type $(6, 11)$, $(20, 10, f)$ −arc of type $(5, 10)$ and $(15, 8, f)$ −arc of type $(3, 8)$. The extensions work of the weighted arcs in $PG(2, 9)$ was investigated by Abass [1] in 2011. He proved there exist$(81, 12, f)$ −arc of type $(9, 12)$ and $(76, 11, f)$ −arc of type $(8, 11)$.In 2012Abudljebar [3], studied the generalization of $(k, n, f)-$ arcs of type $(1, \mathrm{n})$ in $PG(2, q)$ and weighted(k, n) −arcs of type $(n-4, n)$ in $PG(2,8)$.

Most of the authors were used the table of lines and coordinates of points to construct the example of $(k, n;\ f)$-arcs in $PG(2, q)$while we used a new technique depending on the dual of

k-arcs and the envelope in projective plane of order eleven.

2. Preliminaries

Definition 2.1[10]. Let $GF(p) = \mathbb{Z}/p\mathbb{Z}, p$ is prime and let $f(x)$ be irreducible polynomial of degree$h$ over $GF(p)$, then

$GF(q)$ =$GF(p^h)$=$\frac{GF(p)[x]}{f(x)} = \{a_0 + a_1 t + \cdots + a_{h-1}t^{h-1}$:$a_iinGF(P)$;$f(t) = 0\}$

Definition 2.2[10]. A projective plane over $GF(q)$ is 2-dimensional projective space and denoted by $PG(2, q)$ orπ which contains $q^2 + q + 1$ lines, every line contains $q + 1$ points and satisfy the following axioms:

(i) Any two distinct points determine a unique line;

(ii) Any two distinct lines intersect in exactly one point;

(iii) There exist four distinct points such that no three of them are collinear.

Definition2.3[10]. Let $f(x) = x^{r+1} - a_r x^r - \cdots - a_0$ be any monic polynomial, the companion matrix $C(f)$ is given by the following$(r + 1) \times (r + 1)$ matrix**:**

$$C(f) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_0 & a_1 & a_2 & & a_r \end{bmatrix}$$

In particular, when $r = 2$we have$f(x) = x^3 - a_2 x^2 - a_1 x - a_0$, and

$$C(f) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_0 & a_1 & a_2 \end{bmatrix}$$

We choose $f(x) = x^3 - x^2 - x - 3$**.**

Definition 2.4[10]. A t_n-arc is a set of t_n points such that no three points are collinear.

Lemma 2.5[10]. Let $t(p)$ be the number of tangents through p oft_n $-$arcand T_i be the number of$i$ $-$secants of t_n in $PG(2, q)$, then

(i) $t(p) = q + 2 - t_n$;

(ii)$T_2 = \frac{t_n(t_n-1)}{2}$;

(iii) $T_1 = t_n t$, $t = q + 2 - t_n$;

(iv)T_0=$\frac{q(q-1)}{2} + \frac{t(t-1)}{2}$;

(v)$T_0 + T_1 + T_2 = q^2 + q + 1$.

Definition 2.6[10]. The set of t_n lines such that no three are concurrent is called a dual of t_n-arc .

Lemma 2.7. Let $t(l)$be the number of points lie on a line$l$ and let S_i be the number of points which pass through their$i$2-secant, then

(i) $t(l) = q + 2 - t_n$;

(ii) $S_2 = \frac{t_n(t_n-1)}{2}$;

(iii) $S_1 = t_n t$, $t = q + 2 - t_n$;

(iv)S_0=$\frac{q(q-1)}{2} + \frac{t(t-1)}{2}$;

(v) $S_0 + S_1 + S_2 = q^2 + q + 1$.

Proof. According to the principle of duality, so the proof is dual of the proof of Lemma 2.5.

Definition 2.8 [10]. A (k,n)-arc $\mathcal{H}$ is a set of k points such that there are n but no $n+1$ of them are collinear.

Definition 2.9 [8].A line ℓ in $PG(2,q)$is an i – secant of a $(k,\ n)$ – arc $\mathcal{H}$ if $|\ell \cap \mathcal{H}| = i$. Let τ_i denote the total number of i – secants to $\mathcal{H}$ in $PG(2,q)$

Lemma 2.10 [10]. For the (k,n)-arc $\mathcal{H}$, the following equations are hold:

(i) $\sum_{i=0}^{n} \tau_i = q^2 + q + 1$;

(ii) $\sum_{i=1}^{n} i\tau_i = k(q+1)$;

(iii)$\sum_{i=2}^{n} \frac{i(i-1)}{2} \tau_i = \frac{k(k-1)}{2}$,

Definition 2.11 [2]. A point$p$ of $PG(2,q)$is called a point of index zero if not lies on the (k,n) $-$arc$\mathcal{H}$ and not on any n-secants of $\mathcal{H}$.

Theorem2.12 [10]. Let $\mathcal{H}$ be a (k,n) $-$arc in $PG(2,q)$ where q is prime, then

(i) If $n \leq (q+1)/2$, then the maximum size $m_n(2,q)$satisfies the inequality $m_n(2,q) \leq (n-1)q+1$.

(ii) If $n \geq (q+3)/2$, then $m_n(2,q) \leq (n-1)q+n-(q+1)/2$.

Definition 2.13 [5]. Let πbe the projective plane of order q and denoted by$p$ and R are respectively the sets of points and lines of π. Let f be a function from P into the set N of non-negative integers and call the weight of $p \epsilon P$ the value $f(p)$ and the support of $f$the set of points of the plane havenon $-$zero weight. By using $f$we can define a function$F: R \longrightarrow Z^{+}$such that for any $r \epsilon R, F(r) = \sum_{p \epsilon r} f(p)$. $F(r)$ is calledthe weight of the line r.

Definition 2.14[5]. A $(k,n;f)$-arc of the plane π is a subset K of the points of the plane such that

(i) K is the support of f ;

(ii)$k = |K|$;

(iii) $n = max\{F(r): r \epsilon R\}$.

Denote ω =$max_{p \epsilon P} f(p)$, V_i^j to the number of lines of weight i through a point of weight j and $W = \sum_{j=0}^{\omega} \mathcal{H}_j = \sum_{p \epsilon P} f(p)$. For a $(k,n;f)$-arc, we have the following important Lemma:

Lemma 2.15[8]. For the weighted (k,n) $-$arcs in $PG(2,11)$, the following statements are hold:

(i)$\omega \leq 11$;

(ii) If p is any point of the plane, then $\sum_{r \epsilon [p]} F(r) = W + qf(p)$, where$[p]$ denote the set of lines through p;

(iii) The weight W of a $(k,n;f)$ $-$ arc satisfies $(n-11)(q+1) \leq W \leq (n-\omega)q+n$;

(iv) Let K be a$(k,n;f)$ $-$ arc of type $(n-11,n)$, $n-11>0$ and let p be a point having weight s, then V_{n-11}^{s} andV_n^s are determined p and are given by:

$$V_{n-11}^{s} = \frac{q(n-s)-W+n}{11}$$

and

$$V_n^s = \frac{q(s-n+11)+W-n+11}{11};$$

(v) $q \equiv 0 mod(11)$;

(vi) $k = \sum_{j=1}^{2} l_j$;

(vii) The characters of $(k,n;f)-$arc$K$ of type $(n-11,n)$ are given by

$$t_{n-11} = [\frac{q+1}{11}][\frac{n(q^2+q+1)}{q+1} - W]$$

and

$$t_n = [\frac{q+1}{11}][W - \frac{(n-11)(q^2+q+1)}{q+1}]$$

Corollary 2.16[8]. If$W = (n-11)(q+1)$, then $(k,n;f)-$arc is minimal and if $W = (n-\omega)+n$, then $(k,n;f)-$arc is maximal.

Definition 2.17[12]. A $(k,n;f)$-arc is called monoidal if Im$f = \{0,1,m\}$ and $l_m = 1$,with $m \geq 2$.

Principle of Duality 2.18[10]. For any space $S = PG(n,q)$, there is a dual space S^*, whose points and primes are respectively primes and points of S. For any theorem true in S there is an equivalent theorem true in S^*.

Lemma 2.19. The existence of a $(k,n;f)-$arcs of type$(n-11,n)$,in$PG(2,q)$with$q+1 < n < 2q+2$ requires $q \equiv 0 mod(11)$

Proof. Directly, from Lemma 2.15 case (v).■

Lemma 2.20[5]. The existence of a $(k,n;f)-$arcs of type $(n-11,n)$ in $PG(2,q)$ with $q+1 < n < 2q+2$ requires $l_i = 0, i \geq 3$.

We used Lemma 2.15 case (iii) to get

$$(n-11)(q+1) \leq W \leq (n-11)(q+1)+11$$

Lemma 2.21. For a $(k,n;f)-$arcs of type $(n-11,n)$ in $PG(2,q)$ with W minimal

$(W = (q+1)(n-11))$ we have

$$V_{n-11}^0 = q+1, \quad V_{n-11}^1 = \frac{10q+11}{11}, \quad V_{n-11}^2 = \frac{9q+11}{11}$$

$$V_n^0 = 0, \quad V_n^1 = \frac{q}{11}, \quad V_n^2 = \frac{2q}{11}$$

Proof. From Lemma 2.15 case(iv), by substituting $n-11 = q+1$for Im$f = \{0,1,2\}$.■

Corollary 2.22. There is no points of weight 0 on $n-$ weighting lines of $(k,n;f)-$arcs of type $(n-11,n)$.

For the case $l_0 > 0, l_1 > 0, l_2 > 0$ and $l_i = 0$, where $3 \leq i \leq 11$, we have the weight of the points of the $(k,n;f)-$arcs is $\omega = 2$, and by using the minimal case $(W = (n-11)(q+1))$ and by counting the number of lines of $PG(2,q)$ we find the following:

$$t_n + t_{n-11} = q^2+q+1$$

By counting the number of $n-$ weighting lines (t_n) and $(n-11)$- weighting lines (t_{n-11}), and counting the total incidence, it follows that

$$nt_n + (n-11)t_{n-11} = W(q+1) = (n-11)(q+1)^2$$

Consequently, we get

$t_n=\frac{(n-11)q}{11}$(2.1)

$t_{n-11}=\frac{11q^2+(22-n)q+11}{11}$(2.2)

Lemma 2.23. The $n-$ weighting lines of $(k,n;f)-$arcs of type $(n-11,n)$ form a dual of t_n −arc in $PG(2,11)$.

Proof. From Lemma 2.21, we have $V_n^2=2$, this mean that there are no three $n-$ weighting lines are concurrent. Then the number of $n-$ weighting lines t_n form a dual of t_n −arc.■

Suppose that on n −weighting lines there are α points of weight 1 and β points of weight 2. Then counting the points of n −weighting lines, it follows that:

$$\alpha+\beta=q+1$$

And counting the weight of points on n −weighting lines, we have

$$\alpha+2\beta=n$$

Solving these two equations, we obtain

$\alpha=2(q+1)-n$(2.3)

$\beta=n-(q+1)$(2.4)

counting the incidences between the points of weight two and n −weighting lines, we get

$$l_2V_n^2=t_n\beta$$

Making use of Lemma 2.21, equation (2.1) and equation (2.4) we obtain

$l_2=\frac{(n-11)(n-q-1)}{2}$(2.5)

Similarly, counting the incidences between the points of weight one and n −weighting lines, we have

$$l_1V_n^1=t_n\alpha$$

Hence, by using Lemma 2.21, equation (2.2) and equation (2.3), we get

$l_1=(n-11)(2q+2-n)$(2.6)

From equations (2.5) and (2.6), counting the points in the plane, we have

$$l_0+l_1+l_2=q^2+q+1$$

$$l_0=q^2+q+1-(n-11)(2q+2-n)-\frac{(n-11)(n-q-1)}{2}$$

Hence

$2q^2+(35-3n)q+n^2-14n+35-2l_0=0$(2.7)

The solution of equation (2.7) exists with respect to q if $(35-3n)^2-8(n^2-14n+35-2l_0)$ is square, then

$(n-49)^2-(1456-16l_0)=square$(2.8)

Let l be $(n-11)$-weighting lines having on it μ points of weight 2, δ points of weight 1 and γ points of weight 0. Then counting points on l gives

$$\mu+\delta+\gamma=q+1 \qquad (2.9)$$

and summing the weights of points on l gives

$$2\mu+\delta=n-11 \qquad (2.10)$$

3. $(\boldsymbol{k},\boldsymbol{n};\ \boldsymbol{f})$ −arcs of Type $(\boldsymbol{n}-\mathbf{11},\boldsymbol{n})$ in $\boldsymbol{PG}(\mathbf{2},\mathbf{11})$

3.1. The case of l_0 equal 70.

Substitute $l_0 = 70$ and $q = 11$ in the equation (2.7) we get $n = 18$, from the equations (2.1) and (2.2) we get $t_n = 7$ which are represented in Figure 3.1.

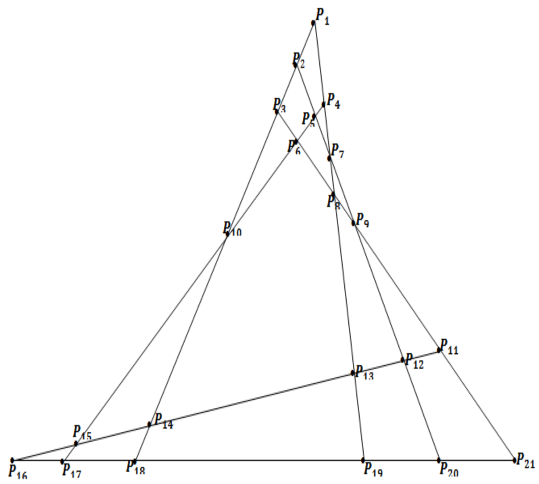

Figure 3.1

Since $V_n^2 = 2$ and $t_n = 7$, so the number of n-weighting lines form a dual of 7-arc. Hence the intersection of any two n-weighting lines are points of weight two, so the number of points of weight two equal $\binom{7}{2} = 21$. The remaining points of any n-weighting lines are points of weight one. From the Figure 3.1, we deduce that the number of points of weight one equal$t_n(q + 1 - 6) = 7 \times 6 = 42$.

Lemma 3.1.1. (i) The number of points of weight zero form $(70, 8)$-arc of type (14,63,42,7,0,0,0,0,7).

(ii)The number of points of weight one form $(42, 7)$-arc of type (7,7,42,0,63,0,14,0).

(iii) The number of points of weight two form $(21, 6)$-arc of type (7,0,0,14,63,42,7).

Proof. Directly, from the equations (2.9) ,(2.10)and Lemma 2.10. ■

Since $k = \sum_{j=1}^{2} l_j = 63$. Hence we deduce the following theorem.

Theorem 3.1.2. There exist a $(63{,}18; f) - arc$ of type (7,18) in $PG(2{,}11)$ with the Imf = $\{0{,}1{,}2\}$ andthe points of weight zero is $(70, 8)$-arc of type (14,63,42,7,0,0,0,0,7).

3.2. The case of l_0 equal $76, 83, 91$ and 100

In this section we discuss the case of l_0 = $76, 83, 91$ and 100. By substitute the values of l_0 and $q = 11$, from equation(2.7) we get $n = 17, 16, 15$ and 14 respectively. By the same argument in the section 3.1 we get $t_n = 6, 5, 4$ and 3 respectively. Since $V_n^2 = 2$, so the n-weighting lines form a t_n-arcs. Since the intersection of any two n-weighting lines is a point of weight two, hence the number of points of weight two (l_2) equal $\binom{t_n}{2}$, t_n = $6, 5, 4$ and 3. By the same argument in 3.1, the remaining points of any n-weighting lines are points of weight one. As in 3.1 we summarize the values of k in the table 3.1 and Figures 3.2 and 3.3.

t_n	l_1	l_2	l_0	k
6	42	15	$(76, 9)$-arc	57
5	40	10	$(83, 9)$-arc	50

4	36	6	$(91, 10)$-arc	42
3	30	3	$(100, 10)$-arc	33

Table 3.1

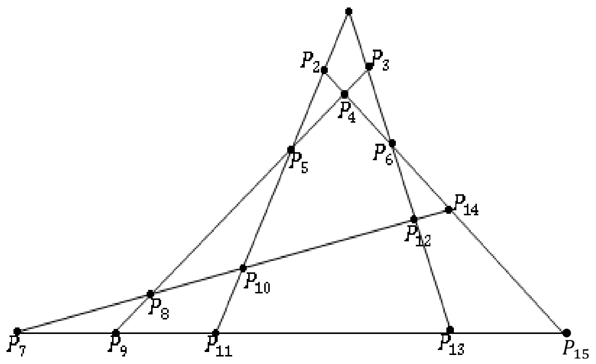

Figure 3.2 ($t_n = 6$)

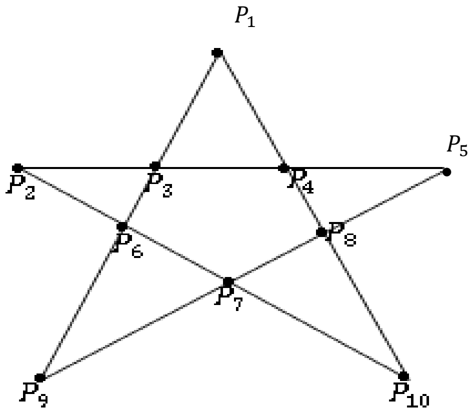

Figure 3.3 ($t_n = 5$)

Since $k = 57, 50, 42$ and 33. Hence we deduce the following theorem.

Theorem 3.2.1. There exist a $(k, n; f) - arc$ of type $(n - 11, n)$ in $PG(2,11)$ with the Im$f = \{0,1,2\}$.

Corollary 3.2.2. The points of weight zero form a maximum $(k, 10)$-arc$\mathcal{H}$ of type $(\tau_{10} = 30, \tau_9 = 100, \tau_0 = 3$ and $\tau_i = 0$ other wise) in $PG(2,11)$.

Proof. Suppose that the points $P_1(1,0,0)$,$P_2(0,1,0)$ and $P_3(0,0,1)$ are points of weight two, since through every point of weight two there pass exactly two n-weighting linesas in Figure 3.4, hence the remaining points of l_1, l_2 and $l_{10}$are points of weight one.

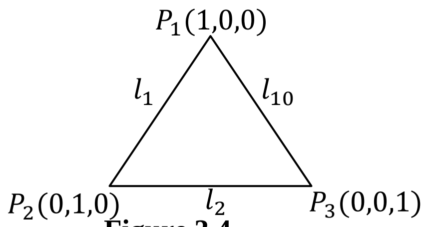

Figure 3.4

Let l be a line of weighting $(n - 11)$. Suppose that there are α points of weight two, β points of weight one and γ points of weight zero, we have

$$\alpha + \beta + \gamma = 12$$

$$2\alpha + \beta = 3$$

The only non-negative integers solutions are given in the table 3.2

α	β	γ
1	1	10
0	3	9

Table 3.2

From the solutions above we get that the points of weight zero form a (100,10)-arc of type $(\tau_{10} = 30, \tau_9 = 100, \tau_0 = 3$), since every point $P \notin$ (100,10)-arc lies on 10-secants of (100,10)-arc. Since there is no points of index zero, therefore, the(100,10)-arc is maximum $\mathcal{H}$ in $PG(2,11)$.

4. Monoidal $(23,13;\, f)$ −arc of type $(2,13)$ in $PG(2,11)$ **Lemma 4.1 [9].** For the monoidal arc of type $(2,13)$ in $PG(2,11)$ with W minimal($W = 24$), we have

$V_2^0 = 12,\quad V_2^1 = 11,\quad V_2^2 = 10,$

$V_{13}^0 = 0,\quad V_{13}^1 = 1,\quad V_{13}^2 = 2$

Lemma 4.2 [9]. For the existence of the monoidal arc of type $(2,13)$ in $PG(2,11)$ with W minimal($W = 24$), we must have the following:

(i) The number of 13 −weighting lines (t_{23}) is 2;

(ii) The number of 2 −weighting lines(t_{12}) is 131;

(iii) The number of points of weight 2 (l_2) is 1;

(iv) The number of points of weight 1 (l_1) is 22.

Corollary 4.3. (i) The number of points of weight one form a $(22,11)$ −arc of type (2,0,0,0,0,0,0,0,0,121,0,10).

(ii)The number of points of weight zero form a $(110,11)$ −arc of type (10,121,0,0,0,0,0,0, 0,0,0,2).

Proof. Directly, from the equations (2.9) ,(2.10) and Lemma 2.10. ■

Corollary 4.4. The points of weight zero form a(110,11) −arcoftype($\tau_{11} = 10,\quad \tau_{10} = 121$, $\tau_0 = 2$ and $\tau_i = 0$ other wise) in $PG(2,11)$.

Since $k = \sum_{j=1}^{2} l_j = 23$. Hence we deduce the following theorem.

Theorem4.5. There exist a monoidal$(23,13;f)$- arc of type (2, 13) in $PG(2,11)$ when the points of weight zero is 110 with Imf={0,1,2}.

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