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On Order - Convergence of Filters in a Riesz Spaces

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Abstract

The main purpose of this paper, is study the ideas of order- Convergence of filters in a Riesz spaces and that is through prove an important theorems related to the some properties Riesz spaces. So we established that the intersection and union all subsets of the collection of all proper filters $F(\mathcal{M})$ were converge to point in Riesz space it's the same convergence point the filter there exist in these subsets and proved that order-convergence for intersection (union) two filters to intersection (union) two different points in Riesz space is equivalent to the set consist of two convergence points we could write it by using these filters .

Keywords: Order - Convergence for filter, Lattice, Riesz space**.**

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التقارب الرتيب للفلاتر في فضاءات ريسز

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الملخص

الهدف الرئيسي من هذا البحث هو دراسة فكرة التقارب الرتيب بالنسبة للفلاتر في فضاء ريسز. وذلك من خلال اثبات مبرهنات مهمة تتعلق ببعض الخواص الرئيسية لهذا الفضاء. وكذلك ثبتنا ان تقاطع واتحاد كل المجموعات الجزئية من المجموعة الفعلية تقترب الى نقطة في فضاء ريسز وهي نفس نقطة التقارب للفلتر الموجود في هذه المجموعات الجزئية, و كذلك برهنا ان التقارب الرتيب لتقاطع (اتحاد) فلترين الى تقاطع (اتحاد) نقطتين مختلفتين في فضاء ريسز يكون مكافئ الى مجموعة تحتوي على نقطتي التقارب التي تكتب بواسطة الفلاتر.

الكلمات الدالة: التقارب الرتيب للفلتر، الحزمة، فضاء ريسز (الحزم المتجه).

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1. Introduction:

The concept of a Riesz space was first introduced by F. Riesz in [1]. Since then many others have developed the subject further. The first contributions to the theory came from L. V. Kantorovich [2] and H. Freudenthal [3]. Most of the spaces encountered in the analysis are Riesz spaces. They play an important role in optimization, problems of Banach spaces, measure theory and operator theory. They have also some applications in economics [4-5].

One of the fundamental ideas in the reading of Riesz spaces is the "Order- Convergence", which leads to the concept of order continuity. In these study, we explored (order - convergence for filters) in a Dedekind complete Riesz space [6].

As a natural consequence, this paper will introduce the idea of Order -Convergence of filters in a Riesz spaces, examining some of its properties and also proving some of its basic realities.

2. Preliminaries:

This section contains of a collection of known notions, and basic information of Riesz spaces and Order- Convergence of filters and a number of its properties, To understand term Riesz space, we refer to [7]. A non-empty set $\mathcal{M}$ with a partially ordered set " $\leq$ " is called a lattice if the infimum and supremum of any twosome of elements in $\mathcal{M}$ exist.

A real vector space $\mathcal{M}$ which is also an ordered space with the linear and order arrangements related by the implications,

i. If $x, y, z \in \mathcal{M}$ and, $x \leq y$ then, $x + z \leq y + z$.

ii. If $x, y \in \mathcal{M}$, $x \leq y$ and $0 \leq \alpha \in \mathbb{R}$ then, $\alpha x \leq \alpha y$.

Thus the set $\mathcal{M}_+ = \{x \in \mathcal{M} : x \geq 0\}$ is a expression the positive cone in $\mathcal{M}$ and its elements are characterized positive.

An ordered vector space who is also a lattice is a Riesz space (also called vector lattice).

Remark 2.1.[9]. Let $\mathcal{M}$ be a Riesz space. For $\mathfrak{I} \subset \mathcal{M}$, we define $\mathfrak{I}_l = \{x : x \leq y \text{ } for\ all \text{ } y \in \mathfrak{I}\}$, $\mathfrak{I}_u = \{x : x \geq y \text{ } for\ all \text{ } y \in \mathfrak{I}\}$, where $\mathfrak{I}_l$ and $\mathfrak{I}_u$ are the sets of all lower and upper bounds of $\mathfrak{I}$, respectively, $\{x\}_l$ and $\{x\}_u$ will be written x_l and x_u respectively. The closed interval notation, $[x, y] = x_u \cap y_l$ ($x \leq y$) will be employed.

Definition 2.2.[8]. A filter $\mathcal{F}$ on a lattice $\mathcal{M}$ is said to be order- Convergent to a point $x \in \mathcal{M}$, in signs $\mathcal{F} \xrightarrow{(o)} x$ if $x = \vee \mathcal{F}_l = \wedge \mathcal{F}_u$.

Lemma 2.3.[9]. Let $y \in \mathcal{M}$, $\mathfrak{I}, Z \subset \mathcal{M}$, $\mathcal{F}, \wp \in F(\mathcal{M})$, where $F(\mathcal{M})$ the collection of all proper filters on $\mathcal{M}$ and $F \subset F(\mathcal{M})$. Then:

a. $\mathfrak{I} \subset Z$ implies $\mathfrak{I}_l \supset Z_l$ and $\mathfrak{I}_u \subset Z_u$,

b. $\mathfrak{I}_l = \{x: Y \subset x_u\},\ \mathfrak{I}_u = \{x: Y \subset x_l\},$

c. $\mathfrak{I} \subset \mathfrak{I}_{ul},\ \mathfrak{I} \subset \mathfrak{I}_{lu},$

d. $\wedge\mathfrak{I} = y$ if and only if $\mathfrak{I}_l = y_l,\ \ \vee\mathfrak{I} = y$ if and only if $\mathfrak{I}_u = y_u,$

e. $y_l = y_{ul},\ y_u = y_{lu},$

f. $\mathcal{F} \subset \wp$ implies $\mathcal{F}_l \subset \wp_l$ and $\mathcal{F}_u \subset \wp_u,$

g. $\mathcal{F}_l = \{x: x_u \in \mathcal{F}\},\ \mathcal{F}_u = \{x: x_l \in \mathcal{F}\},$

h. $\mathcal{F}_{ul} = \bigcap \{x_l: x_l \in \mathcal{F}\},\ \mathcal{F}_{lu} = \bigcap\{x_u: x_u \in \mathcal{F}\},$

i. $\mathcal{F}_u \subset \mathcal{F}_{ulu} \subset \mathcal{F}_{lu},\ \mathcal{F}_l \subset \mathcal{F}_{lul} \subset \mathcal{F}_{ul},$

j. $\mathcal{F}_l = \wp_l$ and $\mathcal{F}_u = \wp_u$ for every base $\wp$ of $\mathcal{F}$. In particular,

$$[y]_l = y_l,\ [y]_u = y_u$$

k. $(\bigcap F)_l = \bigcap\{\mathcal{F}_l : \mathcal{F} \in F\},\ (\bigcap F)_u = \bigcap\{\mathcal{F}_u : \mathcal{F} \in F\},$

l. $y \le z$ for all $y \in \mathcal{F}_l$ and all $z \in \mathcal{F}_u$.

Corollary 2.4.[9]. For $\mathcal{F} \in F(\mathcal{M})$, the following five statements are equivalent.

i. $\mathcal{F} \overset{(o)}{\to} x.$

ii. $\mathcal{F}_{lu} = x_u$ and $\mathcal{F}_{ul} = x_l$.

iii. $\mathcal{F}_{lu} \subset x_u$ and $\mathcal{F}_{ul} \subset x_l$.

iv. $x \in \mathcal{F}_{lul} \cap \mathcal{F}_{ulu}.$

v. $\{x\} = \mathcal{F}_{lul} \cap \mathcal{F}_{ulu}.$

Remark 2.5.[8]. Let $\mathcal{M}$ be a lattice, $\mathcal{F}, \wp \in F(\mathcal{M})$, then.

i. $(\mathcal{F} \vee \wp)_u = \mathcal{F}_u \vee \wp_u.$

ii. $(\mathcal{F} \wedge \wp)_l = \mathcal{F}_l \wedge \wp_l.$

iii. $(\mathcal{F} \wedge \wp)_u \supset \mathcal{F}_u \wedge \wp_u.$

iv. $(\mathcal{F} \vee \wp)_l \supset \mathcal{F}_l \vee \wp_l.$

3. Main Results:

Now, we will be establish basic facts related to the some properties of the order-convergence for filters ina vector lattices.

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Theorem 3.1 A Riesz space $\mathcal{M}$, a filters $\mathcal{F}, \wp \in F(\mathcal{M})$, such that $\mathcal{F} \xrightarrow{(o)} x$ and $\wp \xrightarrow{(o)} y$, we have :

i. $\mathcal{F} \cup \wp \xrightarrow{(o)} x \cup y$,

ii. $\mathcal{F} \cap \wp \xrightarrow{(o)} x \cap y$,

iii. If $\mathcal{F} \xrightarrow{(o)} x$, then $\mathcal{F}_u \xrightarrow{(o)} x$ and $\mathcal{F}_l \xrightarrow{(o)} x$.

Proof:

i. First we note that x is a lower bound of $\mathcal{F}_u$ and y is lower bound of $\wp_u$, then $x \cup y$ is lower bound of $\mathcal{F}_u \cup \wp_u$, implies that $x \cup y = \wedge(\mathcal{F} \cup \wp)_u$. Furthermore, since all elements of $(\mathcal{F} \cup \wp)_u$ are upper bounds of $(\mathcal{F} \cup \wp)_l$, $x \cup y$ is an upper bound of $(\mathcal{F} \cup \wp)_l$. If z is an upper bound of $(\mathcal{F} \cup \wp)_l$, at that moment z is an upper bound of $\mathcal{F}_l \cup \wp_l$, it follows that $z \geq x \cup y$. Thus $x \cup y = \vee(\mathcal{F} \cup \wp)_l$. From the Definition 2.2, $\mathcal{F} \cup \wp \xrightarrow{(o)} x \cup y$.

ii. By Remark 2.5, $(\mathcal{F} \cap \wp)_u \supset \mathcal{F}_u \cap \wp_u$. If z is lower bound of $(\mathcal{F} \cap \wp)_u$ see that, x is a lower bound of $\mathcal{F}_u$ and y alower bound of $\wp_u$, then $x \cap y = \wedge(\mathcal{F}_u \cap \wp_u)$ implies that $z \leq x \cap y$, and thus $x \cap y = \wedge (\mathcal{F} \cap \wp)_u$. On the other hand, since all elements of $(\mathcal{F} \cap \wp)_u$ are upper bounds of $(\mathcal{F} \cap \wp)_l, x \cap y$ is an upper bound of $(\mathcal{F} \cap \wp)_l$, it follows that $x \cap y$ is upper bound of $\mathcal{F}_l \cap \wp_l$, we know x is upper bound of $\mathcal{F}_l$ and $y$an upper bound for $\wp_l$. Consequently $x \cap y = \vee (\mathcal{F} \cap \wp)_l$, hence, $\mathcal{F} \cap \wp \xrightarrow{(o)} x \cap y$.

iii. Suppose $\mathcal{F} \xrightarrow{(o)} x$. Then by Lemma 2.3 ($i$) and Corollary 2.4 ($iii$), see that $\mathcal{F}_l \subset \mathcal{F}_{ul} \subset x_l$, whence, $(\mathcal{F}_u)_l = \mathcal{F}_{ul} \subset x_l$, and by duality $(\mathcal{F}_u)_u = \mathcal{F}_u \subset \mathcal{F}_{lu} \subset x_u$. From Corollary 2.4, deduce that $\mathcal{F}_u \xrightarrow{(o)} x$. The final assertion $\mathcal{F}_l \xrightarrow{(o)} x$. Follows from the same applications in above. Thus,

$(\mathcal{F}_l)_u = \mathcal{F}_{lu} \subset x_u$ and $(\mathcal{F}_l)_l = \mathcal{F}_l \subset \mathcal{F}_{ul} \subset x_l$.

Theorem 3.2 Let $\mathcal{M}$ be a Riesz space, a filters $\mathcal{F}, \wp, \mathfrak{G} \in F(\mathcal{M})$. Then,

i. If $\mathcal{F} \subset \wp$ and $\mathcal{F} \xrightarrow{(o)} x$ and $\wp \xrightarrow{(o)} y$, then $x \subset y$.

ii. Every order-convergent filter on X has a unique limit.

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iii. If $\wp \subset \mathcal{F} \subset \mathfrak{G}$, $\wp \overset{(o)}{\rightarrow} x$ and $\mathfrak{G} \overset{(o)}{\rightarrow} x$ then $\mathcal{F} \overset{(o)}{\rightarrow} x$.

Proof:

i. By Definition 2.2, it follows that $x = \bigvee \mathcal{F}_l = \bigwedge \mathcal{F}_u$ and $y = \bigvee \wp_l = \bigwedge \wp_u$ such that $\mathcal{F} \subset \wp$, then by Lemma 2.3, we have $\mathcal{F}_l \subset \wp_l$and $\mathcal{F}_u \subset \wp_u$. Consequently $\bigvee\mathcal{F}_l \subset \bigvee\wp_l$and $\bigwedge \mathcal{F}_u \subset \bigwedge \wp_u$. Hence $x \subset y$.

ii. Assume that $x \neq y$ where $x, y \in X$. If $\mathcal{F} \overset{(o)}{\rightarrow} x$ and $\mathcal{F} \overset{(o)}{\rightarrow} y$, by Corollary 2.4 implies that $\{x\} = \mathcal{F}_{lul} \cap \mathcal{F}_{ulu}$ and $\{y\} = \mathcal{F}_{lul} \cap \mathcal{F}_{ulu}$, it is contradiction. Thus $x = y$.

iii. By Definition 2.2, then $x = \bigvee \wp_l = \bigwedge \wp_u$ and $x = \bigvee \mathfrak{G}_l = \bigwedge \mathfrak{G}_u$, since $\wp \subset \mathcal{F} \subset \mathfrak{G}$, then by Lemma 2.3 $\wp_l \subset \mathcal{F}_l \subset \mathfrak{G}_l$ and $\wp_u \subset \mathcal{F}_u \subset \mathfrak{G}_u$. Consequently,

$x = \bigvee\wp_l \subset \bigvee\mathcal{F}_l \subset \bigvee\mathfrak{G}_l = x$ and $x = \bigwedge\wp_u \subset \bigwedge\mathcal{F}_u \subset \bigwedge\mathfrak{G}_u = x$.

Thus, $x = \bigvee \mathcal{F}_l = \bigwedge \mathcal{F}_u$. The desired result follows now.

Theorem 3.3 For a Riesz space $\mathcal{M}$, a filter $\mathcal{F} \in F(\mathcal{M})$, $F \subset F(\mathcal{M})$ and $\mathcal{F} \overset{(o)}{\rightarrow} x$,then

i. $\bigcap F \overset{(o)}{\rightarrow} x$.

ii. $\bigcup F \overset{(o)}{\rightarrow} x$.

Proof:

i. By Lemma 2.3,$(\bigcap F)_l = \bigcap\{\mathcal{F}_l : \mathcal{F} \in F\}and(\bigcap F)_u = \bigcap\{\mathcal{F}_u : \mathcal{F} \in F\}$. By Definition 2.2, implies that $x = \bigvee \mathcal{F}_l = \bigwedge \mathcal{F}_u$, it follows that, $x = \bigcap(\bigvee\mathcal{F}_l) = \bigcap(\bigwedge \mathcal{F}_u)$, thus $x = \bigvee(\bigcap\mathcal{F}_l) = \bigwedge(\bigcap \mathcal{F}_u)$, consequently $x = \bigvee(\bigcap F)_l = \bigwedge(\bigcap F)_u$. Hence $\bigcap F \overset{(o)}{\rightarrow} x$.

ii. By Definition 2.2, thereby, $x = \bigvee \mathcal{F}_l = \bigwedge \mathcal{F}_u$. Note that, $(\bigcup F)_l = \bigcup\{\mathcal{F}_l : \mathcal{F} \in F\}$ and $(\bigcup F)_u = \bigcup\{\mathcal{F}_u : \mathcal{F} \in F\}$. From this observe $x = \bigcup(\bigvee\mathcal{F}_l) = \bigcup(\bigwedge \mathcal{F}_u)$, from this we can write $x = \bigvee(\bigcup\mathcal{F}_l) = \bigwedge(\bigcup \mathcal{F}_u)$. Thus $x = \bigvee(\bigcup F)_l = \bigwedge(\bigcup F)_u$. Then the proof is finished.

Theorem 3.4 On a Riesz space $\mathcal{M}$, where a filters $\mathcal{F}, \wp \in F(\mathcal{M})$ are Order- Convergence to $x, y \in \mathcal{M}$ respectively, then,

i. $\mathcal{F} \cup \wp \overset{(o)}{\rightarrow} x \cup y$, if and only if $\{x, y\} = (\mathcal{F} \cup \wp)_{lu} \cap (\mathcal{F} \cup \wp)_{ul}$.

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ii. $\mathcal{F}\cap\wp \overset{(o)}{\rightarrow} x\cap y$, if and only if $\{x,y\} = (\mathcal{F}\cap\wp)_{lu}\cap(\mathcal{F}\cap\wp)_{ul}$.

Proof: ***i.*** By Corollary 2.4 and application of Lemma 2.3, implies that,

$\{x,y\} = (x\cup y)_u\cap(x\cup y)_l = (x_u\cup y_u)\cap(x_l\cup y_l)$,

$= (x_{lu}\cup y_{lu})\cap(x_{ul}\cup y_{ul})$,

$= (\mathcal{F}_{lu}\cup\wp_{lu})\cap(\mathcal{F}_{ul}\cup\wp_{ul})$,

$= (\mathcal{F}\cup\wp)_{lu}\cap(\mathcal{F}\cup\wp)_{ul}$.

Conversely, assume $\{x,y\} = (\mathcal{F}\cup\wp)_{lu}\cap(\mathcal{F}\cup\wp)_{ul}$ and $x\cup y \in (\mathcal{F}\cup\wp)_{lu}$, then $x\cup y$ is an upper bound of $(\mathcal{F}\cup\wp)_l$. If there would be another upper bound z of$(\mathcal{F}\cup\wp)_l$, then z is an upper bound of $\mathcal{F}_l\cup\wp_l$ with $z < x\cup y$ and $x\cup y \in (\mathcal{F}\cup\wp)_{ul}$, would imply $z \in (\mathcal{F}\cup\wp)_{ul}$, thus, $z \in (\mathcal{F}\cup\wp)_{lu}\cap(\mathcal{F}\cup\wp)_{ul} = \{x,y\}$,

it is contradiction. Since $\mathcal{M}$ is a lattice, it satisfies that $x\cup y$ is a least upper bound of $(\mathcal{F}\cup\wp)_l$, i.e., $x\cup y = \vee(\mathcal{F}\cup\wp)_l$. Furthermore, if $x\cup y$ is an lower bound of $(\mathcal{F}\cup\wp)_u$, see that $x\cup y \in (\mathcal{F}\cup\wp)_{ul}$. If there exist another lower bound w of $(\mathcal{F}\cup\wp)_u$, then w is a lower bound of $\mathcal{F}_u\cup\wp_u$ with $w > x\cup y$, then $x\cup y \in (\mathcal{F}\cup\wp)_{lu}$, it satisfies that $w \in (\mathcal{F}\cup\wp)_{lu}$, thereby, $w \in (\mathcal{F}\cup\wp)_{lu}\cap(\mathcal{F}\cup\wp)_{ul} = \{x,y\}$,

it is contradiction, it follows that $x\cup y = \wedge(\mathcal{F}\cup\wp)_u$. The proof of (***ii***) is similar.

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