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The Construction of Complete (k,4)-Arcs in PG(2,8) on projective plane over Galois Field GF (8)

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Abstract

In this work, we construct by an algebraic method the projectively distinct (k,4)-arcs in PG(2,8),where $k \geq 5$ and we prove that the complete (k,4)-arcs do not exist , when $5 \leq k \leq 18$. Also we find that complete (k,4)-arc in PG(2,8) is the (20,4)-arc an complete (k,3)-arc and PG(2,8)is the (19,4)-arc .

المستخلص

في هذا البحث ، نبني بطريقة جبرية الأقواس -(k,4) المختلفة اسقاطيا في المستوي الاسقاطي PG(2,8) ، حيث إن $k \geq 5$ ونبرهن إن الأقواس -(k,4) التامة غير موجودة عندما $5 \leq k \leq 18$. وجدنا قوس-(k,4) تام في PG(2,8) هو القوس -(20,4) . وكذلك القوس -(k,4) تام في PG(2,8) هو القوس -(19,4) .

1. Projective geometry [2]

The history of projective geometry is a remarkable instance of art and science feeding off one another. Based on the optics studies of the Arabic mathematician Alhazen (Ibu Ali al-Hasan ibn al-Haytham) (965–1040), several early Renaissance artists attempted to develop a style of visual depiction that presented the eye with a truer semblance of three dimensional space than did earlier, flatter styles. The discovery of the principle of linear perspective (the idea that all parallel lines appear to converge at a single point) is credited to Filippo Brunelleschi (1377–1446). This led to a flurry of activity, culminating in the work of Girard Desargues (1591–1661), which introduced projective geometry as we now it.

2.Introduction[2,5]

The projective plane consists of the standard Euclidean plane, together with a set of points called points at infinity, one for each collection of parallel lines. We say that a line passes through the point at infinity corresponding to its direction (and no others), and that all of the points at infinity lie on a line at infinity. Note that three parallel lines now indeed have a common point at infinity, which retroactively justifies our calling such lines “concurrent”.

1. A point is that of which there is no part.
2. And a line is a length without breadth.
3. And the extremities of a line are points.
4. A straight-line is whatever lies evenly with points upon itself.
5. And a surface is that which has length and breadth alone.
6. And the extremities of a surface are lines.
7. A plane surface is whatever lies evenly with straight-lines upon itself.
8. And a plane angle is the inclination of the lines,
when two lines in a plane meet one another, and are not laid down straight-on with respect to one another.
9. And when the lines containing the angle are straight then the angle is called rectilinear.
10. And when a straight-line stood upon (another) straight-line makes adjacent angles (which are) equal to one another, each of the equal angles is a right-angle, and the former straight-line is called perpendicular to that. upon which it stands.

A projective plane PG(2,q) over GF(q) is a two – dimensional projective space , which consists of points and lines with incidence relation between them.

PG(2,q) satisfying the following axioms :

A . Any two distinct points are contained in a unique line.
B . Any two distinct lines are intersected in a unique point.
C . There exist at least four points such that no three of them are collinear.

Also , in PG(2,q) :There are

There are $1+q+q^2$ points.
There are $1+q+q^2$ lines.
There are $1+q$ points on every line.
There are $1+q$ lines through every point.

Finally , the points of PG(2,q) can be numerated as follows:

The number of the point (1,0,0) is 1, point (x,1,0) is numerated as x+2, the point (x,y,1) is numbered as x +(y.q) + q + 2.

3. Basic Definitions and Theorems

Definition 3. 1:- [2]

A (k,n) – arc K in PG(2,q) is a set of k points , such that same n, but no n+1 of which are collinear. A (k,4) – arc is a set of k points no five of them are collinear.

Definition 3. 2:- [2]

A (k,n) – arc K is complete if it is not contained in a (k+1,n) – arc .

Theorem 3. 3:- [2]

Let r_i be the total number of i – secants of a (k,n) – arc K in PG(2,q) , then the following equations are hold :-

$$\sum_{i=0}^{n} r_i = q^2 + q + 1 \qquad \text{(1)}$$

$$\sum_{i=1}^{n} i r_i = k(q+1) \qquad \text{(2)}$$

$$\sum_{i=2}^{n} i(i-1) r_i = k(k-1) \qquad \text{(3)}$$

Definition 3.4:- [7]

The rays X =(x_1,x_2,x_3) and X = λ (x_1,x_2,x_3) are the same and
are mapped to the same point m of the plane P, X is the coordinate vector of m, (x_1,x_2,x_3) are its homogeneous coordinates

Definition 3.5:- [6]

The matrix H can be multiplied by an arbitrary non-zero number
without altering the projective transformation
n Matrix H is called a “homogeneous matrix” (only ratios of terms are
important)

Theorem 3. 6:- [7]

A mapping is a projectively if and only if the mapping consists of a linear transformation of homogeneous coordinates x'=Hx with H non singular

The addition's and Multiplication's Operations Of GF(8).

To find the addition and multiplication tables in PG(8) , we have the order triples (x1,x2,x3) such that x1,x2,x3 in GF(2) , as follows :

O $\equiv$(0,0,0) , 1$\equiv$(1,0,0) , 2$\equiv$(0,1,0) , 3$\equiv$(1,1,0),
4 $\equiv$ (0,0,1) , 5$\equiv$(1,0,1) , 6$\equiv$(0,1,1) , 7$\equiv$(1,1,1).

Put these points in one orbit (1,0,0) at the first point and by the principle (1,0,0) Ai , i= 0,1,2,……,7 and A= $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

$$(1,0,0)\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \left| \begin{matrix} (0,1,0) \\ (0,0,1) \\ (1,0,1) \\ (1,1,1) \\ (1,1,0) \\ (0,1,1) \end{matrix} \right.$$

Now, in the left of this table the operation of multiplication m, and in the right the operation of addition n, in multiplication side we write the numeration of points as last and the addition side takes the normal sequence.

M(*)		(+)n = f(m)
1	(1,0,0)	0
2	(0,1,0)	1
3	(1,1,0)	5
4	(0,0,1)	2
5	(1,0,1)	3
6	(0,1,1)	6
7	(1,1,1)	4
Mod 7		

In the addition's table we have the following relation :

(x1,x2,x3)+(y1,y2,y3) =(z1,z2,z3) where zi $\equiv$(yi+xi) mod(2) , for i = 1,2,3

In multiplication table we have the following relation :

m1*m2 = m3 $\Leftrightarrow$ ((1,0,0)Af(m1) A f(m2)) =

m1 * m2 = m3 $\Leftrightarrow$ ((1,0) Af(m1)) A f(m2) = (1,0,0) Af(m1) + f(m2)) mod(7)
= (x1,x2,x3)

For example : 3 * 7 = 4 $\Leftrightarrow$ ((1,0,0)A5)A4 = (1,0,0) A2 = (0,0,1)

Where (0,0,1) equal to 4 in multiplication side.

Now, we have the addition's and multiplication's tables

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	3	2	5	4	7	6
2	2	3	0	1	6	7	4	5
3	3	2	1	0	7	6	5	4
4	4	5	6	7	0	1	2	3
5	5	4	7	6	1	0	3	2
6	6	7	4	5	2	3	0	1
7	7	6	5	4	3	2	1	0

*	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	2	4	6	5	7	1	3
3	3	6	5	1	2	7	4
4	4	5	1	7	3	2	6
5	5	7	2	3	6	4	1
6	6	1	7	2	4	3	5
7	7	3	4	6	1	5	2

Definition 3.7:- [1,3,4]

Let Q_1, Q_2 in PG(2,q)\ K and let $K_1 = K \cup \{Q_1\}$, $K_2 = K \cup \{Q_2\}$, then Q_1 and Q_2 are in the same set if and only if K_1 and K_2 are projectively equivalent under type of lines.
The projective plane PG(2,8) contains 73 points, 73 lines , 9 points on every line and 9 lines through every point. Any line in PG(2,8)

Table: Points and lines of PG(2,8)

I	P_i	L_i
1	1 0 0	2 10 18 26 34 42 50 58 66
2	0 1 0	1 10 11 12 13 14 15 16 17
3	1 1 0	3 10 19 28 37 46 55 64 73
4	2 1 0	8 10 24 27 41 44 54 61 71
5	3 1 0	6 10 22 31 35 49 53 60 72
6	4 1 0	5 10 21 32 39 43 52 65 70
7	5 1 0	9 10 25 29 38 48 51 63 68
8	6 1 0	4 10 20 30 40 47 57 59 69
9	7 1 0	7 10 23 33 36 45 56 62 67
10	0 0 1	1 2 3 4 5 6 7 8 9
11	1 0 1	2 11 19 27 35 43 51 59 67
12	2 0 1	2 16 24 32 40 48 56 64 72
13	3 0 1	2 14 22 30 38 46 54 62 70
14	4 0 1	2 13 21 29 37 45 53 61 69
15	5 0 1	2 17 25 33 41 49 57 65 73
16	6 0 1	2 12 20 28 36 44 52 60 68
17	7 0 1	2 15 23 31 39 47 55 63 71
18	0 1 1	1 18 19 20 21 22 23 24 25

I	P_i	L_i
19	1 1 1	3 11 18 29 36 47 54 65 72
20	2 1 1	8 16 18 33 35 46 52 63 69
21	3 1 1	6 14 18 27 39 45 57 64 68
22	4 1 1	5 13 18 31 40 44 51 62 73
23	5 1 1	9 17 18 30 37 43 56 60 71
24	6 1 1	4 12 18 32 38 49 55 61 67
25	7 1 1	7 15 18 28 41 48 53 59 70
26	0 2 1	1 58 59 60 61 62 63 64 65
27	1 2 1	4 11 21 31 41 46 56 58 68
28	2 2 1	3 16 25 30 39 44 53 58 67
29	3 2 1	7 14 19 29 40 49 52 58 71
30	4 2 1	8 13 23 28 38 43 57 58 72
31	5 2 1	5 17 22 27 36 48 55 58 69
32	6 2 1	6 12 24 35 37 47 51 58 70
33	7 2 1	9 15 20 32 33 45 54 58 73
34	0 3 1	1 42 43 44 45 46 47 48 49
35	1 3 1	5 11 20 33 38 42 53 64 71
36	2 3 1	9 16 19 31 36 42 57 61 70
37	3 3 1	3 14 23 32 41 42 51 60 69
38	4 3 1	7 13 24 30 35 42 55 65 68
39	5 3 1	6 17 21 28 40 42 54 63 67
40	6 3 1	8 12 22 29 39 42 56 59 73
41	7 3 1	4 15 25 27 37 42 52 62 72
42	0 4 1	1 34 35 36 37 38 39 40 41
43	1 4 1	6 11 23 30 34 48 52 61 73
44	2 4 1	4 16 22 28 34 45 51 65 71
45	3 4 1	9 14 21 33 34 44 55 59 72
46	4 4 1	3 13 20 27 34 49 56 63 70
47	5 4 1	8 17 19 32 34 47 53 62 68
48	6 4 1	7 12 25 31 34 43 54 64 69
49	7 4 1	5 15 24 29 34 46 57 60 67

i	P_i	L_i								
50	0 5 1	1	66	67	68	69	70	71	72	73
51	1 5 1	7	11	22	32	37	44	57	63	66
52	2 5 1	6	16	20	29	41	43	55	62	66
53	3 5 1	5	14	25	28	35	47	56	61	66
54	4 5 1	4	13	19	33	39	48	54	60	66
55	5 5 1	3	17	24	31	38	45	52	59	66
56	6 5 1	9	12	23	27	40	46	53	65	66
57	7 5 1	8	15	21	30	36	49	51	64	66
58	0 6 1	1	26	27	28	29	30	31	32	33
59	1 6 1	8	11	25	26	40	45	55	60	70
60	2 6 1	5	16	23	26	37	49	54	59	68
61	3 6 1	4	14	24	26	36	43	53	63	73
62	4 6 1	9	13	22	26	41	47	52	64	67
63	5 6 1	7	17	20	26	39	46	51	61	72
64	6 6 1	3	12	21	26	35	48	57	62	71
65	7 6 1	6	15	19	26	38	44	56	65	69
66	0 7 1	1	50	51	52	53	54	55	56	57
67	1 7 1	9	11	24	28	39	49	50	62	69
68	2 7 1	7	16	21	27	38	47	50	60	73
69	3 7 1	8	14	20	31	37	48	50	65	67
70	4 7 1	6	13	25	32	36	46	50	59	71
71	5 7 1	4	17	23	29	35	44	50	64	70
72	6 7 1	5	12	19	30	41	45	50	63	72
73	7 7 1	3	15	22	33	40	43	50	61	68

The Construction of the projectively Distinct (k,4) – Arcs in PG(2,8).

Let A ={1,2,3,10,19} be the set of the reference and the unite points in PG(2,8) , no three of them are collinear.

The distinct (k,4) – arcs can be constructed by adding to A in each time one point from the remaining 68 points of the projective plane PG(2,8) as follows :

$A_1 = A \cup \{4\}$, $A_2 = A \cup \{5\}$, ……….., $A_{68} = A \cup \{73\}$.

						19	10	3	2	1
Incomplete					19	10	4	3	2	1
Incomplete					19	10	5	3	2	1
Incomplete					19	10	5	4	2	1
Incomplete				19	11	10	4	3	2	1
Incomplete				19	12	10	4	3	2	1
Incomplete				28	19	10	4	3	2	1
Incomplete				29	19	10	4	3	2	1
Incomplete				19	11	10	5	3	2	1
Incomplete				29	19	10	5	4	2	1
Incomplete			19	12	11	10	4	3	2	1
Incomplete			19	18	11	10	4	3	2	1
Incomplete			20	19	11	10	4	3	2	1
Incomplete			22	19	11	10	4	3	2	1
Incomplete			32	19	11	10	4	3	2	1
Incomplete			29	19	12	10	4	3	2	1
Incomplete			28	19	11	10	4	3	2	1
Incomplete			28	19	14	10	4	3	2	1
Incomplete			28	20	19	10	4	3	2	1
Incomplete			38	28	19	10	4	3	2	1
Incomplete			41	29	19	10	4	3	2	1
Incomplete			19	12	11	10	5	3	2	1
Incomplete			36	29	19	10	5	4	2	1
Incomplete		19	18	12	11	10	4	3	2	1
Incomplete		30	19	18	11	10	4	3	2	1
Incomplete		44	19	18	11	10	4	3	2	1
Incomplete		32	20	19	11	10	4	3	2	1
Incomplete		52	22	19	11	10	4	3	2	1
Incomplete		45	32	19	11	10	4	3	2	1
Incomplete		28	19	18	11	10	4	3	2	1
Incomplete		28	21	19	11	10	4	3	2	1
Incomplete		28	23	19	11	10	4	3	2	1
Incomplete		38	28	19	11	10	4	3	2	1
Incomplete		72	28	19	14	10	4	3	2	1
Incomplete		28	20	19	13	10	4	3	2	1
Incomplete		53	38	28	19	10	4	3	2	1
Incomplete		41	29	19	15	10	4	3	2	1
Incomplete		19	18	12	11	10	5	3	2	1
Incomplete		49	36	29	19	10	5	4	2	1
Incomplete	26	19	18	12	11	10	4	3	2	1
Incomplete	31	30	19	18	11	10	4	3	2	1
Incomplete	44	30	19	18	11	10	4	3	2	1
Incomplete	44	33	19	18	11	10	4	3	2	1
Incomplete	45	32	20	19	11	10	4	3	2	1
Incomplete	63	52	22	19	11	10	4	3	2	1
Incomplete	71	45	32	19	11	10	4	3	2	1
Incomplete	30	28	19	18	11	10	4	3	2	1
Incomplete	38	28	19	18	11	10	4	3	2	1
Incomplete	63	28	19	18	11	10	4	3	2	1

Incomplete				38	28	21	19	11	10	4	3	2	1
Incomplete				53	28	23	19	11	10	4	3	2	1
Incomplete				72	39	28	19	14	10	4	3	2	1
Incomplete				28	26	20	19	13	10	4	3	2	1
Incomplete				20	19	18	12	11	10	5	3	2	1
Incomplete				56	49	36	29	19	10	5	4	2	1
Incomplete			27	26	19	18	12	11	10	4	3	2	1
Incomplete			38	31	30	19	18	11	10	4	3	2	1
Incomplete			44	31	30	19	18	11	10	4	3	2	1
Incomplete			44	33	30	19	18	11	10	4	3	2	1
Incomplete			45	44	33	19	18	11	10	4	3	2	1
Incomplete			70	44	33	19	18	11	10	4	3	2	1
Incomplete			63	52	42	22	19	11	10	4	3	2	1
Incomplete			38	30	28	19	18	11	10	4	3	2	1
Incomplete			39	30	28	19	18	11	10	4	3	2	1
Incomplete			63	30	28	19	18	11	10	4	3	2	1
Incomplete			63	38	28	19	18	11	10	4	3	2	1
Incomplete			44	38	28	21	19	11	10	4	3	2	1
Incomplete			53	38	28	23	19	11	10	4	3	2	1
Incomplete			72	56	39	28	19	14	10	4	3	2	1
Incomplete			43	28	26	20	19	13	10	4	3	2	1
Incomplete			26	20	19	18	12	11	10	5	3	2	1
Incomplete			56	49	36	29	19	11	10	5	4	2	1
Incomplete		28	27	26	19	18	12	11	10	4	3	2	1
Incomplete		39	38	31	30	19	18	11	10	4	3	2	1
Incomplete		44	38	31	30	19	18	11	10	4	3	2	1
Incomplete		45	44	31	30	19	18	11	10	4	3	2	1
Incomplete		45	44	33	30	19	18	11	10	4	3	2	1
Incomplete		70	44	33	30	19	18	11	10	4	3	2	1
Incomplete		70	45	44	33	19	18	11	10	4	3	2	1
Incomplete		69	63	52	42	22	19	11	10	4	3	2	1
Incomplete		39	38	30	28	19	18	11	10	4	3	2	1
Incomplete		44	38	30	28	19	18	11	10	4	3	2	1
Incomplete		63	38	30	28	19	18	11	10	4	3	2	1
Incomplete		41	39	30	28	19	18	11	10	4	3	2	1
Incomplete		63	40	38	28	19	18	11	10	4	3	2	1
Incomplete		44	40	38	28	21	19	11	10	4	3	2	1
Incomplete		53	40	38	28	23	19	11	10	4	3	2	1
Incomplete		27	26	20	19	18	12	11	10	5	3	2	1
Incomplete	36	28	27	26	19	18	12	11	10	4	3	2	1
Incomplete	48	44	38	31	30	19	18	11	10	4	3	2	1
Incomplete	70	45	44	31	30	19	18	11	10	4	3	2	1
Incomplete	52	45	44	33	30	19	18	11	10	4	3	2	1
Incomplete	70	45	44	33	30	19	18	11	10	4	3	2	1
Incomplete	70	45	44	33	32	19	18	11	10	4	3	2	1
Incomplete	45	39	38	30	28	19	18	11	10	4	3	2	1
Incomplete	56	39	38	30	28	19	18	11	10	4	3	2	1
Incomplete	63	44	38	30	28	19	18	11	10	4	3	2	1
Incomplete	71	41	39	30	28	19	18	11	10	4	3	2	1
Incomplete	63	44	40	38	28	19	18	11	10	4	3	2	1

Incomplete								63	53	40	38	28	19	18	11	10	4	3	2	1
Incomplete								53	44	40	38	28	21	19	11	10	4	3	2	1
Incomplete								28	27	26	20	19	18	12	11	10	5	3	2	1
Incomplete							38	36	28	27	26	19	18	12	11	10	4	3	2	1
Incomplete							71	48	44	38	31	30	19	18	11	10	4	3	2	1
Incomplete							71	70	45	44	31	30	19	18	11	10	4	3	2	1
Incomplete							70	52	45	44	33	30	19	18	11	10	4	3	2	1
Incomplete							71	52	45	44	33	30	19	18	11	10	4	3	2	1
Incomplete							70	53	45	44	33	32	19	18	11	10	4	3	2	1
Incomplete							71	70	45	44	33	32	19	18	11	10	4	3	2	1
Incomplete							56	45	39	38	30	28	19	18	11	10	4	3	2	1
Incomplete							63	56	39	38	30	28	19	18	11	10	4	3	2	1
Incomplete							71	52	41	39	30	28	19	18	11	10	4	3	2	1
Incomplete							63	44	40	38	33	28	19	18	11	10	4	3	2	1
Incomplete							63	53	44	40	38	28	19	18	11	10	4	3	2	1
Incomplete							53	49	44	40	38	28	21	19	11	10	4	3	2	1
Incomplete							38	28	27	26	20	19	18	12	11	10	5	3	2	1
Incomplete						39	38	36	28	27	26	19	18	12	11	10	4	3	2	1
Incomplete						71	70	52	45	44	33	30	19	18	11	10	4	3	2	1
Incomplete						63	56	52	39	38	30	28	19	18	11	10	4	3	2	1
Incomplete						63	45	44	40	38	33	28	19	18	11	10	4	3	2	1
Incomplete						70	63	44	40	38	33	28	19	18	11	10	4	3	2	1
Incomplete						66	53	49	44	40	38	28	21	19	11	10	4	3	2	1
Incomplete					40	39	38	28	27	26	20	19	18	12	11	10	5	3	2	1
Incomplete					45	39	38	36	28	27	26	19	18	12	11	10	4	3	2	1
Incomplete					70	66	53	49	44	40	38	28	21	19	11	10	4	3	2	1
Incomplete				45	40	39	38	28	27	26	20	19	18	12	11	10	5	3	2	1
Incomplete				53	45	39	38	36	28	27	26	19	18	12	11	10	4	3	2	1
Incomplete			47	45	40	39	38	28	27	26	20	19	18	12	11	10	5	3	2	1
Incomplete			56	53	45	39	38	36	28	27	26	19	18	12	11	10	4	3	2	1
Incomplete		56	47	45	40	39	38	28	27	26	20	19	18	12	11	10	5	3	2	1
Complete		63	56	53	45	39	38	36	28	27	26	19	18	12	11	10	4	3	2	1
Complete	62	56	47	45	40	39	38	28	27	26	20	19	18	12	11	10	5	3	2	1

Results

From the above results , the number of the distinct (k,4)–arcs , k =5,6,7,8,9,10,11,….,18 are All of these arcs are incomplete . Finally , we obtain two complete (19,4) – arcs and(20,4) – arcs.

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