Approximated Methods for Linear Delay Differential Equations Using Weighted Residual Methods

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Abstract:

The main work of this paper is devoted to a new technique of constructing approximated solutions for linear delay differential equations using the basis functions power series functions with the aid of Weighted residual methods (collocations method, Galerkin's method and least square method).

Keywords: linear Delay Differential Equation ; Weighted residual methods.

Introduction:

The delay differential ordinary equation "DDE" is an equation in an unknown function y (t) and some of its derivatives are evaluated at arguments that differ in any of fixed number of values $\tau_1, \tau_2, \dots, \tau_n$.

The general form of the n-th order DDE is given by

$$F(t, y(t), y(t-\tau_1), \dots, y(t-\tau_k), y'(t), y'(t-\tau_1), \dots, y'(t-\tau_k), y^n(t), \dots, y^n(t-\tau_k)) = 0 \quad \dots (1)$$

Where F is a given function and $\tau_1, \tau_2, \dots, \tau_k$ are given fixed positive numbers called the "time delay", [1].

In some literature, equ. (1) is called a differential equation with deviating argument [1,2], or an equation with time lag [3], or a differential difference equation ,or a functional differential equation [4].

The general linear first order delay differential equation with constant coefficients can be presented in the form

$$a_0 y'(t) + a_1 y'(t-\tau) + b_0 y(t) + b_1 y(t-\tau) = g(t) \quad \dots (2)$$

where g(t) is a given continuous function and τ is a positive constant and a_0, a_1, b_0, b_1 are constants.

We can distinguish between three types of delay differential equations, which are:

1. Retarded delay differential equation, which is obtained when: $(b_1 \neq 0, a_1 = 0)$ in equ.(2) the delay comes in y only and (2) takes the Form: $a_0 y'(t) + b_0 y(t) + b_1 y(t-\tau) = g(t)$
2. Neutral delay differential equation, which is obtained when: $(a_1 \neq 0, b_1 = 0)$ in equ.(2) the delay comes in y' only and (2) takes the form: $a_0 y'(t) + a_1 y'(t-\tau) + b_0 y(t) = g(t)$
3. Mixed delay differential equation, (sometimes called advanced type), which is obtained where $(a_1 \neq 0, b_1 \neq 0)$, which means a combination of the previous two types.

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Weighted Residual Methods:

We present these methods by considering the following functional equation [5].

L [y (x)] = g (x)   x $\in$ D   …(3)

Where L denotes an operator which maps a set of functions Y in to a set G such that y$\in$Y, g$\in$G and D is a prescribed domain.

The epitome of the expansion method is to approximate the solution y(x) of equ.(3) by the form

$$y_N(x) = \sum_{i=0}^{N} c_i \psi_i(x) \qquad \ldots. (4)$$

Where the parameters c_i and the functions $\psi_i(x)$ are chosen in our work as:

The Power Series Functions:

This basis function can be defined as

$$\psi_i(x) = x^i$$

An approximate solution $y_N(x)$ given by (4) will not, in general, satisfy equ.(3) exactly and associated with such an approximate solution is the residual defined by

E (x) = L [y_N (x)] - g (x)   …. (5)

The residual E (x) depends on x as well as on the way that the parameter c_i are chosen.

It is obvious that when E (x) = 0, then the exact solution is obtained which is difficult to be achieved, therefore we shall try to minimize E (x) in some sense.

In the weighted residual method the unknown parameters c_i are chosen to minimize the residual E (x) by setting its weighted integral equal to zero, i.e.

$$\int_D w_j E(x) dx = 0 \qquad j = 0,1,\ldots.,N \qquad \ldots. (6)$$

Where w_j is prescribed weighting function, the technique based on (6) is called weighted residual method. Different choices of w_j yield different methods with different approximate solutions.

The weighted residual methods that will be discussed in this work are:

1. **Collocation Method**
2. **Galerkin's Method**
3. **Least Square Method**

1 Collocation Method:

It is a simple method for obtaining a linear approximation y_N [7]

In this method the weighting function is chosen to be

$$w_j = \delta(x - x_j) \qquad \ldots. (7)$$

Where the fixed points $x_j \in D, \quad j = 0,1,\ldots.,N$ are called collocation points. Here Dirac's delta function $\delta(x - x_j)$ is defined as

$$\delta(x - x_j) = \begin{cases} 1 & if \ \ x = x_j \\ 0 & else \end{cases}$$

Inserting equ.(7) in equ.(6) gives

$$\int_D w_j E(x_j) dx = \int_D \delta(x - x_j) E(x_j) dx = \int_{x_j^-}^{x_j^+} \delta(x - x_j) E(x_j) dx$$
$$= E(x_j) \int_{x_j^-}^{x_j^+} \delta(x - x_j) dx = E(x_j) = 0 \qquad j = 0,1,\ldots,N$$

…(8)

Equ.(8) will provide us with (N+1) simultaneous equations to determine the parameters c_i's. Moreover the distribution of collocation points on D is arbitrary, however, in practice we distribute the collocation points uniformly on D.

2 Galerkin' s Method: [7]

Galerkin method is the most efficient of the weighted residual methods. This method makes the residual E (x) of equ. (6) orthogonal to (N+1) given linearly independent function on the domain D.

The weighting function w_j is defined as:

$$w_j(x) = \frac{\partial y_N(x)}{\partial c_j} \qquad j = 0,1,\ldots,N$$

where $y_N(x)$ is the approximate solution of the problem, then equ.(6) becomes.

$$\int_D \frac{\partial y_N(x)}{\partial c_j} E(x)dx = 0 \qquad j = 0,1,\ldots,N$$

...(9)

and will provide (N+1) simultaneous equations for determination of c_0 , c_1 , …. c_N.

3 Least Square Method:[7]

In this method weighting function w_j is defined as:

$$w_j(x) = \frac{\partial E(x)}{\partial c_j} \qquad j = 0,1,\ldots,N$$

and due to equ.(6) we have

$$\int_D \frac{\partial E(x)}{\partial c_j} E(x)dx = 0 \qquad j = 0,1,\ldots.,N$$

…(10)

then the square of the error on the domain D is

$$J = \int_D [E(x)]^2 dx$$

Now, we compute the derivatives

$$\frac{\partial J}{\partial c_j} = 2\int_D E(x)\frac{\partial E(x)}{\partial c_j}dx \qquad j = 0,1,\ldots.,N$$

…(11)

it implies from equ.(10),(11) that

$$\frac{\partial J}{\partial c_j} = 2\int_D E(x)\frac{\partial E(x)}{\partial c_j}dx = 0 \qquad j = 0,1,\ldots.,N$$

Therefore J is stationary and the square of residual E(x) attains its minimum

Solution of Linear DDE' s By Weighted Residual Methods:

The general form nth order non-homogenous non linear delay differential equations is

$$y^n(x) + \sum_{p=0}^{n} a_p y^p(x) + \sum_{m=0}^{n-1} b_m y^m(x-\tau) + \sum_{l=0}^{n} d_l y^l(x-\tau) = g(x)$$

.... (12)

where τ is a prescribed constant and a_p , b_m and d_l are constants with initial condition

$$y(0) = y_0, \quad y'(0) = y_1, \ldots., y^{(n-1)}(0) = y_{n-1}, \quad y^n(0) = y_n$$

Using operator forms this equation can be written as

L [y] = g (x) …(13)

where the operator L is defined as

$$L[y] = y^n(x) + \sum_{p=0}^{n} a_p y^p(x) + \sum_{m=0}^{n-1} b_m y^m(x-\tau) + \sum_{l=0}^{n} d_l y^l(x-\tau)$$

The unknown function y(x) is approximated by the form

$$y_N(x) = \sum_{j=0}^{N} c_j \psi_j(x) \qquad \ldots(14)$$

Substituting equ.(14) in equ.(13) we get

$$L[\,y_N\,] = g(x) + E_N(x)$$

Where

$$L[y_N(x)] = y_N^n(x) + \sum_{p=0}^{n} a_p y_N^p(x) + \sum_{m=0}^{n-1} b_m y_N^m(x-\tau) + \sum_{l=0}^{n} d_l y_N^l(x-\tau)$$

Which we have the residue equation

$$E_N(x) = L[y_N(x)] - g(x) \qquad \dots (15)$$

Substituting the equ.(14) in equ.(15), we get

$$E_N(x) = L\left[\sum_{j=0}^{N} c_j \psi_j\right] - g(x) = \sum_{j=0}^{N} c_j L[\psi_j(x)] - g(x) \qquad \dots (16)$$

Evidently from equ.(6) that

$$\int_D w_j E_N(x) = 0 \qquad \dots (17)$$

Inserting equ.(16) in equ.(17) yields

$$\int_D w_j\left(\sum_{j=0}^{N} c_j L[\psi_j(x)] - g(x)\right) = 0$$

$$\sum_{j=0}^{N} c_j \int_D w_j L[\psi_j(x)] = \int_D w_j g(x) \qquad \dots (18)$$

Where

$$L(\psi_j(x)) = \psi_j^n(x) + \sum_{p=0}^{n} a_p \psi_j^p(x-\tau) + \sum_{m=0}^{n-1} b_m \psi_j^m(x-\tau) + \sum_{l=0}^{n} d_l \psi_j^l(x-\tau)$$

For $j=0,1,\dots,N$ ... (19)

Introducing matrix k and vector h as

$$k_{ji} = \int_D w_j L(\psi_i(x))dx \qquad i = 0,1,\dots,N \qquad \dots (20)$$

$$h_j = \int_D w_j g(x)dx \qquad j = 0,1,\dots,N$$

We can write equ.(18) in compact form

$$K\,C = H \qquad \dots (21)$$

Where

$$C = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_N \end{pmatrix}$$

Finally, we have N+1 linear equations (21) for determing N+1 coefficient c_0, c_1, …., c_N.

1 UsingCollocation Method:

In this method we choose the collocation points x_0, x_1, …., x_N in the closed interval $[t_0, t_0 + \tau]$, such that

$x_i = 1 / (i+1)$ $\qquad i = 0, 1, \dots, N$

Hence we have

$E_N(x_i) = 0$ $\qquad i = 0, 1, \dots, N$

This leads to

$$\sum_{j=0}^{N} L(\psi_j(x_i))c_j = g(x_i) \quad i = 0,1,\dots,N \qquad \dots (22)$$

Hence equ.(22) can be seen as a system of (N+1) equations in the (N+1) unknown c_j , $j = 0, 1, \dots, N$

$$\begin{bmatrix} L(\psi_0(x_0)) & L(\psi_1(x_0)) & \dots & L(\psi_N(x_0)) \\ L(\psi_0(x_1)) & L(\psi_1(x_1)) & \dots & L(\psi_N(x_1)) \\ \vdots & & & \\ L(\psi_0(x_N)) & L(\psi_1(x_N)) & \dots & L(\psi_N(x_N)) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_N \end{bmatrix} = \begin{bmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_N) \end{bmatrix} \qquad \dots (23)$$

A computationally efficient way to calculate the value c_j is by solving the sysetm

KC=H

Where

$$K = \begin{bmatrix} L(\psi_0(x_0)) & L(\psi_1(x_0)) & \dots & L(\psi_N(x_0)) \\ L(\psi_0(x_1)) & L(\psi_1(x_1)) & \dots & L(\psi_{Nn}(x_1)) \\ \vdots & & & \\ L(\psi_0(x_N)) & L(\psi_1(x_N)) & \dots & L(\psi_N(x_N)) \end{bmatrix}, C = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_N \end{bmatrix}, H = \begin{bmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_N) \end{bmatrix} \qquad \dots (24)$$

For the coefficient c_j's which satisfies equ.(14) the approximate solution of equ.(12) will be given.

Algorithm (WC)

The approximate solution for DDE using collocation method can be briefly introduced in following steps.

Step 1: Put $x_i = 1 / (i+1)$ for i = 0, 1, …., N.

Step 2: Select ψ_j for j = 0, 1, …., N.

Step 3: Compute $L(\psi_j(x_i))$ using equ.(19) for i = 0, 1, …., N.

Step 4:Compute the matrices k and H by using equ.(24).

Step 5: Solve the system (23) for coefficient c_j's.

Step 6: Substitute c_j's in transforming form to obtain the approximate solution of y(x).

2 Using Galerkin's Method:

It is a simple technique for obtaining a linear approximation y_N , the weight function w_j in equ.(6) is defined as [7]

$$w_j(x) = \frac{\partial y_N(x)}{\partial c_j} \qquad i = 0,1,....,N$$

now since

$$y_N(x) = \sum_{i=0}^{N} c_i \psi_i$$

Therefore

$$w_j(x) = \frac{\partial}{\partial c_j} \sum_{i=0}^{N} c_i \psi_i = \psi_j \quad , \quad j = 0,1,....,N$$

… (25)

Substituting equ.(25) in equ.(18) we get

$$\sum_{i=0}^{N} c_j \psi_j L(\psi_j) = \int_D \psi_j g(x), \quad j = 0,1,....,N$$

The linear system (21) becomes

$$\begin{bmatrix} \int_D \psi_0 L(\psi_0)dx & \int_D \psi_0 L(\psi_1)dx & .... & \int_D \psi_0 L(\psi_N)dx \\ \int_D \psi_1 L(\psi_0)dx & \int_D \psi_1 L(\psi_1)dx & .... & \int_D \psi_1 L(\psi_N)dx \\ \vdots & & & \\ \int_D \psi_N L(\psi_0)dx & \int_D \psi_N L(\psi_1)dx & .... & \int_D \psi_N L(\psi_N)dx \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_N \end{bmatrix} = \begin{bmatrix} \int_D \psi_0 g dx \\ \int_D \psi_1 g dx \\ \vdots \\ \int_D \psi_N g dx \end{bmatrix}$$

... (26)

Hence solving the system KC = H for coefficient c_j's.

Where

$$K = \begin{bmatrix} \int_D \psi_0 L(\psi_0)dx & \int_D \psi_0 L(\psi_1)dx & .... & \int_D \psi_0 L(\psi_N)dx \\ \int_D \psi_1 L(\psi_0)dx & \int_D \psi_1 L(\psi_1)dx & .... & \int_D \psi_1 L(\psi_N)dx \\ \vdots & & & \\ \int_D \psi_N L(\psi_0)dx & \int_D \psi_N L(\psi_1)dx & .... & \int_D \psi_N L(\psi_N)dx \end{bmatrix}, \quad ... (27)$$

$$C = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_N \end{bmatrix} \quad \& \quad H = \begin{bmatrix} \int_D \psi_0 g dx \\ \int_D \psi_1 g dx \\ \vdots \\ \int_D \psi_N g dx \end{bmatrix}$$

Substitute the coefficient in equ.(13) to obtain approximate solution.

Algorithm (WG)

This method can be summarized in the following steps

Step 1: Select ψ_j for j = 0, 1, …., N.

Step 2: Compute $L(\psi_j)$ by using equ.(19) for j = 0, 1, …., N.

Step 3: Compute the matrices K and H by using equ.(27).

Step 4: Solve the system (26) for coefficients c_j's .

Step 5:Substitute c_j's in transforming form to obtain the approximate solution of y(x).

3Using Least Square Method:

The standard least squares approach is to attempt to minimize the sum of the squares of the residual at each point in the domain D.

Here this minimization of [7]

$$I(\bar{c}) = \frac{1}{2}\int_D [E_N(x)]^2 dx$$

Where $\bar{c} = (c_0, c_1, \ldots, c_N)$

This technique is to solve the (N+1) normal equation given by

$$\frac{\partial I}{\partial c_i} = \int_D E_n(x)\frac{\partial E_n(x)}{\partial c_i}dx = 0 \quad , \quad i = 0,1,\ldots,N$$

Substituting eq.(18), yields

$$\int_D \sum_{j=0}^{N} c_j L(\psi_i(x))L(\psi_j(x))dx = \int_D L(\psi_i(x))g(x)dx$$

$$i = 0,1,\ldots N$$

The linear system (21) becomes

$$\begin{bmatrix} \int_D L(\psi_0)L(\psi_0)dx & \int_D L(\psi_0)L(\psi_1)dx & \ldots & \int_D L(\psi_0)L(\psi_n)dx \\ \int_D L(\psi_1)L(\psi_0)dx & \int_D L(\psi_1)L(\psi_1)dx & \ldots & \int_D L(\psi_1)L(\psi_n)dx \\ \vdots & & & \\ \int_D L(\psi_n)L(\psi_0)dx & \int_D L(\psi_n)L(\psi_1)dx & \ldots & \int_D L(\psi_n)L(\psi_n)dx \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_N \end{bmatrix} = \begin{bmatrix} \int_D L(\psi_0)gdx \\ \int_D L(\psi_1)gdx \\ \vdots \\ \int_D L(\psi_n)gdx \end{bmatrix}$$

... (28)

Where

$$K = \begin{bmatrix} \int_D L(\psi_0)L(\psi_0)dx & \int_D L(\psi_0)L(\psi_1)dx & \ldots & \int_D L(\psi_0)L(\psi_n)dx \\ \int_D L(\psi_1)L(\psi_0)dx & \int_D L(\psi_1)L(\psi_1)dx & \ldots & \int_D L(\psi_1)L(\psi_n)dx \\ \vdots & & & \\ \int_D L(\psi_n)L(\psi_0)dx & \int_D L(\psi_n)L(\psi_1)dx & \ldots & \int_D L(\psi_n)L(\psi_n)dx \end{bmatrix}$$

$$C = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_N \end{bmatrix} \quad \& \quad H = \begin{bmatrix} \int_D L(\psi_0)gdx \\ \int_D L(\psi_1)gdx \\ \vdots \\ \int_D L(\psi_n)gdx \end{bmatrix}$$

… (29)

Hence the system of N+1 linear equations with N+1 unknown N+1 coefficients c_0 , c_1 , …. , c_N will construct.

By solving this system, the values of c_j's will be found and then substituted in equ.(13) to obtain the approximate solution.

Algorithm (WL)

This method can be summarized in the following steps

Step 1: Select ψ_j for j = 0, 1, …., N.

Step 2: Compute $L(\psi_j)$ & $L(\psi_i)$ by using equ.(19) for i = 0,1,….,N , j = 0, 1, …., N.

Step 3: Compute the matrices K and H by using equ.(29).

Step 4: Solve the system (28) for coefficients c_j's .

Step 5: Substitute c_j's in equ.(13) to obtain the approximate solution.

Transforming the Interval:

Sometimes it is necessary to take a problem state on interval [a, b], then we can convert the variable so that the problem is reformulated on [-1, 1], as follows $x = \frac{(b-a)t + b + a}{2}$,

so that $dx = \frac{(b-a)}{2}dt$

Then $\int_a^b g(x)dx = \frac{b-a}{2}\int_{-1}^{1} g\left(\frac{(b-a)t + b + a}{2}\right)dt$

Conversely, transform the limits of integration from [-1, 1] to [a, b] using

$t = 2\left(\frac{x-a}{b-a}\right) - 1$,

So that $dt = \frac{2}{b-a}dx$

Then

$$\int_a^b g(t)dt = \frac{2}{b-a}\int_a^b g\left(2\left(\frac{x-a}{b-a}\right)-1\right)dx$$

Test Example:

Example (1):

Consider the following first order delay differential equation of retarded type:

$$y'(t) = -2y(t) + y(t-\frac{\pi}{2}) + \sin t \qquad t \geq 0$$

With initial function

$$y(t) = -0.25 \qquad -0.25 \leq t \leq 0$$

Which has exact solution

$$y(t) = -\frac{1}{4}\cos t + \frac{1}{4}\sin t \qquad 0 \leq t \leq \frac{1}{4}$$

Assume the approximate solution in the form

$$y_N(t) = \sum_{j=1}^{N} a_j \psi_j(2x-1)$$

Table (1) lists the least square errors obtained by running the programs for algorithms (WC, WG and WL) with different values to find approximate solution of above equation.

Table (1) The approximated values of y(t) against the exact value for N=5 and Δ t=10

t	Exact	Power series function		
		WC	WG	WL
0.0	-0.2500	-0.2500	-0.2500	-0.2500
0.1	-0.2238	-0.2238	-0.2238	-0.2238
0.2	-0.1953	-0.1954	-0.1953	-0.1953
0.3	-0.1650	-0.1650	-0.1650	-0.1650
0.4	-0.1329	-0.1330	-0.1329	-0.1329
0.5	-0.0995	-0.0996	-0.0995	-0.0996
0.6	-0.0652	-0.0653	-0.0652	-0.0652
0.7	-0.0302	-0.0303	-0.0302	-0.0302
0.8	0.0052	0.0051	0.0052	0.0051
0.9	0.0404	0.0403	0.0404	0.0404
1.0	0.0753	0.0752	0.0753	0.0753
L.S.E.		2.3e-3	2.2e-4	3.0e-3

Example (2):

Consider the following first order delay differential equation of neutral type:

$$y'(t) = 1 - y'(t - \frac{y(t)^2}{4})$$

With initial function

$$y(t) = 1 + t \qquad 0 \leq t \leq 1$$

Which has exact solution

$$y(t) = 1 + \frac{t}{2} + \frac{t^2}{4} \qquad 0 \leq t \leq 1$$

Assume the approximate solution as in the form

$$y_N(t) = \sum_{j=1}^{N} a_j \psi_j(2x-1)$$

Table (2) lists the least square errors obtained by running the programs for algorithms (WC, WG and WL) with different values to find approximate solution of above equation.

Table (2)The approximated values of y(t) against the exact value for N=5 and Δ t=10

t	Exact	Power series function		
		WC	WG	WL
0.0	1	1	1	1
0.1	1.0525	1.0525	1.0525	1.0525
0.2	1.1100	1.1100	1.1100	1.1100
0.3	1.1725	1.1725	1.1725	1.1725
0.4	1.2400	1.2400	1.2400	1.2400
0.5	1.3125	1.3125	1.3125	1.3125
0.6	1.3900	1.3900	1.3900	1.3900
0.7	1.4725	1.4725	1.4725	1.4725
0.8	1.5600	1.5600	1.5600	1.5600
0.9	1.6525	1.6525	1.6525	1.6525
1.0	1.7500	1.7500	1.7500	1.7500
L.S.E.		9.8e-32	9.8e-32	9.8e-32

Conclusions:

The approximate solution using two different types of basis function power series functions with the aid of weighted residual methods has been obtained for two examples. Good results were obtained and the following conclusion points are drawn.

1. In terms of the results, Galerkiń s method gives more accurate solution than collocation method and least square method, see tables (1)

The good approximation depends on:
The number of the orthogonal polynomials
The number of the root ́s x_i ́ s of the orthogonal polynomials, i.e. as it increases the L.S.E. approaches to zero.

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الحلول التقريبية للمعادلات التفاضلية التباطؤية الخطية باستخدام طرق البواقي الموزونة

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الخلاصة:

تم تقديم أسلوب جديد لتطوير الحلول تقريبية للمعادلات التفاضلية التباطوية الخطية مستخدما دوال المتسلسلة القوة معتمدا على طرق البواقي الموزون والمتضمنة طريقة التواضيع ،طريقة كالكن وطريقة التقريبات الصغرى لإيجاد المعلمات المتضمنة في الطريقة التوسيعية.