New Condition of Automatic Continuity of Dense Range Homomorphisms on Jordan – Banach Algebras

RUQAYAH N. BALO
Department of Mathematics
College of Education
University of Mosul

NADIA A. ABDULRAZAQ
Department of Mathematics
College of Education
University of Mosul

DUAA F. ABDULLAH
Department of Mathematics
College of Education
University of Mosul

Received 23/12/2012 **Accepted** 06/03/2013

الملخص

المسألة المفتوحة التالية تنص على انه ، اذا كان $T : A \longrightarrow B$ تطبيق متشاكل ذا مستقر كثيف بين جبور باناخ A وB بحيث ان B شبه بسيطة، هل ان T مستمرة تلقائيا؟ (أنظر [1])

في [3] اعطي حلا جزئيا للمسألة اعلاه كالآتي:-

ليكن A وB جبور فريجيت بحيث ان B شبه بسيطة، نصف القطر الطيفي r_B مستمر على B ونصف القطر الطيفي r_A مستمر عند الصفر. اذا كان $T : A \longrightarrow B$ تطبيق متشاكل ذا مستقر كثيف ، عندئذ T مستمرة تلقائيا.

في هذا البحث برهنا النتيجة التالية:

اذا كان $T : A^+ \longrightarrow B^+$ تطبيق متشاكل ذا مستقر كثيف بين جبور– جوردن باناخ A^+ و B^+ بحيث ان B^+ شبه بسيطة ، نصف القطر الطيفي r_B+ مستمر على B^+ ونصف القطر الطيفي r_A+ مستمر عند الصفر، عندئذ T مستمرة تلقائياً.

ABSTRACT

The following open problem stated that, if $T: A \longrightarrow B$ is a dense range homomorphism between Banach algebras A and B such that B is semi-simple. Is T automatically continuous? (see [1]).

In [3] given a partial solution of the above problem as follows:

Let A and B be Fréchet algebras such that B is semi simple, the spectral radius r_B is continuous on B and the spectral radius r_A is continuous at zero. If $T : A \longrightarrow B$ is a dense range homomorphism, then T is automatically continuous.

In this paper, we prove the following result :

If $T : A^+ \longrightarrow B^+$ is a dense range homomorphism between Jordan – Banach algebras A^+ and B^+ such that B^+ is semi simple, the spectral radius r_B+ is continuous on B^+ and the spectral r_A+ is continuous at zero, then T is automatically continuous.

1. Introduction :

If A and B are Banach algebras, B is semi simple and $T: A \longrightarrow B$ is a dense range homomorphism, then the continuity of T is a long – standing open problem.

This is perhaps the most interesting open problem remains in automatic continuity theory for Banach algebras. (see [1]).

We recall that from [2], the radical of an algebra A, denoted by rad A, is the intersection of all maximal left (right) ideals in A. The algebra A is called semi simple if rad $A = \{0\}$. In [3], for the algebra A the spectrum of an element $x \in A$ is the set of all $\lambda \in \mathbb{C}$ such that $\lambda 1 - x$ is not invertible in A and is denoted by $Sp\,(x)$ (or by $SpA\,(x)$). Thus

$$Sp\,(x) = \{\ \lambda \in \mathbb{C} : \lambda\, 1 - x \notin Inv\,(A)\ \}.$$

Also let A be Banach algebra, then the spectral radius of x (with respect to A) is denoted by $r\,(x)$ (or $r_A\,(x)$) and is defined by the formula

$$r\,(x) = Sup\ \{\ |\lambda| : \lambda \in Sp\,(x)\ \}.$$

It is known that for any algebra A we have

$$rad\ A = \{\ x \in A : r_A\,(x\,y) = 0 \ \text{ for every y} \in A\ \}.$$

From [6], for X, Y normed spaces and T a linear mapping from X into Y, then the separating subspace $S\,(T)$ of T is defined as follows :

$$S\,(T) = \{\ y \in Y : \exists\ \{x_n\} \subseteq X, x_n \longrightarrow 0, Tx_n \longrightarrow y, \forall\ \ n \in N\ \}.$$

We recall that a complex Jordan algebra A is a non – associative and the product satisfies the identities $a\ b = b\ a$ and $(a\ b)\ a^2 = a(b\ a^2)$, for all a, b in A. A unital Jordan – Banach algebra is a Jordan algebra with a complete norm satisfying $\|x\ y\| \leq \|x\|\ \|y\|$, for $x, y \in A$, and $\|1\| = 1$. (see [4]). The well – known example of Jordan – Banach algebra is that if we take any Banach algebra A, then A^+ is a Jordan – Banach algebra with a product defined as follows :

$$a.b = \frac{1}{2}(ab+ba) \qquad \forall a,b \in A$$

So $(A^+,.)$ is Jordan – Banach algebra over a field F of characteri - stic ≠ 2. (see [5]).

In this paper, we prove that :

Let A^+ and B^+ be Jordan – Banach algebras such that B^+ is semi-simple, the spectral radius r_{B^+} is continuous on B^+ and the spectral radius r_{A^+} is continuous at zero. If T: $A^+ \longrightarrow B^+$ is a dense range homomorphism, then T is automatically continuous.

This is in fact an extension of the open problem from the associative case to the more general situation of Jordan – Banach algebras.

2. Fundamental Results :

In this section we prove our fundamental following results

Theorem 2.1 :

Let A^+ and B^+ be Jordan – Banach algebras and $T: A^+ \longrightarrow B^+$ a dense range homomorphism. Then the separating subspace $S(T)$ is a closed ideal of B^+.

Proof :

Clearly $S(T)$ is a closed linear subspace of $B+$. Let $y \in S(T)$ and $z \in B^+$. There exists a sequence $\{x_n\}$ in A^+ such that $x_n \longrightarrow 0$ and $Tx_n \longrightarrow y$. Moreover, $z = Tx$ for some $x \in A^+$. Hence if $x\, x_n = x_n\, x$ then $x.\, x_n \longrightarrow 0$ imply that $\frac{1}{2}(x\, x_n + x_n\, x) \longrightarrow 0$ and this imply that $x\, x_n \longrightarrow 0$

$$\text{and } T(x.\, x_n) = \frac{1}{2}\, T(x\, x_n + x_n\, x).$$
$$= \frac{1}{2}\,(Tx\, Tx_n + Tx_n\, Tx).$$
$$= Tx\, Tx_n \longrightarrow z\, y \text{ and so } zy \in S(T).$$

Similarly $yz \in S(T)$. Therefore, $S(T)$ is an ideal in B^+.

Now, $B^+ = \overline{T(A^+)}$, for $y \in S(T)$ and $z \in B+ = \overline{T(A^+)}$, there exist sequences $\{x_n\}$ in A^+ and $\{z_n\}$ in $T(A^+)$ such that $x_n \longrightarrow 0$ in A^+, $z_n \longrightarrow z$ and $Tx_n \longrightarrow y$ in B^+. Since $y\, z_n$, $z_n\, y \in S(T)$ and $y\, z_n \longrightarrow y\, z$, $z_n\, y \longrightarrow z\, y$ it follows that $y\, \overline{z, z\, y} \in S(T) = S(T)$. ■

Theorem 2.2 :

Let A^+ and B^+ be Jordan – Banach algebras such that B^+ is semi-simple, the spectral radius r_{B^+} is continuous on $B+$ and the spectral radius r_A+ is continuous at zero. If $T : A^+ \longrightarrow B^+$ is a dense range homomorphism, then T is automatically continuous.

Proof :

If $xy = yx$ for all x and y in A^+ then

$$T(x.\,y) = T(y.\,x) = \frac{1}{2}\; T(x\,y + y\,x)$$
$$= \frac{1}{2}\;(Tx\,Ty + Ty\,Tx)$$
$$= Tx\,Ty$$
$$= Ty\,Tx$$

If x is an arbitrary quasi – Invertible element of A^+, then there exists an element y in A^+ such that $xy = yx = x+y$. It follows that

$$TxTy = TyTx = Tx+Ty$$

That is Ty is Quasi – invertible element of Tx. Hence, T reduces the spectrum of elements. so,

$$r_B+ (Tx)\; r_A+ (x)$$

For every $y \in S\,(\,T\,)$ there exists a sequence $\{\,x_n\,\}$ in A^+ such that $x_n \longrightarrow 0$ in A^+ and $Tx_n \longrightarrow y$ *in* B^+. Since $r_B+ (Tx) \leq r_A+ (x)$ for every $x \in A^+$ *and* r_A+ is continuous by assumption, we have $r_A+ (x_n) \longrightarrow 0$, then $r_B+ (Tx_n) \longrightarrow 0$. On the other hand, again by continuity of r_B+ we have $r_B+ (Tx_n) \longrightarrow r_B+ (y)$. Hence

$$r_B+ (y) = 0 \;\ldots\ldots\; (1)$$

Since $T : A^+ \longrightarrow B^+$ is a dense range homomorphism, by Theorem (2.1) $S\,(T)$ is an ideal in $B+$. Thus for every $z \in B^+$, $y\,z \in S\,(T)$. By (1) we get $r_B+ (y\,z) = 0$.

Since rad $B^+ = \{y \in B^+ : r_B+ (y\,z) = 0$ for every $z \in B^+\}$, therefore $y \in$ rad B^+. So $S\,(T) \subseteq$ rad B^+. Since B^+ is semi-simple, we have $S\,(T) = \{0\}$ and so T is continuous by the closed graph theorem.

Acknowledgement

I would like to thank the referees for valuable conversations and comments.

REFERENCES

[1] Bachar, J. M., **Radical Banach Algebras and Automatic Continuity,** Lecture Notes in Mathematics 975, Springer – Verlag, Berlin Heidelberg New York, 1983.

[2] Dales, H. G., **Banach Algebras and Automatic Continuity,** London Mathematics Society Monographs 24, Clarendon Press, Oxford, 2000.

[3] Honary.T.GH., Automatic Continuity of Homomorphisms Betwe-en Banach Algebras and Fréchet Algebras,Bull. Iranian Math.Soc., 32 (2006), No.2, PP. 1-11.

[4] Maouche, A., Spectrum Preserving Linear Mappings For Scattered Jordan – Banach Algebras, Proc. Amer. Math. Soc., 127(1999), NO.11, PP.3187 – 3190.

[5] Schafer, R. D., **An Introduction to Non – Associative Algebras**, Massachusetts Institute of Technology, Renewed 1994.

[6] Sinclair, A. M., **Automatic Continuity of Linear Operators,** London Mathematical Society, Lecture Note Series 21, Cambridge University Press, Cambredge, 1976.