Some Statistical Characteristics Depending on the Maximum Variance of Solution of Two Dimensional Stochastic Fredholm Integral Equation contains Two Gamma Processes

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Abstract

In this paper, we find the two solutions of two dimensional stochastic Fredholm integral equations contain two gamma processes differ by the parameters in two cases and equal in the third are solved by the Adomain decomposition method. As a result of the solutions probability density functions and their variances at the time t are derived by depending upon the maximum variances of each probability density function with respect to the three cases. The auto covariance and the power spectral density functions are also derived. To indicate which of the three cases is the best, the auto correlation coefficients are calculated.

Keywords: Two Dimensional Stochastic Fredholm Integral Equations, Gamma Process, Adomain Decomposition Method

Introduction

In this paper, the solution of the Stochastic Fredholm integral equation, for one and two dimensions is found analytically by a new method (beginning of 1980's) for solving linear and nonlinear integral (differential) equations for various kinds has been proposed by G.Adomian, the so called Adomian decomposition method, [1].

After that, in recent years many researchers interested as a final aim either by studying the existence and uniqueness of the solution of one or more dimensional integral equations (Balachandran et al., 2005; Milton et al., 1972) [4,7] or to find by using different methods of the modified quadrature the numerical solutions of this kind of equations on some definite closed interval to study a comparison between the numerical solutions and their exact solutions (Huda, 2007; AL-Sadany, 2008) [5]. While Vahidi and Mokhtari (2008) [9] use the Adomian decomposition method to compare this method with the classical successive method for solving system of linear Fredholm integral equations and Biazar and Rangbar (2007) [3] studied the comparison between Newton's method and Adomian decomposition method for solving special Fredholm integral equations and Wahdan. M.M. (2011) [11] is interested not only in the solution of the supposing stochastic Fredholm integral equation but in concentrating in the derivation of many probability characteristics of the solution.

In this paper, for combining the integral equations as an important branch of mathematics with some Statistical characteristics Stochastic Fredholm integral equations with gamma processes differ by the parameters in two cases and equal in the third. Our aim is not only interesting in the solution of the supposing stochastic Fredholm integral equation but we concentrate ourselves in the derivation of many statistical characteristics, of this solution (mean, variance, autocovariance function and power spectral density function) that is by depending on the maximum variances.

1. Preliminary

The gamma process $\Gamma(w , \alpha ,\beta , t)$ is a lévy process whose marginal distribution at the time $t > 0$ is a gamma distribution with mean = $\frac{\alpha t}{\beta}$ and variance = $\frac{\alpha t}{\beta^2}$

$$\Gamma(\omega,\alpha,\beta,t) = \frac{(\beta)^{\alpha t}}{\Gamma(\alpha t)}\, w^{\alpha t-1}\, e^{-\beta w} \quad ,w > 0, t > 0 \qquad \dots(1)$$

Where the parameter α controls the rate of jump arrivals (shape parameter) and the scaling parameter β inversely the jump size (scale parameter). [10]

Moreover, the gamma process has the following properties:

1. $\Gamma(w,\alpha,\beta,0) = 0$.

2. For any $0 \le t_0 \le t_1 < \dots < t_n < \infty$, $n \ge 1$ $\Gamma(w,\alpha,\beta,t_1) - \Gamma(w,\alpha,\beta,t_0),\cdots,\Gamma(w,\alpha,\beta,t_n) - \Gamma(w,\alpha,\beta,t_{n-1})$ are independent increments.

3. For any $0 \le s < t$; $\Gamma(w,\alpha,\beta,t) - \Gamma(w,\alpha,\beta,s)$ have the same distribution as $\Gamma(w,\alpha,\beta,t-s)$.

Now, we consider the following two dimensional systems of stochastic Fredholm integral Equation 2:

$$Y_i(t,w) = \Gamma_i(\omega,\alpha_i,\beta_i,t) + \int_0^{\infty}\sum_{j=1}^{2} k_{ij}(w,s,t)Y_j(s,t)ds \,, i = 1,2 \qquad \dots(2)$$

where,

- $k_{ij}(w,s,t)$, i, j = 1, 2 are known stochastic kernels defined by $t > 0$, s □S, S is a co

and having respectively the supposing formulas Equation 3:

$$k_{1j}(t,s,w) = e^{-(s+wt)} \,, k_{2j}(t,s,w) = e^{-(s+wt^2)} \,, j = 1,2 \qquad \dots(3)$$

- $Y_j(s,t), j = 1,2$ are scalar functions defined for t>0, s>0. By substituting (1), (3) into (2), we get Equation 4:

$$\left.\begin{aligned} Y_1(t,w) &= \frac{(\beta_1)^{\alpha_1 t}}{\Gamma(\alpha_1 t)}\, w^{\alpha_1 t-1}\, e^{-\beta_1 w} + \int_0^\infty e^{-(s+wt)}\,(Y_1(t,s)+Y_2(t,s))ds \\ Y_2(t,w) &= \frac{(\beta_2)^{\alpha_2 t}}{\Gamma(\alpha_2 t)}\, w^{\alpha_2 t-1}\, e^{-\beta_2 w} + \int_0^\infty e^{-(s+wt^2)}\,(Y_1(t,s)+Y_2(t,s))ds \end{aligned}\right\} \quad \dots (4)$$

Remark (1)
In this study, three cases for the parameters (α, β) should be considered :

$$\left.\begin{aligned} &(1)\ \alpha_1 > \beta_1,\ \alpha_1 = 1,\ \beta_1 = 0.5,\ \alpha_2 > \beta_2,\ \alpha_2 = 2,\ \beta_2 = 1.5 \\ &(2)\alpha_1 = \beta_1 = 0.5,\ \alpha_2 > \beta_2,\ \alpha_2 = 1.5, \beta_2 = 1 \\ &(3)\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 1 \end{aligned}\right\}$$

Now, to find the stochastic solutions of the system (4), Adomian decomposition method should be used which briefly depends on the following steps Equation 5-7,[9]

$$\left.\begin{aligned} Y_{10}(t,w) &= \Gamma_1(\omega,\alpha_1,\beta_1,t) = \frac{(\beta_1)^{\alpha_1 t}}{\Gamma(\alpha_1 t)}\, w^{\alpha_1 t-1}\, e^{-\beta_1 w} \\ Y_{20}(t,w) &= \Gamma_2(\omega,\alpha_2,\beta_2,t) = \frac{(\beta_2)^{\alpha_2 t}}{\Gamma(\alpha_2 t)}\, w^{\alpha_2 t-1}\, e^{-\beta_2 w} \end{aligned}\right\} \quad \dots (5)$$

$$\left.\begin{aligned} Y_{10}(t,s) &= \Gamma_1(s,\alpha_1,\beta_1,t) = \frac{(\beta_1)^{\alpha_1 t}}{\Gamma(\alpha_1 t)}\, s^{\alpha_1 t-1}\, e^{-\beta_1 s} \\ Y_{20}(t,s) &= \Gamma_2(s,\alpha_2,\beta_2,t) = \frac{(\beta_2)^{\alpha_2 t}}{\Gamma(\alpha_2 t)}\, s^{\alpha_2 t-1}\, e^{-\beta_2 s} \end{aligned}\right\} \quad \dots (6)$$

And,

$$\left.\begin{aligned} Y_{1,m+1}(t,w) &= \int_0^\infty e^{-(s+wt)}\,(Y_{1m}(t,s)+Y_{2m}(t,s))\,ds \\ Y_{2,m+1}(t,w) &= \int_0^\infty e^{-(s+wt^2)}\,(Y_{1m}(t,s)+Y_{2m}(t,s))\,ds \end{aligned}\right\},\ m = 0,1,2,\dots \quad \dots (7)$$

That is to get the following two stochastic solutions Equation (8):

$$\left.\begin{aligned} Y_1(t,w) &= Y_{10}(t,w) + \sum_{n=1}^{\infty} Y_{1n}(t,w) \\ Y_2(t,w) &= Y_{20}(t,w) + \sum_{n=1}^{\infty} Y_{2n}(t,w) \end{aligned}\right\} \quad \dots (8)$$

So, for m = 0, by (6) and (7):

$$\left.\begin{aligned} Y_{11}(t,w) &= \int_0^\infty e^{-(s+wt)}(Y_{10}(t,s)+Y_{20}(t,s))ds \\ Y_{21}(t,w) &= \int_0^\infty e^{-(s+wt^2)}(Y_{10}(t,s)+Y_{20}(t,s))ds \end{aligned}\right\}$$

$$\left.\begin{aligned} Y_{11}(t,w) &= \int_0^\infty e^{-(s+wt)}\left(\frac{(\beta_1)^{\alpha_1 t}}{\Gamma(\alpha_1 t)}\, s^{\alpha_1 t-1}\, e^{-\beta_1 s} + \frac{(\beta_2)^{\alpha_2 t}}{\Gamma(\alpha_2 t)}\, s^{\alpha_2 t-1}\, e^{-\beta_2 s}\right)ds \\ Y_{21}(t,w) &= \int_0^\infty e^{-(s+wt^2)}\left(\frac{(\beta_1)^{\alpha_1 t}}{\Gamma(\alpha_1 t)}\, s^{\alpha_1 t-1}\, e^{-\beta_1 s} + \frac{(\beta_2)^{\alpha_2 t}}{\Gamma(\alpha_2 t)}\, s^{\alpha_2 t-1}\, e^{-\beta_2 s}\right)ds \end{aligned}\right\}$$

$$\left.\begin{aligned} Y_{11}(t,w) &= \frac{(\beta_1)^{\alpha_1 t}}{\Gamma(\alpha_1 t)} e^{-wt}\int_0^\infty s^{\alpha_1 t-1} e^{-(\beta_1+1)s} ds + \frac{(\beta_2)^{\alpha_2 t}}{\Gamma(\alpha_2 t)} e^{-wt}\int_0^\infty s^{\alpha_2 t-1} e^{-(\beta_2+1)s} ds \\ Y_{21}(t,w) &= \frac{(\beta_1)^{\alpha_1 t}}{\Gamma(\alpha_1 t)} e^{-wt^2}\int_0^\infty s^{\alpha_1 t-1} e^{-(\beta_1+1)s} ds + \frac{(\beta_2)^{\alpha_2 t}}{\Gamma(\alpha_2 t)} e^{-wt^2}\int_0^\infty s^{\alpha_2 t-1} e^{-(\beta_2+1)s} ds \\ Y_{11}(t,w) &= \frac{(\beta_1)^{\alpha_1 t}}{\Gamma(\alpha_1 t)} e^{-wt}\int_0^\infty s^{\alpha_1 t-1} e^{-\frac{s}{1/\beta_1+1}} ds + \frac{(\beta_2)^{\alpha_2 t}}{\Gamma(\alpha_2 t)} e^{-wt}\int_0^\infty s^{\alpha_2 t-1} e^{-\frac{s}{1/\beta_2+1}} ds \\ Y_{21}(t,w) &= \frac{(\beta_1)^{\alpha_1 t}}{\Gamma(\alpha_1 t)} e^{-wt^2}\int_0^\infty s^{\alpha_1 t-1} e^{-\frac{s}{1/\beta_1+1}} ds + \frac{(\beta_2)^{\alpha_2 t}}{\Gamma(\alpha_2 t)} e^{-wt^2}\int_0^\infty s^{\alpha_2 t-1} e^{-\frac{s}{1/\beta_2+1}} ds \end{aligned}\right\}$$

or

$$\left.\begin{aligned} Y_{11}(t,w) &= \left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right] e^{-wt} \\ Y_{21}(t,w) &= \left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right] e^{-wt^2} \end{aligned}\right\} \quad \ldots(9)$$

and for m = 1 , by (7) and (9):

$$\left.\begin{aligned} Y_{12}(t,w) &= \int_0^\infty e^{-(s+wt)}(Y_{11}(t,s)+Y_{21}(t,s))ds \\ Y_{22}(t,w) &= \int_0^\infty e^{-(s+wt^2)}(Y_{11}(t,s)+Y_{21}(t,s))ds \end{aligned}\right\}$$

$$\left.\begin{aligned} Y_{12}(t,w) &= \int_0^\infty e^{-(s+wt)}\left(\left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right] e^{-st} + \left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right] e^{-st^2}\right)ds \\ Y_{22}(t,w) &= \int_0^\infty e^{-(s+wt^2)}\left(\left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right] e^{-st} + \left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right] e^{-st^2}\right)ds \end{aligned}\right\}$$

$$\left.\begin{aligned} Y_{12}(t,w) &= \left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right] e^{-wt}\int_0^\infty \left(e^{-(1+t)s} + e^{-(1+t^2)s}\right)ds \\ Y_{22}(t,w) &= \left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right] e^{-wt^2}\int_0^\infty \left(e^{-(1+t)s} + e^{-(1+t^2)s}\right)ds \end{aligned}\right\}$$

or

$$\left.\begin{aligned} Y_{12}(t,w) &= \left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right]\left(\frac{t^2+t+2}{(1+t)(1+t^2)}\right) e^{-wt} \\ Y_{22}(t,w) &= \left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right]\left(\frac{t^2+t+2}{(1+t)(1+t^2)}\right) e^{-wt^2} \end{aligned}\right\} \quad \ldots(10)$$

and for m = 2 , by (7) and (10):

$$\left.\begin{aligned} Y_{13}(t,w) &= \int_0^\infty e^{-(s+wt)}\big(Y_{12}(t,s)+Y_{22}(t,s)\big)ds \\ Y_{23}(t,w) &= \int_0^\infty e^{-(s+wt^2)}\big(Y_{12}(t,s)+Y_{22}(t,s)\big)ds \end{aligned}\right\}$$

$$\left.\begin{aligned} Y_{13}(t,w) &= \int_0^\infty e^{-(s+wt)}\left(\begin{aligned}&\left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right]\left(\frac{t^2+t+2}{(1+t)(1+t^2)}\right) e^{-st} + \\ &\left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right]\left(\frac{t^2+t+2}{(1+t)(1+t^2)}\right) e^{-st^2}\end{aligned}\right)ds \\ Y_{23}(t,w) &= \int_0^\infty e^{-(s+wt^2)}\left(\begin{aligned}&\left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right]\left(\frac{t^2+t+2}{(1+t)(1+t^2)}\right) e^{-st} + \\ &\left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right]\left(\frac{t^2+t+2}{(1+t)(1+t^2)}\right) e^{-st^2}\end{aligned}\right)ds \end{aligned}\right\}$$

$$\left.\begin{aligned} Y_{13}(t,w) &= \left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right]\left(\frac{t^2+t+2}{(1+t)(1+t^2)}\right) e^{-wt}\int_0^\infty \left(e^{-(1+t)s}+e^{-(1+t^2)s}\right)ds \\ Y_{23}(t,w) &= \left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right]\left(\frac{t^2+t+2}{(1+t)(1+t^2)}\right) e^{-wt^2}\int_0^\infty \left(e^{-(1+t)s}+e^{-(1+t^2)s}\right)ds \end{aligned}\right\}$$

or

$$\left.\begin{aligned} Y_{13}(t,w) &= \left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right]\left(\frac{t^2+t+2}{(1+t)(1+t^2)}\right)^2 e^{-wt} \\ Y_{23}(t,w) &= \left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right]\left(\frac{t^2+t+2}{(1+t)(1+t^2)}\right)^2 e^{-wt^2} \end{aligned}\right\} \quad \ldots(11)$$

and by repeating iterations for m=3,4,… and adding (9) ,(10) and(11) we get :

$$\left.\begin{aligned} \sum_{n=1}^{\infty} Y_{1n}(t,w) &= \left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right] e^{-wt}\sum_{k=0}^{\infty}\left(\frac{t^2+t+2}{(1+t)(1+t^2)}\right)^k \\ \sum_{n=1}^{\infty} Y_{2n}(t,w) &= \left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right] e^{-wt^2}\sum_{k=0}^{\infty}\left(\frac{t^2+t+2}{(1+t)(1+t^2)}\right)^k \end{aligned}\right\}$$

Where $\sum_{k=0}^{\infty}\left(\frac{t^2+t+2}{(1+t)(1+t^2)}\right)^k$ is a geometric series converges for t> 1, hence Equation (12) :

$$\left.\begin{aligned} \sum_{n=1}^{\infty} Y_{1n}(t,w) &= \left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right] e^{-wt}\left[\frac{(1+t)(1+t^2)}{t^3-1}\right] \\ \sum_{n=1}^{\infty} Y_{2n}(t,w) &= \left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right] e^{-wt^2}\left[\frac{(1+t)(1+t^2)}{t^3-1}\right] \end{aligned}\right\} \quad \ldots(12)$$

Finally, by substituting (5) and (12) into (8) which represents the two stochastic solutions of (4), we get Equation (13):

$$\left.\begin{aligned} Y_1(t,w) &= \frac{(\beta_1)^{\alpha_1 t}}{\Gamma(\alpha_1 t)}\, w^{\alpha_1 t-1}\, e^{-\beta_1 w} + \left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right] e^{-wt} \left[\frac{(1+t)(1+t^2)}{t^3-1}\right] \\ Y_2(t,w) &= \frac{(\beta_2)^{\alpha_2 t}}{\Gamma(\alpha_2 t)}\, w^{\alpha_2 t-1}\, e^{-\beta_2 w} + \left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right] e^{-wt^2} \left[\frac{(1+t)(1+t^2)}{t^3-1}\right] \end{aligned}\right\} \quad \ldots (13)$$

where, $w > 0$, t □□□ , $\alpha_i, \beta_i > 0$, $i = 1,2$

Moreover, the stochastic solutions (13) can be considered as a stochastic solutions over the interval $0<w<1$ for some $t>1$.

2. Statistical Characteristics of the Stochastic Solutions

In order to derive the statistical characteristics of the two stochastic solutions (13) over the interval $0<w<1$, $t>1$, it must be that each of them is a probability density function (p.d.f.) of (t,w). So, we multiply them respectively by A and B and equate their integrals by one that is to find A and B which make each stochastic solution is a p.d.f., i.e., we write:

$$\int_0^1 A\, Y_1(t,w)\, dw = 1 \quad , \quad \int_0^1 B\, Y_2(t,w)\, dw = 1$$

$$\int_0^1 A \left[\frac{(\beta_1)^{\alpha_1 t}}{\Gamma(\alpha_1 t)}\, w^{\alpha_1 t-1}\, e^{-\beta_1 w} + \left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right]\left[\frac{(1+t)(1+t^2)}{t^3-1}\right] e^{-wt}\right] dw = 1$$

$$\int_0^1 B \left[\frac{(\beta_2)^{\alpha_2 t}}{\Gamma(\alpha_2 t)}\, w^{\alpha_2 t-1}\, e^{-\beta_2 w} + \left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right]\left[\frac{(1+t)(1+t^2)}{t^3-1}\right] e^{-\frac{w}{t^2}}\right] dw = 1$$

$$A\frac{(\beta_1)^{\alpha_1 t}}{\Gamma(\alpha_1 t)} \int_0^1 w^{\alpha_1 t-1}\left(1 - \frac{\beta_1 w}{1!} + \frac{(\beta_1 w)^2}{2!} - \frac{(\beta_1 w)^3}{3!} + \cdots\right) dw$$
$$+ A\left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right]\left[\frac{(1+t)(1+t^2)}{t^3-1}\right] \int_0^1 e^{-\frac{w}{t}}\, dw = 1$$

$$B\frac{(\beta_2)^{\alpha_2 t}}{\Gamma(\alpha_2 t)} \int_0^1 w^{\alpha_2 t-1}\left(1 - \frac{\beta_2 w}{1!} + \frac{(\beta_2 w)^2}{2!} - \frac{(\beta_2 w)^3}{3!} + \cdots\right) dw$$
$$+ B\left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right]\left[\frac{(1+t)(1+t^2)}{t^3-1}\right] \int_0^1 e^{-wt^2}\, dw = 1$$

$$A\frac{(\beta_1)^{\alpha_1 t}}{\Gamma(\alpha_1 t)} \int_0^1 \sum_{n=0}^{\infty} \frac{(-\beta_1)^n}{n!}\, w^{\alpha_1 t+n-1} dw + A\left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right]\left[\frac{(1+t)(1+t^2)}{t^3-1}\right] \int_0^1 e^{-wt}\, dw = 1$$

$$B\frac{(\beta_2)^{\alpha_2 t}}{\Gamma(\alpha_2 t)} \int_0^1 \sum_{n=0}^{\infty} \frac{(-\beta_2)^n}{n!}\, w^{\alpha_2 t+n-1} dw + B\left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right]\left[\frac{(1+t)(1+t^2)}{t^3-1}\right] \int_0^1 e^{-wt^2}\, dw = 1$$

$$A\frac{(\beta_1)^{\alpha_1 t}}{\Gamma(\alpha_1 t)} \sum_{n=0}^{\infty} \frac{(-\beta_1)^n}{n!\,(\alpha_1 t+n)} + A\left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right]\left[\frac{(1+t)(1+t^2)}{t(t^3-1)}\right](1-e^{-t}) = 1$$

$$B\frac{(\beta_2)^{\alpha_2 t}}{\Gamma(\alpha_2 t)} \sum_{n=0}^{\infty} \frac{(-\beta_2)^n}{n!\,(\alpha_2 t+n)} + B\left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right]\left[\frac{(1+t)(1+t^2)}{t^2(t^3-1)}\right](1-e^{-t^2}) = 1$$

Hence

$$A = \frac{1}{\frac{(\beta_1)^{\alpha_1 t}}{\Gamma(\alpha_1 t)} \sum_{n=0}^{\infty} \frac{(-\beta_1)^n}{n!\,(\alpha_1 t+n)} + \left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right]\left[\frac{(1+t)(1+t^2)}{t(t^3-1)}\right](1-e^{-t})}$$

$$B = \frac{1}{\frac{(\beta_2)^{\alpha_2 t}}{\Gamma(\alpha_2 t)} \sum_{n=0}^{\infty} \frac{(-\beta_2)^n}{n!\,(\alpha_2 t + n)} + \left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right]\left[\frac{(1+t)(1+t^2)}{t^2(t^3-1)}\right](1-e^{-t^2})}$$

and let

$\zeta_1(t,w) = A\,Y_1(t,w)$

$\zeta_2(t,w) = B\,Y_2(t,w)$

Hence, the two stochastic solutions (13) are probability density functions of (t,w) that is when:

$$\zeta_1(t,w) = \frac{\frac{(\beta_1)^{\alpha_1 t}}{\Gamma(\alpha_1 t)} w^{\alpha_1 t-1} e^{-\beta_1 w} + \left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right]\left[\frac{(1+t)(1+t^2)}{(t^3-1)}\right] e^{-wt}}{\frac{(\beta_1)^{\alpha_1 t}}{\Gamma(\alpha_1 t)} \sum_{n=0}^{\infty} \frac{(-\beta_1)^n}{n!\,(\alpha_1 t + n)} + \left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right]\left[\frac{(1+t)(1+t^2)}{t(t^3-1)}\right](1-e^{-t})}$$

…(14.a)

$$\zeta_2(t,w) = \frac{\frac{(\beta_2)^{\alpha_2 t}}{\Gamma(\alpha_2 t)} w^{\alpha_2 t-1} e^{-\beta_2 w} + \left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right]\left[\frac{(1+t)(1+t^2)}{(t^3-1)}\right] e^{-wt^2}}{\frac{(\beta_2)^{\alpha_2 t}}{\Gamma(\alpha_2 t)} \sum_{n=0}^{\infty} \frac{(-\beta_2)^n}{n!\,(\alpha_2 t + n)} + \left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right]\left[\frac{(1+t)(1+t^2)}{t^2(t^3-1)}\right](1-e^{-t^2})}$$

…(14.b)

For both Equation (14.a) and (14.b) $0 < w \le 1, t$ □□□ $, \alpha_i > 0, \beta_i > 0, \; i = 1,2$.

or

$$\zeta_1(t,w) = \theta_1(t)\, w^{\alpha_1 t-1} e^{-\beta_1 w} + \delta_1(t) e^{-wt}$$
$$\zeta_2(t,w) = \theta_2(t)\, w^{\alpha_2 t-1} e^{-\beta_2 w} + \delta_2(t) e^{-wt^2}$$

where

$$\theta_1(t) = \frac{\frac{(\beta_1)^{\alpha_1 t}}{\Gamma(\alpha_1 t)}}{\frac{(\beta_1)^{\alpha_1 t}}{\Gamma(\alpha_1 t)} \sum_{n=0}^{\infty} \frac{(-\beta_1)^n}{n!\,(\alpha_1 t + n)} + \left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right]\left[\frac{(1+t)(1+t^2)}{t(t^3-1)}\right](1-e^{-t})}$$

$$\delta_1(t) = \frac{\left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right]\left[\frac{(1+t)(1+t^2)}{t^3-1}\right]}{\frac{(\beta_1)^{\alpha_1 t}}{\Gamma(\alpha_1 t)} \sum_{n=0}^{\infty} \frac{(-\beta_1)^n}{n!\,(\alpha_1 t + n)} + \left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right]\left[\frac{(1+t)(1+t^2)}{t(t^3-1)}\right](1-e^{-t})}$$

$$\theta_2(t) = \frac{\frac{(\beta_2)^{\alpha_2 t}}{\Gamma(\alpha_2 t)}}{\frac{(\beta_2)^{\alpha_2 t}}{\Gamma(\alpha_2 t)} \sum_{n=0}^{\infty} \frac{(-\beta_2)^n}{n!\,(\alpha_2 t + n)} + \left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right]\left[\frac{(1+t)(1+t^2)}{t^2(t^3-1)}\right](1-e^{-t^2})}$$

$$\delta_2(t) = \frac{\left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right]\left[\frac{(1+t)(1+t^2)}{t^3-1}\right]}{\frac{(\beta_2)^{\alpha_2 t}}{\Gamma(\alpha_2 t)} \sum_{n=0}^{\infty} \frac{(-\beta_2)^n}{n!\,(\alpha_2 t + n)} + \left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right]\left[\frac{(1+t)(1+t^2)}{t^2(t^3-1)}\right](1-e^{-t^2})}$$

3. First and Second Means of $\zeta_1(t,w)$ and $\zeta_2(t,w)$

First Mean:

$$\left.\begin{aligned} E_{\zeta_1}(t,w) &= \int_0^1 w[\zeta_1(t,w)]dw \\ E_{\zeta_2}(t,w) &= \int_0^1 w[\zeta_2(t,w)]dw \end{aligned}\right\}$$

we start by the first mean of $\zeta_1(t,w)$:

$$E_{\zeta_1}(t,w) = \int_0^1 w\,[\theta_1(t)\,w^{\alpha_1 t-1}\,e^{-\beta_1 w} + \delta_1(t)e^{-wt}]\,dw$$

$$= \theta_1(t)\int_0^1 w^{\alpha_1 t}\left(1 - \frac{\beta_1 w}{1!} + \frac{(\beta_1 w)^2}{2!} - \frac{(\beta_1 w)^3}{3!} + \cdots\right)dw + \delta_1(t)\int_0^1 w\,e^{-wt}dw$$

$$= \theta_1(t)\int_0^1 \sum_{n=0}^{\infty}\frac{(-\beta_1)^n}{n!}\,w^{\alpha_1 t+n}dw + \delta_1(t)\int_0^1 w\,e^{-wt}dw$$

$$= \theta_1(t)\sum_{n=0}^{\infty}\frac{(-\beta_1)^n}{n!\,(\alpha_1 t+n+1)} + \delta_1(t)\left[\frac{1-(1+t)\,e^{-t}}{t^2}\right]$$

or

$$E_{\zeta_1}(t,w) = \frac{\frac{(\beta_1)^{\alpha_1 t}}{\Gamma(\alpha_1 t)}\sum_{n=0}^{\infty}\frac{(-\beta_1)^n}{n!\,(\alpha_1 t+n+1)} + \left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right]\left[\frac{(1+t)(1+t^2)}{t^3-1}\right]\left[\frac{1-(1+t)\,e^{-t}}{t^2}\right]}{\frac{(\beta_1)^{\alpha_1 t}}{\Gamma(\alpha_1 t)}\sum_{n=0}^{\infty}\frac{(-\beta_1)^n}{n!\,(\alpha_1 t+n)} + \left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right]\left[\frac{(1+t)(1+t^2)}{t(t^3-1)}\right](1-e^{-t})}$$

…(15)

while, the first mean of $\zeta_2(t,w)$:

$$E_{\zeta_2}(t,w) = \int_0^1 w\,[\theta_2(t)\,w^{\alpha_2 t-1}\,e^{-\beta_2 w} + \delta_2(t)e^{-wt^2}]\,dw$$

$$= \theta_2(t)\int_0^1 w^{\alpha_2 t}\left(1 - \frac{\beta_2 w}{1!} + \frac{(\beta_2 w)^2}{2!} - \frac{(\beta_2 w)^3}{3!} + \cdots\right)dw + \delta_2(t)\int_0^1 w\,e^{-wt^2}dw$$

$$= \theta_2(t)\int_0^1 \sum_{n=0}^{\infty}\frac{(-\beta_2)^n}{n!}\,w^{\alpha_2 t+n}dw + \delta_2(t)\int_0^1 w\,e^{-wt^2}dw$$

$$= \theta_2(t)\sum_{n=0}^{\infty}\frac{(-\beta_2)^n}{n!\,(\alpha_2 t+n+1)} + \delta_2(t)\left[\frac{1-(t^2+1)e^{-t^2}}{t^4}\right]$$

or

$$E_{\zeta_2}(t,w) = \frac{\frac{(\beta_2)^{\alpha_2 t}}{\Gamma(\alpha_2 t)} \sum_{n=0}^{\infty} \frac{(-\beta_2)^n}{n!\,(\alpha_2 t+n+1)} + \left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right] \left[\frac{(1+t)(1+t^2)}{t^3-1}\right] \left[\frac{1-(t^2+1)e^{-t^2}}{t^4}\right]}{\frac{(\beta_2)^{\alpha_2 t}}{\Gamma(\alpha_2 t)} \sum_{n=0}^{\infty} \frac{(-\beta_2)^n}{n!\,(\alpha_2 t+n)} + \left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right] \left[\frac{(1+t)(1+t^2)}{t^2(t^3-1)}\right] (1-e^{-t^2})}$$

(16)...

Second Mean:

$$\left.\begin{aligned} E_{\zeta_1}(t,w^2) &= \int_0^1 w^2[\zeta_1(t,w)]dw \\ E_{\zeta_2}(t,w^2) &= \int_0^1 w^2[\zeta_2(t,w)]dw \end{aligned}\right\}$$

we start by the first mean of $\zeta_1(t,w)$:

$$E_{\zeta_1}(t,w^2) = \int_0^1 w^2\,[\theta_1(t)\,w^{\alpha_1 t-1}\,e^{-\beta_1 w} + \delta_1(t)e^{-wt}]\,dw$$

$$= \theta_1(t)\int_0^1 w^{\alpha_1 t+1}\left(1-\frac{\beta_1 w}{1!}+\frac{(\beta_1 w)^2}{2!}-\frac{(\beta_1 w)^3}{3!}+\cdots\right)dw + \delta_1(t)\int_0^1 w^2\,e^{-wt}dw$$

$$= \theta_1(t)\int_0^1 \sum_{n=0}^{\infty}\frac{(-\beta_1)^n}{n!}\,w^{\alpha_1 t+n+1}dw + \delta_1(t)\int_0^1 w^2\,e^{-wt}dw$$

$$= \theta_1(t)\sum_{n=0}^{\infty}\frac{(-\beta_1)^n}{n!\,(\alpha_1 t+n+2)} + \delta_1(t)\left[\frac{2-(t^3+2t^2+2)\,e^{-t}}{t^3}\right]$$

or

$$E_{\zeta_1}(t,w^2) = \frac{\frac{(\beta_1)^{\alpha_1 t}}{\Gamma(\alpha_1 t)} \sum_{n=0}^{\infty} \frac{(-\beta_1)^n}{n!\,(\alpha_1 t+n+2)} + \left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right] \left[\frac{(1+t)(1+t^2)}{t^3-1}\right] \left[\frac{2-(t^3+2t^2+2)\,e^{-t}}{t^3}\right]}{\frac{(\beta_1)^{\alpha_1 t}}{\Gamma(\alpha_1 t)} \sum_{n=0}^{\infty} \frac{(-\beta_1)^n}{n!\,(\alpha_1 t+n)} + \left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right] \left[\frac{(1+t)(1+t^2)}{t(t^3-1)}\right] (1-e^{-t})}$$

…(17)

while, the first mean of $\zeta_2(t,w)$:

$$E_{\zeta_2}(t,w^2) = \int_0^1 w^2\,[\theta_2(t)\,w^{\alpha_2 t-1}\,e^{-\beta_2 w} + \delta_2(t)e^{-wt^2}]\,dw$$

$$= \theta_2(t)\int_0^1 w^{\alpha_2 t+1}\left(1-\frac{\beta_2 w}{1!}+\frac{(\beta_2 w)^2}{2!}-\frac{(\beta_2 w)^3}{3!}+\cdots\right)dw + \delta_2(t)\int_0^1 w^2\,e^{-wt^2}dw$$

$$= \theta_2(t)\int_0^1 \sum_{n=0}^{\infty}\frac{(-\beta_2)^n}{n!}\,w^{\alpha_2 t+n+1}dw + \delta_2(t)\int_0^1 w^2\,e^{-wt^2}dw$$

$$= \theta_2(t)\sum_{n=0}^{\infty}\frac{(-\beta_2)^n}{n!\,(\alpha_2 t+n+2)} + \delta_2(t)\left[\frac{2-(t^4+2t^2+2)e^{-t^2}}{t^6}\right]$$

or

$$E_{\zeta_2}(t,w^2) = \frac{\frac{(\beta_2)^{\alpha_2 t}}{\Gamma(\alpha_2 t)} \sum_{n=0}^{\infty} \frac{(-\beta_2)^n}{n!\,(\alpha_2 t+n+2)} + \left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right] \left[\frac{(1+t)(1+t^2)}{t^3-1}\right] \left[\frac{2-(t^4+2t^2+2)e^{-t^2}}{t^6}\right]}{\frac{(\beta_2)^{\alpha_2 t}}{\Gamma(\alpha_2 t)} \sum_{n=0}^{\infty} \frac{(-\beta_2)^n}{n!\,(\alpha_2 t+n)} + \left[\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1 t} + \left(\frac{\beta_2}{\beta_2+1}\right)^{\alpha_2 t}\right] \left[\frac{(1+t)(1+t^2)}{t^2(t^3-1)}\right] (1-e^{-t^2})}$$

…(18)

Calculations of the variances of the probability density functions (p.d.f's) (14.a), (14.b)by the formula $Var_{\zeta}(t,w) = E_{\zeta}(t,w^2) - \left(E_{\zeta}(t,w)\right)^2$ for the three cases (Remark 1) when t>1in table 1 and 2 . It is noted that, these variances in the three cases are slowly decreasing .

Table (1) Variances of $\zeta_1(t,w)$

t	$\alpha_1 > \beta_1$, α_1=1, β_1=0.5 $\alpha_2 > \beta_2$, α_2=2, β_2=1.5	$\alpha_1 = \beta_1$ =0.5, $\alpha_2 > \beta_2$, α_2=1.5, β_2=1	$\alpha_1 = \beta_1 = \alpha_2 = \beta_2$=1
1.1	0.0789515	0.0794946	0.0785977
1.6	0.0804149	0.0777177	0.0788867
2.1	0.0848566	0.0791683	0.0836413
2.6	0.0892143	0.081976	0.0898906
3.1	**0.0909379**	0.0848037	0.0962628
3.6	0.0877991	0.0875569	0.1011318
4.1	0.079171	0.0905387	**0.1023229**
4.6	0.0667281	0.094079	0.0979512
5.1	0.0534102	0.0984085	0.0876862
5.6	0.0416486	0.1036144	0.0733108
6.1	0.0324904	0.1096082	0.057837
6.6	0.0258588	0.1160917	0.0439211
7.1	0.0211867	0.1225278	0.0329128
7.6	0.0178657	0.1281411	0.0249311
8.1	0.0154254	0.1319801	0.0194327
8.6	0.0135534	**0.1330655**	0.0157145
9.1	0.012057	0.1306163	0.0131734
. . .			
180.6	0.0000307	0.0000307	0.0000307
. . .			
500	0	0	0

Table 3.2.1 Variances of $\zeta_1(t,w)$

t	$\alpha_1>\beta_1$, $\alpha_1=1$, $\beta_1=0.5$ $\alpha_2>\beta_2$, $\alpha_2=2$, $\beta_2=1.5$	$\alpha_1=\beta_1=0.5$, $\alpha_2>\beta_2$, $\alpha_2=1.5$, $\beta_2=1$	$\alpha_1=\beta_1=\alpha_2=\beta_2=1$
1.1	0.0789953	0.0782779	0.0777596
1.6	0.0885412	0.0800432	0.0779569
2.1	0.1049919	0.0878936	0.0910475
2.6	**0.1145189**	**0.0901711**	0.1048118
3.1	0.103725	0.078143	0.1160848
3.6	0.0720713	0.0544222	0.1270881
4.1	0.0379876	0.0305826	0.1394045
4.6	0.0161455	0.0145272	0.1524344
5.1	0.0061193	0.0062519	0.163068
5.6	0.0023558	0.0026571	**0.1661121**
6.1	0.0010656	0.0012327	0.1565736
6.6	0.0006066	0.000673	0.1335063
7.1	0.0004104	0.0004323	0.1019399
. . .			
100	0	0	0

The maximum variances from table 1 and 2 are presented in the following tables

Table (3) Presents the maximum variances from both Table 1and 2

p.d.f	t	$\alpha_1>\beta_1$, $\alpha_1=1$, $\beta_1=0.5$, $\alpha_2>\beta_2$, $\alpha_2=2$, $\beta_2=1.5$	$\alpha_1=\beta_1=0.5$, $\alpha_2>\beta_2$, $\alpha_2=1.5$, $\beta_2=1$	$\alpha_1=\beta_1=\alpha_2=\beta_2=1$
$\zeta_1(t,w)$	3.1 8.6 4.1	0.0909379 … …	… 0.1330655 …	… … 0.1023229
$\zeta_2(t,w)$	2.6 5.6	0.1145189 …	0.0901711 …	… 0.1661121

Furthermore, the probability density functions (14a) and (14b) with respect to the three cases of Remark 1) can be rewritten respectively as follows :(

1. when $\alpha_1>\beta_1:\alpha_1=1,\beta_1=0.5,\alpha_2>\beta_2:\alpha_2=2,\beta_2=1.5$

$$\zeta_1(3.1,w)=1.1331142\,w^{2.1}\,e^{-0.5w}+2.4294245\,e^{-3.1\,w} \quad \dots(19.a)$$

$$\zeta_2(2.6,w)=5.5033493\,w^{4.2}\,e^{-1.5w}+4.6848782\,e^{-6.76\,w} \quad \dots(19.b)$$

where t =3.1,2.6 ,0 < w < 1.

2. when $\alpha_1=\beta_1=0.5,\alpha_2>\beta_2,\alpha_2=1.5,\beta_2=1$

$$\zeta_1(8.6,w)=2.7582866\,w^{3.3}\,e^{-0.5w}+4.9123986\,e^{-8.6\,w} \quad \dots(19.c)$$

$$\zeta_2(2.6,w)=1.9151574\,w^{2.9}\,e^{-w}+5.2464038e^{-6.76\,w} \quad \dots(19.d)$$

. where t = 8.6,2.6 ,0 < w <1

3. when $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 1$

$\zeta_1(4.1, w) = 2.7357416\, w^{3.1}\, e^{-w} + 2.9068758\, e^{-4.1\, w}$ ... (19.e)

$\zeta_2(5.6, w) = 5.6779497\, w^{4.6}\, e^{-w} + 17.626968\, e^{-31.36\, w}$ ... (19.f)

where t = 4.1,5.6 ,0 < w <1

Figures (1) - (3) represent the graphs of two pairs of p.d.f's $\zeta_1(t, w)$, $\zeta_2(t, w)$.

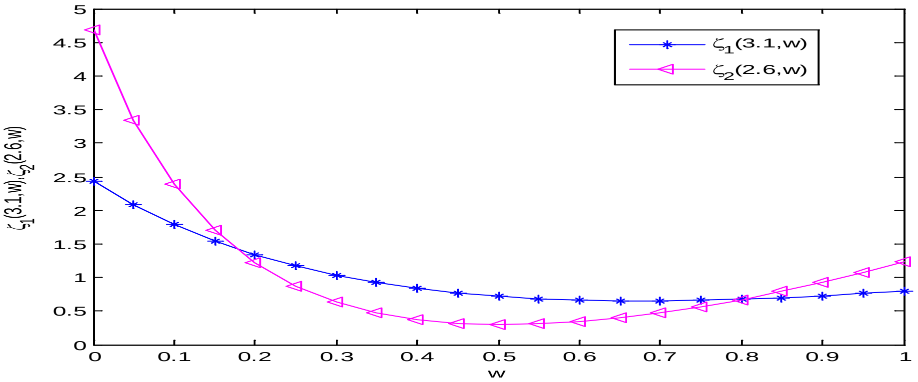

Figure (1) The Curve of ζ_1(3.1, w), ζ_2(2.6, w) when
$\alpha_1 > \beta_1 : \alpha_1 = 1, \beta_1 = 0.5, \alpha_2 > \beta_2 : \alpha_2 = 2, \beta_2 = 1.5$

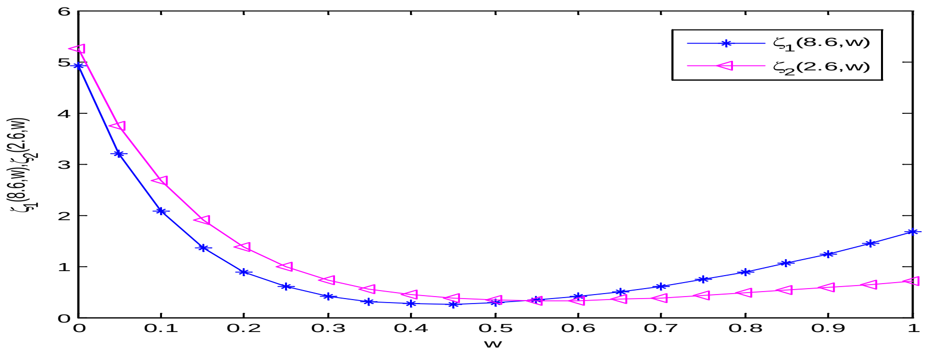

Figure (2) The Curve of ζ_1(8.6, w), ζ_2(2.6, w) when
$\alpha_1 = \beta_1 = 0.5, \alpha_2 > \beta_2, \alpha_2 = 1.5, \beta_2 = 1$

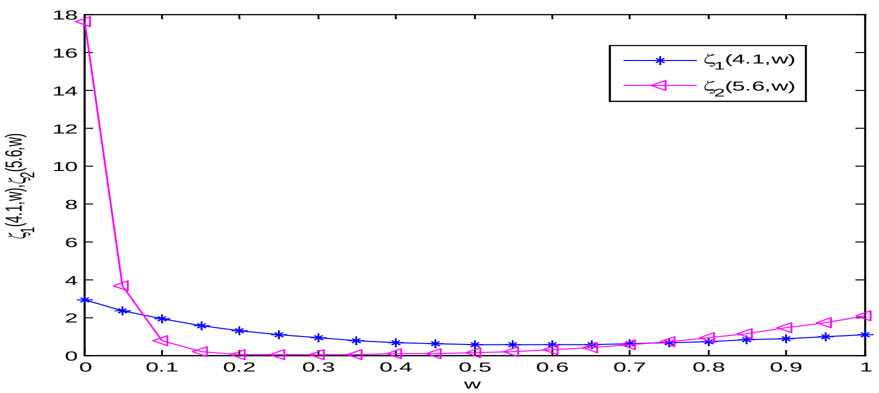

Figure (3) The Curve of ζ_1(4.1, w), ζ_2(5.6, w) when
$\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 1$

1.4 Autocovariance Function

For any s>t , s – t = τ , the autocovariance function h(τ) of the Independent p.d.f functions $\zeta(t,w)$, ζ (t+τ,w) is an even function deepens on the difference |τ| = |s – t| = |t – s| and can be found as follows :[2]

$$h(\tau) = E_\zeta[(t,w)(t+\tau,w)] - E_\zeta(t,w)E_\zeta(t+\tau,w)$$
$$= E_\zeta(t,w^2) + E_\zeta[(t,w)(t+\tau,w)] - E_\zeta(t,w^2) - E_\zeta(t,w)E_\zeta(t+\tau,w)$$
$$= E_\zeta(t,w^2) + E_\zeta(t,w)E_\zeta(t+\tau,w) - E_\zeta(t,w^2) - E_\zeta(t,w)E_\zeta(t+\tau,w)$$
$$= E_\zeta\,(t,w^2) + E_\zeta(t,w)\big[E_\zeta(t+\tau,w) - E_\zeta(t,w)\big] - E_\zeta(t,w)E_\zeta(t+\tau,w)$$

$$= E_\zeta(t,w^2) + E_\zeta(t,w)E_\zeta(t+\tau,w) - \{E_\zeta(t,w)\}^2 - E_\zeta(t,w)E_\zeta(t+\tau,w)$$

or,

$$h(\tau) = E_\zeta(t,w^2) - \{E_\zeta(t,w)\}^2 = \mathrm{var}_\zeta(t,w), \tau > 0 \qquad \ldots(20)$$

So, by **Table 3** and Equation (20) either when $\zeta_1(t,w)$, t = 3.1,8.6,4.1 or $\zeta_2(t,w)$,t = 2.6,5.6 the autocovariance functions for them with respect to the three cases (Remark 1) are presented in **Table 4**.

Table (4) Presents the Autocovariance Functions of $\zeta(t,w)$, and $\zeta(t+\tau,w)$,

p.d.f	t	$\alpha_1 > \beta_1$, α_1=1, β_1=0.5, $\alpha_2 > \beta_2$, α_2=2, β_2=1.5	$\alpha_1 = \beta_1$ =0.5, $\alpha_2 > \beta_2$, α_2=1.5, β_2=1	$\alpha_1 = \beta_1 = \alpha_2 = \beta_2$=1
$\zeta_1(t,w)$	3.1 8.6 4.1	0.0909379 … …	… 0.1330655 …	… … 0.1023229
$\zeta_2(t,w)$	2.6 5.6	0.1145189 …	0.0901711 …	… 0.1661121

1.5 Power Spectral Density Function

let{X(t) , t ϵ T} be a stationary process with autocovariance function h(τ). The power spectral density function $f_\zeta(\lambda)$ (p.s.d.f) of any probability density function $\zeta(t,w)$ is an even function represents the average power in this p.d.f at the angular frequency 0 < λ ≤ 2nπ , n ϵ I^+ and can be found by Khinchin's formula as follows[6]:

$$f_\zeta(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(\tau)e^{-i\lambda\tau}\,d\tau \qquad \ldots(21)$$

$$= \frac{h(\tau)}{2\pi}\int_{-\infty}^{\infty}(\cos\lambda\tau - i\sin\lambda\tau)\,d\tau$$

and for τ = s – t > 0.

$$= \frac{h(\tau)}{\pi}\int_{0}^{s-t}(\cos\lambda\tau - i\sin\lambda\tau)\,d\tau$$

or

$$f_\zeta(\lambda) = \frac{h(\tau)}{\pi}\frac{\sin[\lambda(s-t)]}{\lambda}, 0 < \lambda \le 2n\pi, n \in I^+ \qquad \ldots(22)$$

1. $\boldsymbol{\alpha_1 > \beta_1, \alpha_1 = 1, \beta_1 = 0.5, \alpha_2 > \beta_2, \alpha_2 = 2, \beta_2 = 1.5}$

$$f_{\zeta_1}(\lambda) = \frac{0.0909379}{\pi}\frac{\sin[\lambda(s-3.1)]}{\lambda}, 0 < \lambda < 2\pi \qquad \ldots(22.a)$$

$$f_{\zeta_2}(\lambda) = \frac{0.1145189}{\pi}\frac{\sin[\lambda(s-2.6)]}{\lambda}, 0 < \lambda < 2\pi \qquad \dots(22.b)$$

2. $\boldsymbol{\alpha_1 = \beta_1 = 0.5, \alpha_2 > \beta_2, \alpha_2 = 1.5, \beta_2 = 1}$

$$f_{\zeta_1}(\lambda) = \frac{0.1330655}{\pi}\frac{\sin[\lambda(s-8.6)]}{\lambda}, 0 < \lambda < 2\pi \qquad \dots(22.c)$$

$$f_{\zeta_2}(\lambda) = \frac{0.0901711}{\pi}\frac{\sin[\lambda(s-2.6)]}{\lambda}, 0 < \lambda < 2\pi \qquad \dots(22.d)$$

$\boldsymbol{\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 1}$

$$f_{\zeta_1}(\lambda) = \frac{0.1023229}{\pi}\frac{\sin[\lambda(s-4.1)]}{\lambda}, 0 < \lambda < 2\pi \qquad \dots(22.e)$$

$$f_{\zeta_2}(\lambda) = \frac{0.1661121}{\pi}\frac{\sin[\lambda(s-5.6)]}{\lambda}, 0 < \lambda < 2\pi \qquad \dots(22.f)$$

Figures (4) – (6) represent respectively the graphs of two pairs ($f_{\zeta_1}(\lambda)$, $f_{\zeta_2}(\lambda)$) corresponding the three cases (Remark1) and s =(9.1,5.6) ,(9.1,5.6) and (6,10)respectively just be chosen to complete the figures .

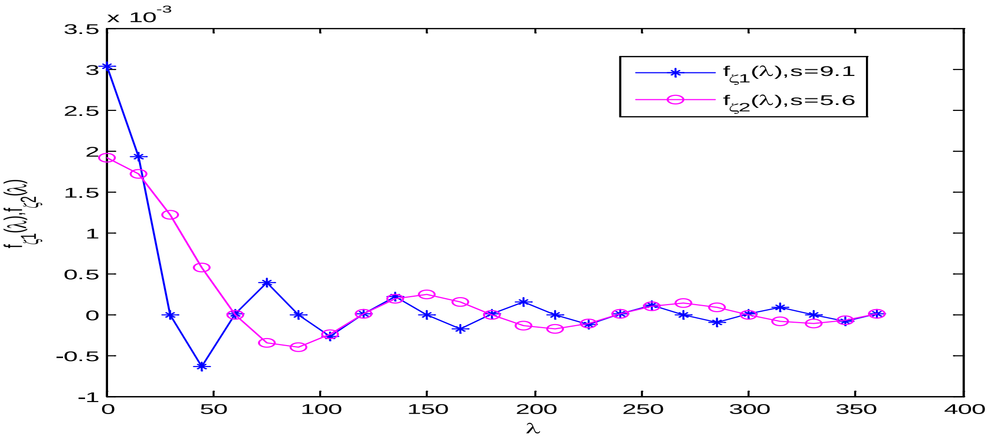

Figure (4) The Curve of $f_{\zeta_1}(\lambda), f_{\zeta_2}(\lambda)$ for $0 < \lambda \le 2\pi$, when s = (9.1,5.6) ,
$\boldsymbol{\alpha_1 > \beta_1 : \alpha_1 = 1, \beta_1 = 0.5, \alpha_2 > \beta_2 : \alpha_2 = 2, \beta_2 = 1.5}$

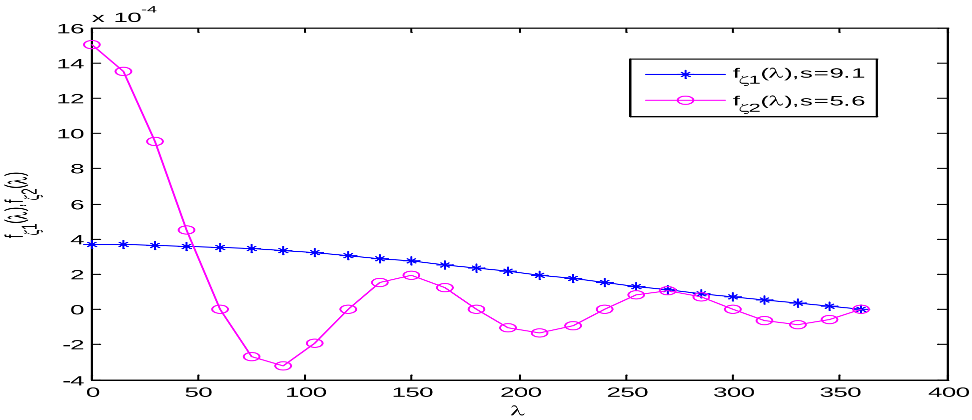

Figure (5) The Curve of $f_{\zeta_1}(\lambda), f_{\zeta_2}(\lambda)$ for $0 < \lambda \le 2\pi$, when s = (9.1,5.6) ,
$\boldsymbol{\alpha_1 = \beta_1 = 0.5, \alpha_2 > \beta_2, \alpha_2 = 1.5, \beta_2 = 1}$

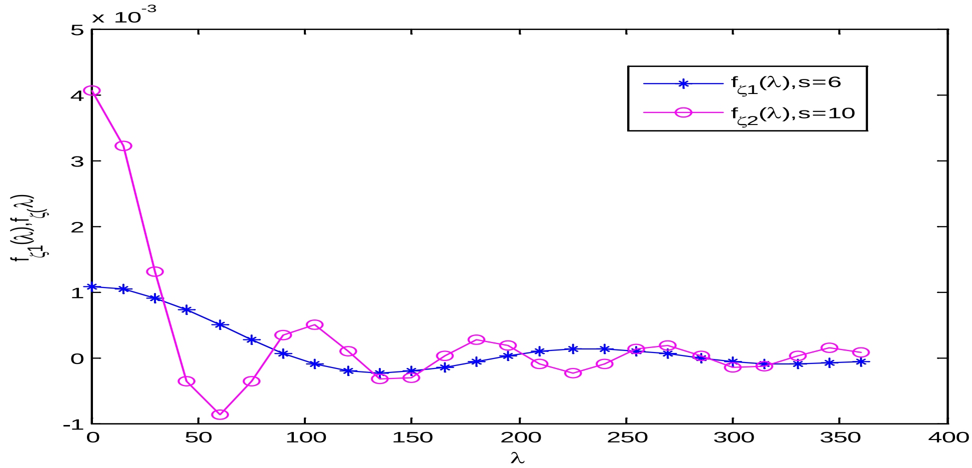

Figure (6) The Curve of $f_{\zeta_1}(\lambda), f_{\zeta_2}(\lambda)$ for $0 < \lambda \le 2\pi$, when s = (6,10) , $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 1$

1.6 Correlation Coefficients

By the known Pearson's correlation coefficient formula for dependent two random variables X, Y:[8]

$$\rho_{x,y} = \frac{E(XY) - E(X)E(Y)}{\sqrt{var(X)var(Y)}}$$

and since each pair of any two p.d.f's $\zeta_1(t,w)$,$\zeta_2(t,w)$ (22.a) to 22.f) with respect to the three cases(Remark 1) are also dependent. By writing:

E(XY) = Expectation of the product of $\zeta_1(t,w)$ by $\zeta_2(t,w)$

E(X) E(Y) = Product of expectation of $\zeta_1(t,w)$ by $\zeta_2(t,w)$

Var(X) = Maximum variance of $\zeta_1(t,w)$

Var(Y) = Maximum variance of $\zeta_2(t,w)$

Where,

1. $\alpha_1 > \beta_1 : \alpha_1 = 1, \beta_1 = 0.5, \alpha_2 > \beta_2 : \alpha_2 = 2, \beta_2 = 1.5$

$$\rho_{\zeta_1\zeta_2} = \frac{0.179068 - (0.3916165)(0.3499049)}{\sqrt{(0.0909379)(0.1145189)}} = 0.4119515$$

2. $\alpha_1 = \beta_1 = 0.5, \alpha_2 > \beta_2, \alpha_2 = 1.5, \beta_2 = 1$

$$\rho_{\zeta_1\zeta_2} = \frac{0.173058 - (0.4088372)(0.2859645)}{\sqrt{(0.1330655)(0.09017106)}} = 0.5125611$$

3. $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 1$

$$\rho_{\zeta_1\zeta_2} = \frac{0.2672499 - (0.3931854)(0.38145)}{\sqrt{(0.1023229)(0.1661121)}} = 0.8994918$$

Conclusion

With respect to the considering three cases (Remark 1), the maximum variances of the resulting probability density function $\zeta_1(t,w)$ (14a) is the greatest with respect to the in the second case $(\boldsymbol{\alpha_1 = \beta_1 = 0.5\,, \alpha_2 > \beta_2\,, \alpha_2 = 1.5, \beta_2 = 1})$ when **(t = 8.6)** while the probability density function $\zeta_2(t,w)$ (14b) is the greatest with respect to the third case $(\boldsymbol{\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 1})$ when **(t = 5.6)** Furthermore, the correlation coefficient between any pair of the probability density functions $\zeta_1(t,w)$ **,** $\zeta_2(t,w)$ is the greatest with respect to the third case $(\boldsymbol{\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 1})$. As a recommendation, it is possible to use the numerically or analytically solutions for any kind of integral equations to study some statistical properties of those solutions that is firstly by deriving the p.d.f's. For that, we suggest to consider other cases of the values of the parameters of the gamma processes that differ by the cases which are studied in this article.

References

1. Adomian, G., (1994). "Solving Frontier Problems of Physics, The Decomposition Method, Kluwer, Dordrecht, Holland.
2. Basu, A.K.,(2003). "Introduction to Stochastic Process", Alph International Ltd, Pangbourne, England.
3. Biazar, J. and Rangbar, A. (2007)."A comparison between newton's method and A.D.M for solving special Fredholm integral equations. Int. Math. Forum, 2: 215-222.
4. Balachandran, K., Sumathy, K. and Kuo, H.H. (2005). "Existence of solutions of general nonlinear Stochastic Voltera Fredholm integral equations". Stochastic Anal. Applic., 23: 827-851.DOI: 10.1081/SAP-2000644879.
5. Huda, H.O., (2007)." Solutions for the Generalized Multi-Dimensional Voltera Integral and Integro-Differential Equations". M.Sc. Thesis, College of Education, for pure science Ibn Al-Haitham, Baghdad University.
6. Krishnan V. (2006). "probability and random processes " A John Wiley & Sons , Inc., Publication,
7. Milton, J.S., Padgett ,W.J. and Tsokos, C.P. (1972). "On the existence and uniqueness of a random solution to a perturbed random integral equation of the Fredholm type". SIAM J. Applied Math., 22: 194-208. DOI: 10.1137/0122022.
8. Rodgers, J. L. and Nicewander, W. A. (1988). "Thirteen ways to look at the correlation coefficient", The American Statistician, 42(1):59–66, February.
9. Vahidi, A.R. and Mokhtari, M. (2008). "The Decomposition Method for system of linear Fredholm integral equations of the second kind". Applied Math. Sci., 2: 57-62.
10. Vladimir K.K. and Dimitrina, S.D. (2009). "Gamma process pricing path –Dependent Option ", Management Science, 55(3): pp 483 496.
11. Wahdan, M. (2011). "Some Probability Characteristic Functions of the Solution of a Stochastic Linear Fredholm Integral Equation of the Second Kind", Baghdad Science Journal,8(2):

بعض المزايا الاحصائيه بالاعتماد على اكبر تباين لحل معادله فريدهولم التكامليه العشوائيه ذات البعد الثاني الحاويه على عمليات گاما

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الخلاصه

في هذا البحث ،تم ايجاد حلين لمعادلة فريدهولم التكاملية العشوائية ذات البعد الثاني المتضمنة عمليات گاما المختلفة بالنسبة للمعلمات في حالتين والمتساوية في حالة ثالثة باستخدام طريقة ادوميان التحليلية وكنتيجة للحلول، اشتقت دوال الكثافة الاحتماليه والتباينات لها عند الزمن t بالاعتماد على اكبر التباينات لكل دالة كثافة احتمالية بالنسبة الى الحالات الثلاثة. اشتقت كل من التباين المشترك الذاتي ودوال الكثافة الطيفية بالاضافة الى ذالك احتسبت معاملات الارتباط – كمؤشر لبيان افضلية الحالات الثلاث.

الكلمات المفتاحيه: معادله فريدهولم التكامليه العشوائيه ذات البعد الثاني، عمليه گاما، طريقه ادوميان التحليليه