An Approximate Solution For Two-Points Boundary Value Problem Corresponding To Some Optimal Control Problem

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1. Abstract

This paper presents a novel method for solving nonlinear optimal control problems of regulator type via its equivalent two point's boundary value problems using the non-classical variational approach. A new procedure for solving some type of optimal control problems has been discussed and developed. The effectiveness of this method has been demonstrated by solving some optimal control problem taken from literates. .

2. Introduction

Important classes of optimization problems, namely optimal control problems exist for which the equality constraints are ordinary differential equations. Since the solution of optimal control problems involves so many theoretical and practical features, here a non-classical variational approach to solve such type of problems has been considered. Consider the following optimal control problem:

For $n \geq 1$, let x^n denote the set of mappings x from $[t_0, t_f]$ to R^n so that x is continuous and has a continuous derivative except for possibly many points and

$$Min\ J(u) = \frac{1}{2} x^T(t_f) S x(t_f) + \frac{1}{2}\int_{t_o}^{t_f} \left[x^T(t)Q(t)x(t)\right] + \left[u^T(t)R(t)u(t)\right] dt \quad ....(1)$$

Subject to

$$\dot{x} = A(t)x(t) + B(t)u(t), \quad t \in [t_o, t_f] \quad ....(2)$$

$$x(t_0) = x_0 \in R^n \quad ....(3)$$

For arbitrary positive definite symmetric matrix S, now, by using the known result of control theory [7], one can show that the original problem (1)-(3) is equivalent to problem of finding a vector valued functions $\lambda(t)$ and x(t) where the Hamiltonian function is defined as:

$$H = \frac{1}{2} x^T(t)Q(t)x(t) + \frac{1}{2} u^T(t)R(t)u(t) + \lambda^T(t)A(t)x(t) + \lambda^T(t)B(t)u(t) \quad .(4)$$

$$\frac{\partial H}{\partial u} = 0 \Rightarrow u(t) = R^{-1}(t)B^T(t)\lambda(t) \quad ....(5)$$

$$\frac{\partial H}{\partial x} = -\dot{\lambda} \Rightarrow Q(t)x(t) + A^T(t)\lambda(t) = -\dot{\lambda} \quad ..(6)$$

$$\frac{\partial H}{\partial \lambda} = \dot{x} \Rightarrow \dot{x} = A(t)x(t) + B(t)u(t) \quad ......(7)$$

$$x(t_0) = x_0 \quad ....(8)$$

$$\lambda(t_f) = Sx(t_f) \quad ....(9)$$

From (4)-(9), the following two points boundary value problem which is equivalent to the original

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optimal control problem (1)-(3) has been obtained.

$$\dot{x} = A(t)x(t) - B(t)R^{-1}B^T(t)\lambda(t) \quad .... (10)$$

$$\dot{\lambda} = -Q(t)x(t) - A^T(t)\lambda(t) \quad ..(11)$$

$$x(t_o) = x_o \quad ....(12)$$

$$\lambda(t_f) = Sx(t_f) \quad ....(13)$$

Now, Let

$$z = (x, \lambda)^T$$

Then from (10)-(14), we have that:

$$\dot{z} = M(t)z(t) \quad ......(15)$$

$$M(t) = \begin{bmatrix} A(t) & -B(t)R^{-1}(t)B^T(t) \\ -Q(t) & -A^T(t) \end{bmatrix} \quad .... (16)$$

where (13)-(14) are satisfied. Because of suitability of problem (13)-(16) in solving the optimal control problem via the proposed method, the aim of this paper is to solve problem (13)-(16) using the present method and to solve the original problem (1)-(3).

3. Non-Classical Variational Approach:

The non-classical variational approach is an effective one for solving general initial-boundary value problems, where the governing operator is not symmetric relative to the usual classical bilinear form. The effectiveness of this approach is demonstrated by solving number of mathematical problems including ordinary differential system, partial differential system, integral and integro-differential equations, moving boundary value problems (Stefan problems) where the governing operator is of parabolic partial deferential one, one dimensional phase change moving partial deferential problem with non uniform initial temperature, and some control problems. For details, one can see our previous work in [1], [4], [3] and [5].

In this paper, a new strategy which is different from [5] has been developed and adapted to solve the optimal control problems. The optimal control problem here first is converted into to two point′s boundary value problem and then the solution to last equivalent one using the non-classical variational approach will effectively obtained. A brief description of this approach is discussed as follows:

4.1 A Brief Description of Non-Classical Variational Approach

Consider the linear equation

$$Lw = f \quad ....(17)$$

Where w denotes scalar or vector valued function and L denotes a linear operator with domain D(L) contained in a linear space U and range R(L) in a second linear space V. The basic theorem of classical variational formulation reads as follows:

THEOREM

If the given linear operator L is symmetric with respect to a certain bilinear form $\langle w_1, w_2 \rangle$, then the solution of (17) are critical points of the functional

$$F[w] = \frac{1}{2}\langle Lw, w \rangle - \langle f, w \rangle \quad ......(18)$$

Moreover, if the chosen bilinear form $\langle w_1, w_2 \rangle$ is a non degenerate on D(L) and R(L), it is also true that the critical points of the functional F[w] are solutions of the given equation (17). [5]

Remarks 1:

1. The bilinear form $\langle w_1, w_2 \rangle$ is called non degenerate on V and U if the following two conditions are satisfied:

$$\forall \bar{u} \in U, \left\langle \bar{u}, v \right\rangle = 0 \Rightarrow v = 0$$

$$\forall \bar{v} \in V, \left\langle u, \bar{v} \right\rangle = 0 \Rightarrow u = 0$$

Many examples of non degenerate bilinear form can be found in [1].

2. In fact that, if w is a critical point of f[w] defined in (18), then for every $\delta w \in D(L)$, it follows that $\delta F[w] = \langle Lw - f, \delta w \rangle = 0$ and by the no degenerate conditions implies that, $Lw - f = 0$, for proof and applications see [1].
3. It is always possible to find a bilinear form that makes a linear operator (17) symmetric to suitable bilinear form which is not necessary the classical bilinear form, and from here the first theoretical hint to non-classical vraitional approach was started [6].

4.2 Non- classical Variational Formulation:

In ordered to find a non classical variational formulation corresponding to (17), one can do as follows:

Step1: Choose an arbitrary symmetric, non degenerate bilinear form (w_1, w_2) defined on U where U may stands for the space of admissible functions corresponding to the given problem (17).

Step 2: Construct a new bilinear form $\langle w_1, w_2 \rangle = (w_1, Lw_2)$ defined for every pair of elements $w_1 \in D(L)$ and $w_2 \in V$. This definition renders the given linear operator L of problem (17) to be symmetric with respect to the new bilinear form $\langle w_1, w_2 \rangle$ whatever the chosen symmetric bilinear form (w_1, w_2) may be, see [1] and [6].

Step3: Using the main theorem of classical variational formulation and the non-classical bilinear form and from above, define the following functional:

$$F[w] = \frac{1}{2}\langle Lw, w \rangle - \langle f, w \rangle$$

$$\equiv \frac{1}{2}(Lw, Lw) - (f, Lw)$$

Step 4: Find the critical points of the functional defined in step 3 in direct or indirect way in ordered to get the solution to original problem provided that the non degenerate conditions are satisfied on the chosen bilinear form $\langle w_1, w_2 \rangle$.

Remarks 2:

1. The difficulties for solving the initial boundary value problem (13)-(16) and hence the original one (1)-(3) have lead in the present work to search for variational problem equivalent to (13)-(16) in the sense that the solution of the (13)-(16) is a critical point of the suitable functional.
2. The operator L of problem (13)-(16) is not symmetric with respect to the classical bilinear form, and hence the motivation of step 2 of section 4.2 is recommended by using the non classical variational formulation. .

5. Numerical Procedure For Solving Problem (1)-(3)

1. Consider the optimal control problem defined by (1)-(3).
2. Find the equivalent two point boundary value problem to (1)-(3) by using principal of optimality and Hamiltonian equation using (4)-(10) and from them we have

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{d\lambda}{dt} \end{pmatrix} = \begin{pmatrix} A & -B^T R^{-1} B^T \\ -Q & -A^T \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix}$$

....(19)

Subject to the initial and boundary conditions

$x(t_0) = x_0, \lambda(t_f) = Sx(t_f)$ ...(20)

3. Define the following linear differential operator of (19)

$$L = \begin{pmatrix} \frac{d.}{dt} \\ \frac{d..}{dt} \end{pmatrix} - \begin{pmatrix} A & -B R B \\ -Q & -A \end{pmatrix} \begin{pmatrix} . \\ .. \end{pmatrix} \quad ......(21)$$

Then rewrite problem (19)-(20) into the following operator form

$Lw = f$,

Where $w = (x, \lambda)^T$;

$f = 0,\ x(t_0) = x_0,$

$\lambda(t_f) = Sx(t_f)$

4. Let a suitable symmetric classical bilinear form be defined by

$$(w_1, w_2) = \int_{t_0}^{t_f} w_1(t) w_2(t) dt \quad ......(22)$$

5. Define a new (non classical) bilinear form using (22) as follows:

$$\langle w_1, w_2 \rangle = (w_1, Lw_2) = \int_{t_0}^{t_f} w_1(t) Lw_2(t) dt \quad ...(23)$$

Where L is the linear differential defined in (21).

6. As in section 4.1 and 4.2, the following functional is defined

$$F[w] = \frac{1}{2}\langle Lw, w \rangle - \langle f, w \rangle$$

$$= \frac{1}{2}(Lw, Lw) - (f, Lw) \quad ....(22)$$

$$= \frac{1}{2}\int_{t_0}^{t_f} [Lw.Lw - f.Lw] dt$$

$$= \frac{1}{2}\int_{t_0}^{t_f} \begin{bmatrix} \left[\frac{dx}{dt} - Ax - C\lambda\right]^2 + \\ \left[\frac{d\lambda}{dt} + Qx + A^T\lambda\right]^2 \end{bmatrix} dt \quad ....(24)$$

$x \in R^n, A \in R^n x R^n, B \in R^m x R^n, C = -B^T R^{-1} B^T, \lambda \in R^n$

7. The main problem is then leads to find the critical points of the functional (24) and hence the solution to (21) and (1)-(3) see step4, section 4.2.

8. To get direct solution to the variational problem (24), the unknown functions x(t), $\lambda(t)$ can be written as a linear combination of complete sequence of admissible functions as:

$$x_i(t) = \sum_{j=1}^{N_i} a_j^i G_j^i(t) \quad ; i = 1,2,...,n$$

$$\lambda_k(t) = \sum_{s=1}^{M_k} b_s^k H_s^k(t) \; ; k = 1,2,...,m$$

....(25)

N_i and M_k are suitable selection numbers depending on the nature of basis and problem. The admissibility of basis functions $G_j^i(t)$ and $H_j^k(t)$ are adjusted by satisfying them to the initial and boundary conditions. Then substitute the expression (25) back into the functional (24); then the critical points of the functional (24) can be found by equating the derivative of this functional with respect to unknown variables and as follows:

$$F[w] \cong F\begin{bmatrix} a_1^1, a_2^1, \dots, a_{N_1}^1, \dots, a_1^n, a_2^n, \dots, \\ a_{N_n}^n, b_1^1, b_2^1, \dots, b_{M_1}^1, \dots, b_1^m, b_2^m, \dots, b_{M_m}^m \end{bmatrix}$$
……………………………………(26)

With respect to the unknown coefficients a_j^i and b_j^k to get the following linear system of (n x m) algebraic equations:

$$\frac{\partial F}{\partial a_j^i} = 0 \ , \quad \frac{\partial F}{\partial b_s^k} = 0$$

For i=1,2,…,n ; k=1,2,….,m ; j=1,….,N_i ; s=1,…,M_k .Then, the solution $w(t) = (x(t), \lambda(t))$ can be found by using (21) and (25) and then the optimal control can be calculated using (4).

Illustration

Consider the following optimal control problem

$$\text{Minimize} f[u] = \frac{1}{2}\int_0^2 \ddot{\theta}(t)\, dt \quad ....(27)$$

Subject to

$$\ddot{\theta}(t) = u(t)$$
$$\theta(0) = 1, \ \theta(2) = 0 \qquad ....(28)$$
$$\dot{\theta}(0) = 1, \ \dot{\theta}(2) = 0$$

Now, let $x_1(t) = \theta(t)$ and $x_2(t) = \dot{\theta}(t)$, then (28) is converted to the following optimal control problem

$$\text{Min } f[u] = \frac{1}{2}\int_0^2 u(t)^2\, dt \quad ......(29)$$

Subject to

$$\dot{x}_1(t) = x_2(t) \quad , \quad \dot{x}_2(t) = u(t)$$
$$x_1(0) = 1.0 \quad , \quad x_1(2) = 0.0$$
$$x_2(0) = 1.0 \quad , \quad x_2(2) = 0.0$$
……………………………………(30)

Adjoin the Lagrange multipliers to have

$$Min\ F[u] = \frac{1}{2}\int_0^2 \begin{matrix} u^2(t) + \\ \lambda_1(t)\left[x_2(t) - \dot{x}_1\right] \end{matrix} + \lambda_2(t)\left[u(t) - \dot{x}_2\right] dt$$
……………………………………(31)

Define the Hamiltonian function of section 1: [7]

$$H(t,x,x,u,\lambda,\lambda) = u(t) + \lambda[x] + \lambda[u(t)] \qquad ....(32)$$

The necessary conditions for optimality using (30)-(32) are found by:

$$\frac{\partial H}{\partial u} = 0 \Rightarrow u(t) = -\lambda_2(t) \quad ....(33)$$

$$\frac{\partial H}{\partial \lambda_1} = \dot{x}_1 \Rightarrow \dot{x}_1 = x_2 \quad ....(34)$$

$$\frac{\partial H}{\partial \lambda_2} = \dot{x}_2 \Rightarrow \dot{x}_2 = u(t) \quad ....(35)$$

$$\frac{\partial H}{\partial x_1} = -\dot{\lambda}_1 \Rightarrow \dot{\lambda}_1 = 0 \quad ....(36)$$

$$\frac{\partial H}{\partial x_2} = -\dot{\lambda}_2 \Rightarrow \dot{\lambda}_2 = -\lambda_1(t) \quad ....(37)$$

$$\lambda(2) \equiv Sx(2) = 0\ , \ \lambda(2) = 0 \quad ......(38)$$

Then from (32)-(38) and (30) we have that:

$$\frac{dx_1(t)}{dt} = x_2(t) \quad ....(39)$$

$$\frac{dx_2(t)}{dt} = -\lambda_2(t) \quad ......(40)$$

$$\frac{d\lambda_1(t)}{dt} = 0 \qquad \text{...............(41)}$$

$$\frac{d\lambda_2(t)}{dt} = -\lambda_1(t) \qquad \text{.......(42)}$$

$$x_1(0)=1\ ,\ x_1(2)=0\ ;$$
$$x_2(0)=1\ ,\ x_2(2)=0; \qquad \text{........(43)}$$
$$\lambda_1(2)=0\ ,\ \lambda_2(2)=0;$$

The next step is then to solve the two points boundary value problem (39)-(43) by using the non-classical variational approach which having the following assumptions

1. Set

$$(w_1, w_2) = \int_0^2 w_1(t) w_2(t) dt$$

2.Define

$$F[w] = \frac{1}{2}\langle Lw, w\rangle - \langle f, w\rangle \equiv$$
$$\frac{1}{2}(Lw, Lw) - (f, Lw)$$

$$L = \begin{bmatrix} \frac{dx_1}{dt} - x_2 \\ \frac{dx_2}{dt} + \lambda_2 \\ \frac{-d\lambda_1}{dt} \\ \frac{d\lambda_2}{dt} + \lambda_1 \end{bmatrix},\ w = \begin{bmatrix} x_1 \\ x_2 \\ \lambda_1 \\ \lambda_2 \end{bmatrix},\ f = 0$$

3. From step 2 above, we have the following functional

$$F = \frac{1}{2}\left[\int_0^2 \left[\left[\frac{dx_1}{dt} - x_2\right]^2 + \left[\frac{dx_2}{dt} + \lambda_2\right]^2 + \left[\frac{-d\lambda_1}{dt}\right]^2 + \left[\frac{d\lambda_2}{dt} + \lambda_1\right]^2 \right] dt\right]$$
....(44)

4. Let the basis functions that satisfying the initial and terminal conditions be assumed as follows:

$$x_1(t) = 1 - 0.5t + a_1\left(t^2 - 2t\right) + a_2\left(t^3 - 4t\right)$$
$$x_2(t) = 1 - 0.5t + a_3\left(t^2 - 2t\right)$$
$$\lambda_1(t) = b_1$$
$$\lambda_2(t) = b_2 + b_3 t$$
....(45)

1. Using (44) and (45) and (26) of step 6, the following linear algebraic system have been obtained:

$$\frac{\partial F}{\partial a_1} = 0,\ \frac{\partial F}{\partial a_2} = 0,\ \frac{\partial F}{\partial a_3} = 0,\ \frac{\partial F}{\partial b_1} = 0,\ \frac{\partial F}{\partial b_2} = 0,\ \frac{\partial F}{\partial b_3} = 0 \qquad \text{......(46)}$$

And thus from (46) and (44) as well as (45), we have obtained that:

$$\begin{bmatrix} 2.666 & 8.000 & 0.000 & 0.00 & 0.00 & 0.00 \\ 8.000 & 25.60 & -0.53 & 0.00 & 0.00 & 0.00 \\ 0.000 & -0.53 & 3.733 & 0.00 & 0.00 & 1.33 \\ 0.000 & 0.000 & 0.000 & 2.00 & 0.00 & 2.00 \\ 0.000 & 0.000 & 0.000 & 0.00 & 2.00 & 2.00 \\ 0.000 & 0.000 & 1.330 & 2.00 & 2.00 & 4.00 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} -0.6666 \\ -2.0000 \\ 1.3333 \\ 0.0000 \\ 1.0000 \\ 1.0000 \end{bmatrix}$$

2. On solving the above (6 x 6) system of algebraic equations we have that

$$a_1 = -1.750118,\ a_2 = 0.5000000,\ a_3 = 1.5005500$$
$$b_1 = 3.0001390,\ b_2 = 3.5001390,\ b_3 = -3.000139$$
....(47)

To get an approximate solution to the two point boundary value problem (39)-(43), we have used the result of (47) in (45). And hence using (33)-(38) and (4) to get the original solution in term of control function u(x) and $x_1(t)$ and $x_2(t)$. The comparison between the exact solution where available in the literature and the approximate one obtaining by present method have been shown in the following Table(1), where the approximate sates and control functions using the non classical variational approach are representing by

$x_1^*(t) = \theta(t),\ x_2^*(t) = \dot{\theta}(t)$ and $u^*(t) \equiv R^{-1}B^T\lambda = \lambda_2$

. $x_1^{\#}(t), x_2^{\#}(t)$ and $u^{\#}(t)$

are standing for the corresponding exact solution taken from literature. The objective function F[u] =3.25 where R=1, $B = [0,1]^T$.

t	$x^*(t)$	$x^*(t)$	$x^*(t)$	$x^*(t)$	$u^*(t)$	$u^*(t)$
0.0	1.000	1.000	1.0000	1.000	-3.5001	-3.5
0.1	1.083	1.083	0.6649	0.665	-3.2001	-3.2
0.2	1.134	1.134	0.3599	0.360	-2.9001	-2.9
0.3	1.156	1.156	0.0849	0.084	-2.6001	-2.6
0.4	1.152	1.154	-0.1600	-0.16	-2.3008	-2.3
0.5	1.125	1.125	-0.3750	-0.37	-2.0007	-2.0
0.6	1.078	1.078	-0.5600	-0.56	-1.7000	-1.7
0.7	1.014	1.014	-0.7150	-0.71	-1.4000	-1.4
0.8	0.936	0.936	-0.8400	-0.84	-1.1000	-1.1
0.9	0.847	0.846	-.935	-.935	-.80001	-.79
1.0	.7501	.7500	-1.000	-1.00	-.50000	-.50
1.1	.6481	.6450	-1.035	-1.03	-.19990	-.19
1.2	.5411	.5440	-1.004	-1.00	100627	.100
1.3	0.441	0.440	-1.01	-1.01	400041	400
1.4	.3420	.3420	-.960	-.960	-.70005	-.69
1.5	.2500	.2500	-.875	-.875	1.00069	1.00
1.6	.1680	.1680	-.7600	-.760	1.30008	1.30
1.7	.0906	.0980	-.615	-.614	1.60000	1.60
1.8	.0460	.0459	-.4400	-.439	1.90010	1.90
1.9	.0120	.0120	-.235	-.235	2.20012	2.20
2.0	0.000	0.00	0.0	0.00	2.50013	2.50

Table (1): The comparison between exact & present approximate solution.

Conclusion:

The effectiveness of present approach is shown to be very good even for a very small number of selected basis functions. Hence the present approach is recommended to solve some optimal control problems.

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حل تقريبي لمسألة القيمة الحدودية ذات الشرطين الابتدائي والحدودي المكافئة لمسألة سيطرة دينامية مثلى

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الخلاصة:

يتضمن البحث طريقة مبتكرة لحل مسائل السيطرة المثلى اللاخطية من خلال حل مسائل القيم الحدودية ذات الشرطين الابتدائي والحدودي المكافئة لها وباستخدام طريقة التغاير غير الكلاسيكية.

لقد ناقشنا وطورنا ووجدنا اسلوب جديد لحل بعض هذه المسائل المقترحة. ثم طبقنا هذه الطريقة في ايجاد الحل التقريبي لبعض من مسائل السيطرة الدينامية المثلى غير الخطية.