Existence and Uniqueness of Mild Solution for Mixed type of Integro-Differential Equation

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الملخص

في هذا البحث ندرس الوجود والوحدانية للحل المعتدل للمعادلات التكاملية التفاضلية المختلطة من نوع فولترا – فريدهولم غير الخطية مع الشروط غير المحلية في فضاء باناخ. بالإضافة الى ذلك ندرس الاعتماد المستمر للحل المعتدل. تحليلنا يستند على نظرية شبه الزمرة ومبرهنة باناخ للنقطة الصامدة.

Abstract

In this paper, we study the existence and uniqueness of a mild solution of a nonlinear mixed Volterra – Fredholm integro-differential equation with nonlocal condition in Banach space. Furthermore, we study continuouce dependence of mild solution. Our analysis is based on semigroup theory and Banach fixed point theorem.

Keywords

Existence and uniqueness ; mild Solution ; C_0 semigroup ; mixed Volterra – Fredholm ; integrodifferential equation ; continuous dependence ; nonlocal conditions.

1. Introduction

Byszewski [2] has studied the existence and uniqueness of mild, strong and classical solutions of the differential nonlocal Cauchy problem of the form

$$\frac{du(t)}{dt} + Au(t) = f\big(t, u(t)\big), t \in [\,0, a\,].$$

$$u\,(\,t_0\,) + g\,\big(\,t_1, \dots, t_p, u\,(.)\,\big) = u_0$$

Where $0 \le t_0 < t_1 < \cdots < t_p \le a, a > 0, (\,p \in \mathbb{N}\,), -A$ is the infinitesimal generator of a C_0 semigroup $T(t),\ t \ge 0$ in a Banach space $X . u_0 \in X,\ and\ f : [0, a] \times X \to X, g : [0, a\,]^p \times X \to X$ are given functions. Several authors have investigated the same type of problem to a differential classes of abstract differential equations in Banach spaces

[1,3,5,6,7,8,9]. The purpose of this paper is to prove the existence, uniqueness and continuous depenence of mild solution of a nonlinear mixed Volterra – Fredholm integrodifferential equation with nonlocal condition of the form

$$x'(t) + Ax(t) = f\left(t, x_t, \int_0^t u(t,s)k(s,x_s)ds, \int_0^a v(t,s)h(s,x_s)ds\right), t \in [0,a] \quad (1.1)$$

$$x(t) + \left[g\left(x_{t_1}, \dots, x_{t_p}\right)\right](t) = \emptyset(t), t \in [-r,0] \quad (1.2)$$

Where $0 \le t_0 < t_1 < \cdots < t_p \le a, a > 0, (p \in \mathbb{N}), -A$ is the infinitesimal generator of a C_0 semigroups $T(t),\ t \ge 0$ on a Banach space E, and $f : [0,a] \times X \times X \times X \to E,\ k,h : [0,a] \times X \to X,\ g : X^p \to X$, $u,v : [0,a] \times [0,a] \to [0,a], \emptyset \in X$.

2. Preliminaries and Hypotheses :

Let E be a Banach space with norm $\|.\|$. Let $X = C([-r,0],E)$, $0 < r < \infty$, be a Banach space of all continuous functions $\psi : [-r,0] \to E$ endowed with the supremum norm

$$\|\psi\|_x = sup\{\|\psi(s)\| : -r \le s \le 0\}.$$

Let $Y = C([-r,a],E), a > 0$, be the Banach space of all continuous functions $x : [-r,a] \to E$ with the supremum norm $\|x\|_Y = sup\{\|x(t)\| : -r \le t \le a\}$. For any $x \in Y$ and $t \in [0,a]$, we denote x_t the element of X given by

$$x_t(s) = x(t+s)\ for\ s \in [-r,0].$$

Definition 2.1[3]

A function $x \in Y$ satisfying

i. $x(t) = T(t)\emptyset(0) - T(t)\left[g\left(x_{t_1}, \dots x_{t_p}\right)\right](0) + \int_0^t T(t-s)f\left(s, x_s, \int_0^s u(s,\tau)k(\tau,x_\tau)d\tau, \int_0^a v(s,\tau)h(\tau,x_\tau)d\tau\right)ds$, $t \in [0,a]$

ii. $x(t) = \emptyset(t) - \left[g\left(x_{t_1}, \dots, x_{t_p}\right)\right](t) \quad , t \in [-r,0]$

Is called the mild solution of the nonlocal Cauchy problem (1.1) - (1.2)

Theorem 2.2 [6]

Let T be an operator from normed space E to a normed space S, then T is continuous iff T is bounded.

We need the following integral inequality, often referred to as Gronwall - Bellman inequality [4].

Lemma 2.3

Let u and f be continuous functions defined on $\mathbb{R}_+$ and c be nonegative constant. If $u(t) \le c + \int_0^t f(s)u(s)ds, for\ t \in \mathbb{R}_+$,then $u(t) \le c\ exp\left(\int_0^t f(s)ds\right), for\ t \in \mathbb{R}_+$.

Assume that $M = sup_{t\in[0,a]}\|T(t)\|_{B(E)}$. In this sequel the operator norm $\|.\|_{B(E)}$ will be denoted by $\|.\|$.

We list the following hypotheses :

.(K_1) For every $u, v, w \in Y$ and $t[0,a], f(.,u_t,v_t,w_t) \in C([0,a],E)$.

(K_2) There exists a constant $L > 0$ such that

$$\|f(t,x_t,y_t,z_t) - f(t,u_t,v_t,w_t)\| \le L\big(\|x-u\|_{C([-r,t],E)} + \|y-v\|_{C([-r,t],E)} + \|z-w\|_{C([-r,t],E)}\big)$$

$$for\ x,y,z,u,v,w \in Y, \quad t \in [0,a].$$

(K_3) There exists a constant $K > 0,\ H > 0$ such that

$$\|k(s,x_s) - k(s,y_s)\| \le K\|x-y\|_{C([-r,s],E)}$$

$$\|h(s,x_s) - h(s,y_s)\| \le H\|x-y\|_{C([-r,s],E)}$$

$$for\ x,y \in Y, \quad s \in [0,a].$$

(K_4) There exists a constant $J > 0,\ N > 0$ such that

$$|u(t,s)| \le J\ , \quad |v(t,s)| \le N \qquad for \quad t,s \in [0,a]$$

(K_5) There exists a constant $G > 0$ such that

$$\left\|\left[g\left(x_{t_1},\dots,x_{t_p}\right)\right](t) - \left[g\left(y_{t_1},\dots,y_{t_p}\right)\right](t)\right\| \le G\|x-y\|_Y$$

for $x, y \in Y, t \in [-r,0]$.

(K_6) $MG + MLa[1 + aJK + aNH] < 1$

(K_7) There exists a constants G_1, H_1, K_1, L_1 such that

$$L_1 = \max_{t\in[0,a]}\|f(t,0,0,0)\|\ , \qquad K_1 = \max_{t\in[0,a]}\|k(t,0)\|$$

$$H_1 = \max_{t\in[0,a]}\|h(t,0)\|\ , \qquad G_1 = \max_{t\in[-r,0]}\left\|\left[g\left(x_{t_1},\dots,x_{t_p}\right)\right](t)\right\|$$

3. Existence of mild solution

Theorem (3.1)

Suppose that the hypotheses $[K_1] - [K_7]$ holds, then the nonlocal Cauchy problem $(1.1) - (1.2)$ has a unique solution.

Proof:

Define an operator F on the Banach space Y by the formula

$$(Fx)(t) = \begin{cases} \emptyset(t) - \left[g\left(x_{t_1},\dots,x_{t_p}\right)\right](t) \quad if\ t \in (-r,0) & (3.1) \\ T(t)\emptyset(0) - T(t)\left[g\left(x_{t_1},\dots,x_{t_p}\right)\right](0) & \\ + \int_0^t T(t-s)f\big(s,x_s,\int_0^s u(s,\tau)k(\tau,x_\tau)d\tau, & \\ \int_0^a v(s,\tau)h(\tau,x_\tau)d\tau\big)ds \quad if\ t \in [0,a] & (3.2) \\ \qquad where\ x \in Y & \end{cases}$$

Now, We show that F maps Y into itself. let $x(t) \in Y$ and by hypotheses (K_7) and from (3.1), We have

$$\|(Fx)(t)\| = \left\|\emptyset(t) - \left[g\left(x_{t_1},\dots,x_{t_p}\right)\right](t)\ \right\|$$

$$\leq \|\emptyset(t)\| + \left\|\left[g\left(x_{t_1},\dots,x_{t_p}\right)\right](t)\right\|$$
$$\leq \|\emptyset\|_X + G_1 \quad , \quad \text{for} \quad t \in [-r,0] \tag{3.3}$$

From (3.2) and hypotheses $(K_1)\_(K_7)$, we get

$$\|(Fx)(t)\| = \|T(t)\|\|\emptyset(0)\| + \|T(t)\|\left\|\left[g\left(x_{t_1},\dots,x_{t_p}\right)\right](0)\right\| +$$
$$+\int_0^t \|T(t-s)\|\left\|f\left(s,x_s,\int_0^s u(s,\tau)\,k(\tau,x_\tau)d\tau,\int_0^a v(s,\tau)h(\tau,x_\tau)d\tau\right)\right\|ds$$
$$\|(Fz)(t)\| \leq M\|\emptyset\|_X + MG_1 +$$
$$+M\int_0^t\left[\left\|f\left(s,x_s,\int_0^s u(s,\tau)\,k(\tau,x_\tau)d\tau,\int_0^a v(s,\tau)h(\tau,x_\tau)d\tau\right)-\right.\right.$$
$$\left.\left.-f(s,0,0,0)\right\| + \|f(s,0,0,0)\|\right]ds$$
$$\leq M\|\emptyset\|_X + MG_1 +$$
$$+M\int_0^t\left[L\left(\|x-0\|_{C([-r,s],E)} + \left\|\int_0^s u(s,\tau)\,k(\tau,x_\tau)d\tau - 0\right\| +\right.\right.$$
$$\left.\left.+\left\|\int_0^a v(s,\tau)h(\tau,x_\tau)d\tau - 0\right\| + L_1\right)\right]ds$$
$$\leq M\|\emptyset\|_X + MG_1 + ML\int_0^t\left[\left(\|x\|_{C([-r,s],E)} + \int_0^s|u(s,\tau)|\,\|k(\tau,x_\tau)-\right.\right.$$
$$k(\tau,0)+k(\tau,0)\|d\tau + \int_0^a|v(s,\tau)|\|h(\tau,x_\tau)-h(\tau,0)+h(\tau,0)\|d\tau\Big) +$$
$$\left.+L_1\right]ds$$
$$\|(Fz)(t)\| \leq M\|\emptyset\|_X + MG_1 + M\int_0^t\left[L(\|x\|_{C([-r,s],E)} + J\int_0^s\|k(\tau,x_\tau)-\right.$$
$$k(\tau,0)\|d\tau + J\int_0^s\|k(\tau,0)\|d\tau + N\int_0^a\|h(\tau,x_\tau)-h(\tau,0)\|d\tau +$$
$$\left.+N\int_0^a\|h(\tau,0)\|d\tau) + L_1\right]ds$$
$$\leq M\|\emptyset\|_X + MG_1 + M\int_0^t\left[L\left(\|x\|_{C([-r,s],E)} + aJK\|x\|_{C([-r,\tau],E)} +\right.\right.$$
$$\left.\left.aJK_1 + aNH\,\|x\|_{C([-r,\tau],E)} + aNH_1\right) + L_1\right]ds$$
$$\|(Fz)(t)\| \leq M\|\emptyset\|_X + MG_1 + M[La\|x\|_Y + a^2LJK\|x\|_Y + a^2LJK_1 +$$
$$+a^2LNH\|x\|_Y + a^2NH_1 + aL_1]$$
$$\leq M[\|\emptyset\|_X + G_1 + \|x\|_Y(La + a^2LJK + a^2LJK_1 + a^2LNH + LNH_1)$$
$$+ aL_1] \tag{3.4}$$

From (3.3)and (3.4), we have

$\|(Fx)(t)\|$ is bounded by using the Theorem (2.2)

This shows that $Fx \in Y$, thus the operator F maps Y into itself. Now, we shall show that F is a contraction on Y.

Let $x(t), y(t) \in Y$, and from the hypotheses $(K_1)-(K_7)$,we get

$$(Fx)(t)-(Fy)(t) = \emptyset(t) - \left[g\left(x_{t_1},\dots,x_{t_p}\right)\right](t) - \emptyset(t) +$$
$$+\left[g\left(y_{t_1},\dots,y_{t_p}\right)\right](t)$$
$$= -\left[g\left(x_{t_1},\dots,x_{t_p}\right)\right](t) + \left[g\left(y_{t_1},\dots,y_{t_p}\right)\right](t) \tag{3.5}$$
$$for\ x,y \in Y \quad , \quad t \in [-r,0]$$

From (3.2) we have

$$(Fx)(t)-(Fy)(t) = T(t)\emptyset(0) - T(t)\left[g\left(x_{t_1},\dots,x_{t_p}\right)\right](0) +$$
$$+\int_0^t T(t-s)\left[f\left(s,x_s,\int_0^s u(s,\tau)k(\tau,x_\tau)d\tau,\int_0^a v(s,\tau)h(\tau,x_\tau)d\tau\right)\right]ds -$$

$T(t)\emptyset(0) + T(t)\left[g\left(y_{t_1}, \dots, y_{t_p}\right)\right](0) -$
$\int_0^t T(t-s)\left[f\left(s, y_s, \int_0^s u(s,\tau)k(\tau, y_\tau)d\tau, \int_0^a v(s,\tau)h(\tau, y_\tau)d\tau\right)\right]ds$
$(Fx)(t) - (Fy)(t) =$
$T(t)\left(-\left[g\left(x_{t_1}, \dots, x_{t_p}\right)\right](0) + \left[g\left(y_{t_1}, \dots, y_{t_p}\right)\right](0)\right) +$
$\int_0^t T(t-s)\left[f\left(s, x_s, \int_0^s u(s,\tau)k(\tau, x_\tau)d\tau, \int_0^a v(s,\tau)h(\tau, x_\tau)d\tau\right) -\right.$
$\left.f\left(s, y_s, \int_0^s u(s,\tau)k(\tau, y_\tau)d\tau, \int_0^a v(s,\tau)h(\tau, y_\tau)d\tau\right)\right]ds$ (3.6)
$for\ x, y \in Y \quad , \qquad t \in [0, a]$
From (3.5) and hypotheses (K_5)we get
$\|(Fx)(t) - (Fy)(t)\| \leq G\|x - y\|_Y$ (3.7)
$for\ x, y \in Y \quad , \qquad t \in [-r, 0]$
From (3.6) and hypotheses $(K_2) - (K_6)$ we get
$\|(Fx)(t) - (Fy)(t)\| \leq$
$\|T(t)\| \left\|\left[g\left(x_{t_1}, \dots, x_{t_p}\right)\right](0) - \left[g\left(y_{t_1}, \dots, y_{t_p}\right)\right](0)\right\| +$
$\int_0^t \|T(t-s)\| \left\|f\left(s, x_s, \int_0^s u(s,\tau)k(\tau, x_\tau)d\tau, \int_0^a v(s,\tau)h(\tau, x_\tau)d\tau\right) -\right.$
$\left.f\left(s, y_s, \int_0^s u(s,\tau)k(\tau, y_\tau)d\tau, \int_0^a v(s,\tau)h(\tau, y_\tau)d\tau\right)\right\|ds$
$\leq MG\|x - y\|_Y + ML\int_0^t \left[\|x - y\|_{C[(-r,s),E]} + \int_0^s |u(s,\tau)| \|k(\tau, x_\tau) -\right.$
$\left.k(\tau, y_\tau)\|d\tau + \int_0^a |v(s,\tau)| \|h(\tau, x_\tau) - h(\tau, y_\tau)\|d\tau\right]ds$
$\leq MG\|x - y\|_Y + ML[\|x - y\|_Y(a + a^2JK + NHa^2)]$
$\leq [MG + MLa(1 + aJK + aNH)]\|x - y\|_Y$ (3.8)
From (3.7)and (3.8), we get the inequality
$\|(Fx)(t) - (Fy)(t)\| \leq q\|x - y\|_Y$ (3.9)
$where\ q = [MG + MLa(1 + aJK + aNH)]$. Since, $q < 1$

The inequality (3.9) shows that F is a contraction on Y.Consequently, the operator F satisfies all the assumptions of the Banach contraction theorem. Therefore, in space Y there is a unique fixed point for F and this point is the mild solution of the nonlocal Cauchy problem $(1.1) - (1.2)$.

4. Continuous dependence of a mild solution

Theorem (4.1)

Suppose that the functions f, g, k, u and v satisfy the hypothesis $[K_1] - [K_7]$.Then for each $\emptyset_1, \emptyset_2 \in X$ and for the corresponding mild solution x_1, x_2 of the problem
$x' + Ax(t) =$
$f\left(t, x_t, \int_0^t u(t,s)k(s, x_s)ds, \int_0^a v(t,s)h(s, x_s)ds\right), t \in [0, a]$ (4.1)
$x(t) + \left[g\left(x_{t_1}, \dots, x_{t_p}\right)\right](t) = \emptyset_i(t), t \in [-r, 0], (i = 1,2)$ (4.2)
The following inequality
$\|x_1 - x_2\|_Y \leq \frac{Me^{aML(1+aJK)}}{\left[1 - M(G + a^2LNH)e^{aML(1+aJK)}\right]} \|\emptyset_1 - \emptyset_2\|_X$ (4.3)

is true, if $M(G + a^2LH)e^{aML(1+aJK)} < 1$.

Proof :

Let $\emptyset_i$, $(i = 1,2)$ be arbitrary functions belonging to X and let $x_i, (i = 1,2)$ be the corresponding mild solutions of problems $(4.1) - (4.2)$. Then,

$$x_1(t) - x_2(t) = T(t)[\emptyset_1(0) - \emptyset_2(0)]$$
$$-T(t)\left(\left[g\left((x_1)_{t_1}, \dots, (x_1)_{t_p}\right)\right](0) - \left[g\left((x_2)_{t_1}, \dots, (x_2)_{t_p}\right)\right](0)\right) +$$
$$\int_0^t T(t-s)\Big[f\big(s, (x_1)_s, \int_0^s u(s,\tau)k(\tau,(x_1)_\tau)d\tau, \int_0^a v(s,\tau)h(\tau,(x_1)_\tau)d\tau\big)$$
$$-f\big(s, (x_2)_s, \int_0^s u(s,\tau)k(\tau,(x_2)_\tau)d\tau, \int_0^a v(s,\tau)h(\tau,(x_2)_\tau)d\tau\big)\Big]ds \quad (4.4)$$
$$t \in [0, a]$$

and for $t \in [-r, 0]$, we have

$$x_1(t) - x_2(t) = [\emptyset_1(t) - \emptyset_2(t)] - \left(\left[g\left((x_1)_{t_1}, \dots, (x_1)_{t_p}\right)\right](t) - \left[g\left((x_2)_{t_1}, \dots, (x_2)_{t_p}\right)\right](t)\right) \quad (4.5)$$

By hypotheses (K_2)_(K_7) and (4.4) we have

$$\|x_1(\theta) - x_2(\theta)\| \le M\|\emptyset_1 - \emptyset_2\|_X + MG\|x_1 - x_2\|_Y + ML\int_0^\theta \big[\|x_1 - x_2\|_{C([-r,s],E)} + \int_0^s |u(s,\tau)|\|k(\tau,(x_1)_\tau - k(\tau,(x_2)_\tau\|d\tau + \int_0^a |v(s,\tau)|\|h(\tau,(x_1)_\tau) - h(\tau,(x_2)_\tau)\|d\tau\big]ds$$
$$\le M\|\emptyset_1 - \emptyset_2\|_X + MG\|x_1 - x_2\|_Y + ML\int_0^\theta \Big[\|x_1 - x_2\|_{C([-r,s],E)} + JK\int_0^s \|x_1 - x_2\|_{C([-r,\tau],E)}d\tau + NH\int_0^a \|x_1 - x_2\|_{C([-r,\tau],E)}d\tau\Big]ds$$
$$\le M\|\emptyset_1 - \emptyset_2\|_X + MG\|x_1 - x_2\|_Y + MLNHa^2\|x_1 - x_2\|_Y + ML\int_0^\theta \Big[\|x_1 - x_2\|_{C([-r,s],E)} + aJK\|x_1 - x_2\|_{C([-r,\tau],E)}\Big]ds$$
$$\le \|\emptyset_1 - \emptyset_2\|_X + M(G + LNHa^2)\|x_1 - x_2\|_Y + ML(1 + aJK)\int_0^\theta \|x_1 - x_2\|_{C([-r,s],E)}ds \;\; for\; 0 \le \tau \le s \le \theta \le t \le a.$$

$Therefore,$

$$\sup_{\theta\in[0,t]} \|x_1(\theta) - x_2(\theta)\| \le M\|\emptyset_1 - \emptyset_2\|_X + M(G + LNHa^2)\|x_1 - x_2\|_Y + ML(1 + aJK)\int_0^\theta \|x_1 - x_2\|_{C([-r,s],E)}ds \;, t \in [0,a] \quad (4.6)$$

By hypotheses (K_5) and (4.5), we have

$$\|x_1(t) - x_2(t)\| \le \|\emptyset_1 - \emptyset_2\|_X + G\|x_1 - x_2\|_Y, t \in [-r, s] \quad (4.7)$$

Since, $M \ge 1$, (4.6) and (4.7) we have

$$\|x_1(t) - x_2(t)\|_{C([-r,t],E)} \le M\|\emptyset_1 - \emptyset_2\|_X + M(G + LNHa^2) \|x_1 - x_2\|_Y + ML(1 + aJK)\int_0^t \|x_1 - x_2\|_{C([-r,s],E)}ds \quad (4.8)$$

For $t \in [0, a]$.Therefore by Gronwalls inequality we get

$$\|x_1 - x_2\|_Y \le [M\|\emptyset_1 - \emptyset_2\|_X + M(G + LNHa^2)\|x_1 - x_2\|_Y]e^{aML(1+aJK)}$$

$$\|x_1 - x_2\|_Y \left[1 - M(G + LNHa^2)e^{aML(1+aJK)}\right] \leq M\|\emptyset_1 - \emptyset_2\|_X \, e^{aML(1+aJK)}$$

$$\|x_1 - x_2\|_Y \leq \frac{Me^{aML(1+aJK)}}{\left[1 - M(G + a^2LNH)e^{aML(1+aJK)}\right]} \|\emptyset_1 - \emptyset_2\|_X$$

Hence the proof is complete.

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