Distal in Topological Transformation Group

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Abstract:

Abstract. In this paper we define distal in a topological transformation group and it will given necessary condition for a function to be a distal, and we obtain strongly distal by using the concept of the automomorphism as well as some of it property is studied.

المستخلص

في هذا البحث قدمنا التباعد في زمرة التحويل التبولوجية و الشرط الضروري لكي تكون الدالة متباعدة كما قدمنا التباعد القوي بالأعتماد على فكرة الأوتومورفزم بالأضافة الى دراسة بعض من خصائصها .

1- Introduction

Let (X,T,π) be a right topological transformation group whose phase space X is compact hausdorff. The distal of (X,T,π) was first defended by Ellis ([1]), The strongly distal of (X,T,π) was introduced and studied intensively it relation some properties dynamics . Woo ([5],[6]) define distal in function which is continuous for certain properties ,In this paper ,we introduce semi distal point by using concept of the syndetic set Finally introduced the $\alpha-$homomorphism between two transformation group on a basis of the notions in the distal function and strongly distal .W use symbol Δ to indicate the end.

2- Basic definitions

In this section we recall the basic definitions needed in this work.

Definition 2-1 [3]:

A topological group is a set T with tow structures :
1- T is a group
2- T is a topological space

Such that the two structures are compatible i.e. the multiplication map $f: T\times T \to T$ the inversion map are both continuous. $\nu : T \to T$ and

Definition 2-2 [2]:

A subset A of T is said to be {left}{right} syndetic in T provided that
for some compact subset K of T. $\{T = AK\}\{T = KA\}$

Definition 2-3 [1]:

A right topological transformation group is a triple (X,T,π) where X is a topological space called the phase space, T is a topological group called the phase group and $\pi : X \times T \to X, \pi(x,t) \to xt$ is a continuous mapping such that

1- $xe = x$ $(x \in X)$, where e is the identity of T .

2- $(xt)s = x(ts)$ $(x \in X, t, s \in T)$.

Definition 2-4 :

Let (X,T,π) be a topological transformation group and let $\pi : X \times X \times T \to X \times X$ define by $\pi((x,y),t) = (\pi(x,t),\pi(y,t))$ for all $(x,y) \in X \times X$ and $t \in T$ then $(X \times X,T,\pi)$ be a topological transformation group.

Definition 2-5:

Let (X,T,π) be a topological transformation group.

1) A subset $A \subset X$ is said to be invariant set if $AT = A$.

2) A non-empty closed invariant set $A \subset X$ is minimal if it contains no non-empty, proper, closed invariant subset. (X,T,π) is minimal if X itself is a minimal set.

Definition 2-6 [1]:

Let (X,T,π) be a topological transformation group and $x \in X$ .Then the set $xT = \{xt : t \in T\}$ is called the orbit of x and the set $\overline{xT}$ the orbit closure of x.

Remark 2-7 [2]:

1- If $x \in X$,then the orbit of x under T is the least T- invariant subset of X which contains the point x.
2- If $x \in X$,then the orbit- closure of x under T is the least closed T- invariant subset of X which contains the point x.

Definition 2-8 [2]:

Let (X,T,π) be a topological transformation group.

1. Let $x \in X$ .the period of T at x or the period of x under T is defined to be greatest subset P of T such that $xP = x$.
2. $x \in X$, the transformation group T is said to be periodic at x and the point x is said to be periodic under T provided that the period of T at x is a syndetic subset of T.
3. Terms of $(X \times X,T,\pi)$ this means if $x \neq y$ then $(xt, yt) = (x, y)t = (x, y)$ $(t \in P)$

Definition 2-9 [2]:

Let $x \in X$, The transformation group (X,T,π) is said to be transitive at x and the point x is said to be transitive under (X,T,π) provided that if U is a neighborhood open subset of X , then there exists $t \in T$ such that $xt \in U$.

Definition 2-10 :

Let (X,T,π) be a topological transformation group is said to be free effective if there exist $x \in X$ with $xt = x$, then $t = e$ for each $t \in T$.

Definition 2-11:

A continuous map π from (X,T) to (Y,T) with π (xt) = π(x) t ($x \in X$) is called a homomorphism. if Y is minimal , π is always onto .A homomorphism π from (X,T) onto itself is called an endomorphism of (X,T), and an isomorphism $\pi:(X,T) \to (X,T)$ is called an automorphism of (X,T) .we denote the group of automorphism of X by A(X) .

3-Main Results:

In this section, we introduce distal in topological transformation group and continuous map as the generalized notions of semi distal point and transitive transformation group.

Definition 3-1 [4]:

A topological transformation group (X,T,π) is called distal if given $x, y \in X$ with $x \neq y$ there exist an index α of X with $(xt, yt) \notin \alpha (t \in T)$.

1-In general (X,T,π) is distal iff $xt_\alpha \to z$ and $yt_\alpha \to z$ implies $x = y (x, y, z \in X$ and (t_α) a net in T).

2- Terms of $(X \times X, T)$ this mean that if $x \neq y$, then $\overline{(x, y)T} \subset \Delta$, and Δ are the diagonal of $X \times X$.

Definition 3-2:

A topological transformation group (X,T,π) is called strongly distal if for a given automorphism h of (X,T) , there exist an index α of X such that $(xt, yt) \notin \alpha (t \in T)$.

Remark 3-3:

Let (X,T,π) be a distal topological transformation group, then (X,T) is strongly distal transformation group (take h to be the identity).

Definition 3-4:

Let (X,T,π) be a topological transformation group, $x \in X$ is called semi distal point if for each index α of X , there exists syndetic subset A of T such that $xA \subset \alpha$

The set of all semi distal points is denoted by D where $D = \{xa \in \alpha : a \in A\}$.

Lemma 3-5

Let (X,T,π) be a topological transformation group, $y\in X$ .The followings hold:

1- If x be semi distal point then $\varphi(x)$ is semi distal point.
2- If x be semi distal point then $xT\subset\alpha$.
3- If (X,T,π) be a distal and x be semi distal point then y is semi distal point.

Theorem 3-6:

Let (X,T,π) be a topological transformation group and A be a syndetic set then $(x,y)T=(x,y)A$.
Proof:
Let $x,y\in X$ since A be a syndetic subset of T then there exist compact subset K of T with $T=AK$,Moreover $(x,y)A=(x,y)TK^{-1}=(xTK^{-1},yTK^{-1})$ since T syndetic and K^{-1} compact set .Thus we obtain $(x,y)A=(xT,yT)=(x,y)T$. Δ

Theorem 3-7:

Let (X,T,π) be a topological transformation group, and let A be a syndetic subgroup of T then following hold:

1. If I be an invariant under T then I be an invariant under A .
2. If T is free effective then (x,y) periodic points.
3. If A be a period of (x,y) then there exist compact set K with $(x,y)T=(x,y)$K.
4. If A be a period of (x,y) then there exist compact set K with $(x,y)k=(x,y)$ (k $\in K$).

Proof:
(1) Clearly $I\subset IA$. Let I be an invariant under T then $IT=I$, since A syndetic then there exist compact subset K of T such that $I=IAK$, since T is group there exist K^{-1} such that $IA=IK^{-1}\subset IT=I$ Thus we obtain $IA\subset I$, therefore I be an invariant under A . Δ
(3) Let A be a period of (x,y) and $A\subset T$,then $(x,y)A\subset(x,y)T$ since T be a syndetic there exist compact subset K of T such that $T=KT$ hence $(x,y)AK\subset(x,y)T$ and A be a period ,thus $(x,y)K\subset(x,y)T$, since A be a syndetic ,for each $t\in T$ there exist $k\in K$ and $a\in A$ such that $t=ak$ and $(x,y)t=(x,y)ak$ implies $(x,y)t=(x,y)k$
thus $(x,y)T\subset(x,y)K$ Then we obtain $(x,y)T=(x,y)K$. Δ

Theorem 3-8 :

Let (X,T,π) be a transitive of x and let I be an invariant set then following hold:
1. (X,T,π) transitive of y .
2. There exist compact set K with $U\cap UK\neq\phi$.
3. U be an invariant set.
4. There exit $t\in T$ with $xt=x$. .
5. (X,T,π) be free effective.

Proof:

(2) Let (X,T,π) be transitive of x ,then for each U neighborhood open of x there exist $t \in T$ such that, $xt \in U$ and $xT \subset U$,since T is syndetic then there exist compact subset K of T such that $TK = T$ and $xTK \subset UK$, implies $xT \subset UK$, Moreover $x \in UK$ (2-7 (1))then $U \cap UK \neq \phi$. Δ

(3) Let I be an invariant set and $x \in I$, since (X,T,π) be transitive of x then for each U neighborhood open of x there exist $t \in T$ such that, $xt \in U$ and $xT \subset U$ .Moreover $I \cap xT \neq \varphi$ ((2-7 (number (1) , $x \in UT$ thus $IT \cap xT \subset IT \cap UT$ and $I \cap UT \neq \phi$,then $xt = x.\Delta$

Theorem 3-9:

Let (X,T,π) be a transformation group, and T is abelain group then (X,T) distal if and only if $(X, T/P)$ distal.

Proof:

Let P be syndetic normal subgroup of T. and $x \neq y$,since (X,T) distal .Then there exist an index α of X with $(xt, yt) \notin \alpha(t \in T)$. $(x,y)T \not\subset \alpha, (x,y)TP \not\subset \alpha P$, and $(x,y)TPT \not\subset \alpha PT$ .since P normal subgroup of T. We obtain $(x,y)T^2P \not\subset \alpha PT$,since $P \subset T$ implies $\alpha P \subset \alpha T$ .There exist $p \in P$ such that $\alpha p = \alpha t, \alpha pt^{-1} = \alpha$, so $\alpha PT \subset \alpha$ therefore $(x,y)TP \not\subset \alpha PT \subset \alpha$,implies $(xtp, ytp) \notin \alpha$ thus $(X, T/P)$ distal. Conversely let $(X, T/P)$ distal. There exist an index α of X with $(xtp, ytp) \notin \alpha(t \in T, p \in P)$ .From Theorem (3-6),we have obtain $(x,y)P = (x,y)T$,since T is abelain group .Then $(x,y)PT = (x,y)TP$ then $(x,y)TP = (x,y)T^2$ and $(x,y)T \not\subset \alpha$ so $(xt, yt) \notin \alpha$ therefore (X,T) distal. Δ

Theorem 3-10

Let (X,T,π) be a transformation group, P period of $(x,y), (x,y) \notin \alpha$ then (X,T) be distal transformation group.

Proof:

We show that (X,T) be distal .Let P is a period of (x,y) in X ,and $P \subset T$,since T be syndetic there exist compact subset K of T such that $T = TK, P \subset TK$, for each $t \in T$ there exist $p \in P$ and $k \in K$ such that $p = tk$,since T group there exist $t^{-1} \in T$ with $pt^{-1} = k, (x,y)PT \subset (x,y)K$ by hypothesis $(x,y)T \subset (x,y)K$ .From theorem (3-7 number (3)) we obtain $(x,y)T = (x,y)K,$ sine P is a period of (x,y) then $(x,y) \in (x,y)K$ (3-7 number (4)) then for each $t \in T$, there exist $\mathrm{k} \in \mathrm{K}$ such that $(x,y)t = (x,y) \notin \alpha$ .Thus we have $(x,y)T \not\subset \alpha$ .Therefore (X,T) be distal transformation group. Δ

Theorem 3-11

Let (X,T,π) be a topological transformation group, and x be is a semi distal point then following hold :

1- D is an invariant set

2- α is an invariant set

3- (D,T) is minimal transformation group.

1-Clearly $D \subseteq DT$ .Let x be semi distal point. then if for each index α of X , there exists syndetic subset A of T such that $xA \subset \alpha$ and $x \in D$ .From remark (2-7 number 1)) we have $x \in xT$ .Therefore $D \subset xT$ and $DT \subset xT$, since $A \subset T$ From lemma (3-5 number (2)) we have $DT \subset \{xt \in \alpha : t \in T\}$ implies $DT \subset D$ .Thus we have D is an invariant set . Δ

2- Let x be semi distal point . then if for each index α of X , there exists syndetic subset A of T such that $xA \subset \alpha$ .Since T syndetic set there exist compact set K of T such that $A \subset TK, xA \subset xTK$,since T be a belain then $xA \subset xKT$ put $xK \subset \alpha$ Then $xA \subset xKT \subset \alpha T$ by hypothesis we obtain $\alpha \cap \alpha T \neq \varphi$ then for each $t \in T$ such that $\alpha t = \alpha$ and $\alpha T \subset \alpha$ Thus we obtain α is an invariant set. Δ

Theorem 3-12

Let P be period of (x, y) , then (X, T) distal transformation group iff $(x, y)P \cap \Delta \neq \varphi.$

Proof : Let $(x, y)T$ be a least an invariant subset of $X \times X$ contain (x,y) and $P \subset T$.
Thus we have $(x, y)P \subset (x, y)T$ and $(x, y)T \subset \overline{(x, y)T}$ implies $(x, y)P \subset \overline{(x, y)T}$ shows that $\overline{(x, y)T} \cap \Delta \neq \varphi.$ Hence $\overline{(x, y)T} \subset \Delta.$ therefore $(x, y)P \subset \Delta$ .Hence we have $(x, y)P \cap \Delta \neq \varphi.$ Conversely .Let x,y $\in X$ and $(x, y)P \cap \Delta \neq \varphi$, $(x, y)P \subset \Delta$,since $P \subset T.$ then $(x, y)P \cap \overline{(x, y)T} \neq \varphi$ so $\overline{(x, y)T} \cap \Delta \neq \varphi.$ Therefore $x = y$ Thus (X, T) distal. Δ

Theorem 3-13

Let (X, T, π) be a transitive of x , then (X, T) distal transformation group iff $xT = yT.$

Proof:

We show that $xT = yT.$ Let $x, y \in X$, since (X, T) be transitive of x ,then for each U be an invariant neighborhood of x there exist $t \in T$ such that, $xt \subset U$ .From theorem (3-8 number (1)) we obtain (X, T) be transitive of y and for each $u \in U$ there exsit $t_1 \in T$ such that $xt = u, yt_1 = u, xt = yt_1$ implies $xT \subset yT$ shows that $(\overline{(x, y)T} \cap \Delta \neq \varphi$ .Hence we have $yT \subset xT$ .Therefore we obtain $xT = yT.$ Conversely

Let $(x, y) \in \overline{(x, y)T} \cap \Delta$. by hypothesis there exist $t \in T$ such that $xt = yt$ and $\pi(xt, t^{-1}) = \pi(yt, t^{-1}).\pi(x, tt^{-1}) = \pi(y, tt^{-1})$ *and* $\pi(x, e) = \pi(y, e)$ implies $x = y$ Hence (X,T) be a distal .Δ

Corollary 3-14

Let (X, T, π) be transitive of x , and $\varphi : (X, T) \to (Y, T)$ be homomorphism then (Y, T) be distal.

Theorems 3-15

Let $\varphi : (X, T) \to (Y, T)$ be homomorphism and (X, T) be distal then following hold:

1. $\overline{xT}$ is a minimal set
2. If X be a minimal then $(\overline{xT}, X)$ be a distal.

Proof:

We prove only (1).Let $x \in yT$ and (X,T) is distal then $\overline{(x,y)T} \cap \Delta \neq \varphi$. implies $x = y$ and $xt = yt$ for some $t \in T$ .Hence $xT \subset yT \subset \overline{yT}$,moreover $xT \subset \overline{xT}$, $\overline{yT} \cap \overline{xT} \neq \phi$ then $\overline{yT} \subset \overline{xT}$,by hypothesis , $\overline{xT} \subset \overline{yT}$,therefore $\overline{xT} = \overline{yT}$,since $\overline{xT}$ non-empty, closed and invariant and dose not have any proper subset with these three properties. Then $\overline{xT}$ is a minimal set. Δ

<u>3-16 Definition</u> [5] [6]

The homomorphism $\varphi : (X,T) \to (Y,T)$ is said to be distal provided $\varphi(x) = \varphi(y)$ and $x \neq y$.

<u>3-17 Definition</u>

Let (X,T) and (Y,T) be a transformation group .An endomorphism $\varphi : X \to Y$ is called an $\alpha -$ homomorphism if a given automorphism h of (X,T) such that $\varphi h = \varphi$.

<u>Remark 3-18</u>

1-If x be semi distal point then $h(x)$ be semi distal point.

2- $\varphi^n h = \varphi^n$ for $n = 1,2,3,...$

<u>Theorem 3-19</u>

Let (X,T,π) be a strongly distal topological transformation group then $\varphi : X \to X$ be $\alpha -$ homomorphism if and only if φ is distal.

Proof:

Let $x, y \in X$ and (X,T) be a strongly distal .Then for each automorphism h of (X,T) there exist index α of X such that $(h(x)t, yt) \notin \alpha$ .Hence $\overline{(h(x),y)T} \subset \Delta$.
implies $h(x) = y$ since φ be $\alpha -$ homomorphism we obtain $\varphi(x) = \varphi(y)$ .Thus we have φ is distal. Conversely Let $\varphi(x) = \varphi(y)$ we show that φ be $\alpha -$ homomorphism .By hypothesis we have $h(x) = y$ and $\varphi h(x) = \varphi(y) = \varphi(x)$ .Thus φh and φ agree at a point of (X,T) .Therefore $\varphi h = \varphi$ and φ be $\alpha -$ homomorphism. Δ

<u>Theorem 3-20</u>

Let (X,T,π) be a strongly distal and $\varphi : X \to X$ be $\alpha -$ homomorphism ,and P be a period of x the followed hold :

1- there exist $y \in X$ then $x = yP$.

2- (X,T) be a distal transformation group .

Proof:

We need prove only (1) .Let P be period of x under T .Since (X,T) be a distal strongly distal .Hence we obtain h(x)= y and φ be $\alpha -$ homomorphism. Then we have $\varphi(x)P = \varphi h(x)P = \varphi(y)P$ and $\varphi(x) = \varphi(xP) = \varphi(yP)$, we obtain $x = yP$.

Theorem 3-21

If $\varphi:(X,T)\to(Y,T)$ homomorphism and (X,T) be a transitive of x then following hold:

1. If $\varphi:(X,T)\to(Y,T)$ distal then (X,T) be a distal.

2. If I invariant set then φ distal.

Proof:
We need prove only (2).We shows that φ distal. Since (X,T) be a transitive of x then for each U neighborhood of x there exist $t\in T$,such that $xt\in U$ .From theorem (3-18 number (1))we obtain (X,T) be a transitive of y , therefore $xT\cap yT\neq\phi$,then $xt=yt$ for all $t\in T$ and $\varphi(xt)=\phi(yt)$ so $\varphi(x)t=\varphi(y)t$ exist $t^{-1}\in T$ such that $\varphi(x)tt^{-1}=\varphi(y)tt^{-1}$ thus φ distal . Δ

3-22 Example

If $\varphi:(X,T)\to(Y,T)$ constant homomorphism then φ is distal .in general the converse dose not holds.

3-23 Theorem

Let $\varphi:(X,T)\to(Y,T)$ be distal homomorphism and $y\in xT$ then

1- $\varphi\pi^{t}=\varphi$.

2-If (X,T) be free effective then $\varphi(x)$ is periodic point.

Proof:
Let $y\in xT$,then there exist $t\in T$ such that $y=xt$ now we have $\varphi\pi^{t}(x)=\varphi(xt)=\varphi(y)$ since $\varphi:(X,T)\to(Y,T)$ be distal then $\varphi\pi^{t}(x)=\varphi(x)$ thus $\varphi\pi^{t}$ and φ agree at a point of (X,T) .Therefore we have $\varphi\pi^{t}=\varphi$. Δ

3-24 Remarks:

1- If (X,T) distal transformation group then φ^{n} be a distal.
2- If $\varphi:(X,T)\to(Y,T)$ distal then (X,T) be a transitive of x.
3- If $\varphi:(X,T)\to(X,T)$ be a *automorphism* then φ^{n} be a distal.

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