On Additive Mapping with Period 3 on Rings and Near-Rings

Shaima'a B. Yass
shai.bader@yahoo.com
Al – Iraqia University

Abstract: *In this research we introduced the definition of a map with period 3 on a ring R and on right (left) ideal $\bar{I}$ of R, then we prove that, when R is a prime ring with char (R) $\neq 2$, and $\bar{I} \neq 0$, $\bar{I}$ is right ideal on R, if đ is a derivation with period 3 in R,then either đ=0, or $u^2=0$ $\forall u \in \bar{I}$. Also we proved, when R is a domain with 1, and char (R) $\neq 6$, If δ is a right generalized derivation on R with period 3, then δ is the identity map. Lastly, we define a map with Period 3 on near-rings, and gived results for prime left near-rings with maps acts as an anti-homomorphism (or homomorphism), with period 3, to obtain commuatative rings.*

Keywords: prime ring, derivation, right generalized derivation, prime near-ring, semiprime near-ring, mapping of period 2, homomorphisms, anti –homomorphisms

1. Introduction

Through this paper, R is associative ring and C is the center of R. Recall R is prime if uRv= (0) implies u= 0 or v= 0. An additive map đ:R⟶R is derivation (respe. Jordan derivation) if đ(uw) =đ(u)w + uđ(w) (respe. đ(w^2) =đ(w)w + wđ(w)) holds $\forall$w, u$\in$ R, see[7]. Following [6], an additive map δ: R⟶R is called generalized derivation if there is a derivation đ: R⟶R, s.t. δ(uw) =δ(u)w + uđ(w), $\forall$u, w$\in$R. An additive map δ: R → R is called right generalized derivation if δ(uw)= δ(u)w + uđ(w), ∀u, w∈Ɍ, and it is called left generalized derivation if δ(uw)= đ(u)w + uδ(w), ∀u, w∈Ɍ. Obviously δ is generalized derivation if it is both right and left generalized derivation. The concept of a map of period 2 was introduced first in [5], from this concept we introduce the concept of a map with period 3 on a ring Ɍ, as a map g :Ɍ⟶Ɍ is named with period 3 in Ɍ, if $g^3(u) = u$,∀u∈Ɍ. And, let Ī be right (left) ideal of Ɍ, a map g:Ī⟶Ī is named with period 3 on Ī if $g^3(u) = u$, ∀u ∈Ī. Also we work replacing of a map of period 2 by a map with period 3 in the results in recent reference. We know a left near-ring is a set LŇ with "two operations + and · " s.t. (LŇ, +) is a group & (LŇ, ·) is a semigroup holds the left distributive law u.(v+h) = u · v + u ·h, ∀u, v, h∈ LŇ, LŇ is called zero symmetric left near-rings if 0 · u = 0,∀u ∈LŇ, also, we put uv= u.v. An additive map đ : Ň →Ň is called left derivation if satisfy đ(uw) = uđ(w) + đ(u)w, ∀u, w ∈Ň, and is called right derivation [10], if đ(uw) = đ(u)w +uđ(w), ∀u, w ∈Ň. In [4], a near-ring Ň is named prime if uŇw= 0,∀u, w ∈Ň gives u = 0 or w = 0. Let ∅ ≠ Ī⊆Ň, then Ī is semigroup left ideal (semigroup right ideal) if ŇĪ⊆Ī (ĪŇ⊆Ī), and Ī is called semigroup ideal if it is semigroup left ideal and semigroup right ideal. For terminologies concerning near-rings, we refer to Pilz [9]. An additive map h:Ň→ Ň is named homomorphism (anti-homomorphism) if h(uv)=h(u)h(v) holds,∀u,v∈Ň (h(uv) = h(v)h(u) ∀u,v∈Ň). Let ∅ ≠ Ī⊆Ň, an additive map f:Ň→Ň is named homomorphism (anti-homomorphism) on Ī, if f(uv)=f(u)f(v) ,∀u,v∈Ī (f(uv) = f(v)f(u) ∀u,v∈Ī).For more information about commutative near-ring with with some conditions see ([1],[2],[8]) . In the second results in this paper, we introduce a map f: Ň→Ň is with period 3 on Ň, if $f^3(u)$ = u for all u∈Ň, and if Ī is non empty subset of Ň, we call a map f with period 3 on Ī if $f^3(u) = u$, for all u∈ Ī. Also, we give results in prime left near-rings with maps acts as a homomorphism (an anti-homomorphism), with period 3, to get commuatative rings.

2. Main Results

Theorm 2-1:

Let R be a prime ring with char (R) $\neq$ 2, and $\bar{I} \neq 0$, $\bar{I}$ right ideal in R. If đ is a derivation with period 3 in R, then either đ=0, or $u^2=0$ $\forall u \in \bar{I}$.

Proof : Suppose $\exists$ derivation đ on R s.t.

$đ^3(u) = u, \forall u \in \bar{I}$ (1)

For $u,v \in \bar{I}$, $uđ(v) \in \bar{I}$, we have:

$đ^3(uđ(v) = uđ(v)$ (2)

$đ^2(đ(uđ(v)) = uđ(v)$

$đ^2(đ(u)đ(v) + uđ^2(v)) = u đ(v)$

$đ[đ(đ(u)đ(v)) + đ(u đ^2(v))] = u đ(v)$

$đ(đ^2(u)đ(v) + đ(u)đ^2(v) + đ(u)đ^2(v) + uđ^3(v)) = uđ(v)$

$đ[đ^2(u)đ(v)] + đ[đ(u)đ^2(v)] + đ[đ(u)đ^2(v)] + đ[uđ^3(v)]) = uđ(v)$

$đ^3(u)đ(v) + đ^2(u)đ^2(v) + đ^2(u)đ^2(v) + đ(u)đ^3(v) + đ^2(u)đ^2(v) + đ(u)đ^3(v) + đ(u)đ^3(v) + uđ^4(v) = uđ(v)$

$3đ^2(u)đ^2(v) + 3đ(u)đ^3(v) + uđ^4(v) = 0$

$3đ^2(u)đ^2(v) + 3đ(u)v + uđ(đ^3(v)) = 0$

That is:

$3đ^2(u)đ^2(v)+3đ(u)v+uđ(v)=0, \forall u,v \in \bar{I}$ (3)

Also, we have:

$đ^4(uv)=đ(đ^3(uv))=đ(uv)=đ(u)v+uđ(v), \forall u,v \in \bar{I}$ (4)

On the other hand:

$đ^4(uv) = đ^3(đ(uv))$

$= đ^2[đ(đ(u)v + uđ(v))]$

$= đ^2[(đ(đ(u)v) + đ(uđ(v))]$

$= đ^2[đ^2(u)v + đ(u)đ(v) + đ(u)đ(v) + uđ^2(v)]$

$= đ^2[đ^2(u)v + 2đ(u)đ(v) + uđ^2(v)]$

$=đ[đ^3(u)v+đ^2(u)đ(v)+2đ^2(u)đ(v)+ 2đ(u)đ^2(v) + đ(u)đ^2(v) + uđ^3(v)]$

$= đ^4(u)v+ đ^3(u)đ(v) + đ^3(u)đ(v) +đ^2(u)đ^2(v) + 2đ^3(u)đ(v) +2đ^2(u)đ^2(v) +2đ^2(u)đ^2(v)+ 2đ(u)đ^3(v) + đ^2(u)đ^2(v) +đ(u)đ^3(v) +đ(u)đ^3(v) + uđ^4(v)$

$= đ(u)v + 4uđ(v) +6đ^2(u)đ^2(v) + 4đ(u)v + uđ(v), \forall u,v \in \bar{I}$ (5)

From (4), (5):

$4uđ(v)+6đ^2(u)đ^2(v)+4đ(u)v=0, \forall u,v \in \bar{I}$ (6)

From (3), (6):

$3uđ(v)+3đ^2(u)đ^2(v)+đ(u)v=0,\forall u,v\in \bar{I}$ (7)

From (3), (7):

$3uđ(v) + đ(u)v – 3đ(u)v – uđ(v) = 0$

That is:

$2uđ(v) – 2đ(u)v=0,\forall u,v\in \bar{I}$ (8)

Putting vr instead of v in (8):

$2uđ(v)r+2uvđ(r)–2đ(u)vr=0,\forall u,v\in \bar{I}, r\in R$ (9)

From (8), (9):

$2uvđ(r)=0,\forall u,v\in \bar{I}, r\in R$ (10)

Since char (R) $\neq$ 2:

$uvđ(r) = 0,\forall u,v\in \bar{I}, r\in R$ (11)

Putting rs instead of r in (11):

$uvđ(r)s + uvrđ(s) = 0,$

That is:

$uvrđ(s)=0,\forall u,v\in \bar{I}, r,s\in R$ (12)

Since R is prime:

Either $đ(s) = 0, \forall s \in R$, that is đ=0

or, $uv = 0\ \ \forall u,v\in \bar{I}$, that is $u^2 = 0, \forall u \in \bar{I}$.

Theorm 2-2:

Let R be a prime ring with char (R) $\neq$ 6 and let đ be a derivation on R. If δ is right generalized derivation given by $\delta(u) = u + đ(u)\ \forall u \in \bar{I}$ is with period 3 on R. Then δ is the identity map.

Proof: Consider:

$\delta(u)=u+đ(u),\forall u\in R$ (13)

Since δ is with period 3:

$\delta^2(\delta(u) = u + đ(u))$

$\delta^3(u) = \delta^2 (u + đ(u))$

$= \delta[\delta(u + đ(u)]$

$= \delta[(u + đ(u) + đ(u+đ(u))]$

$= \delta[(u + đ(u) + đ(u) + đ^2(u)]$

$= \delta(u) + \delta(đ(u)) + \delta(đ(u)) +\delta(đ^2(u))$

$= u + đ(u) + đ(u) + đ^2(u) + đ(u) + đ^2(u) + đ^2(u) + đ^3(u)$

From (13):

$u=u+3đ(u)+3đ^2(u)+đ^3(u),\forall u\in R$ (14)

That is:

$$3đ(u)+3đ^2(u)+đ^3(u)=0,\forall u\in R \quad (15)$$

Putting uv instesd of u in (15):

$$3đ(uv) + 3đ^2(uv) + đ^3(uv) = 0$$

$$3đ(u)v + 3uđ(v) + 3đ[đ(u)v + uđ(v)]+ đ^2[đ(u)v +uđ(v)] = 0$$

$$3đ(u)v + 3uđ(v) + 3đ[đ(u)v] +3đ[uđ(v)] +đ^2[đ(u)v] + đ^2[uđ(v)] = 0$$

$$3đ(u)v + 3uđ(v) + 3đ^2(u)v + 3đ(u)đ(v) +3đ(u)đ(v) + 3uđ^2(v) + đ[đ^2(u)v +đ(u) đ(v)]+ đ[đ(u)đ(v) + uđ^2(v)] = 0$$

$$3đ(u)v + 3uđ(v) + 3đ^2(u)v + 6đ(u)đ(v) + 3uđ^2(v) + đ^3(u)v + đ^2(u)đ(v) +đ^2(u)đ(v)+ đ(u)đ^2(v) +đ^2(u) đ(v) +đ(u)đ^2(v) +đ(u)đ^2(v) + uđ^3(v)=0,\forall u,v\in R \quad (16)$$

From (15) and (16):

$$6đ(u)đ(v)+3đ^2(u)đ(v)+3đ(u)đ^2(v)=0,\forall u,v\in R \quad (17)$$

Putting vr instesd of v in (17):

$$6đ(u) đ(vr) + 3đ^2(u) đ(vr)+ 3đ(u)đ^2(vr) = 0$$

$$6đ(u) đ(v)r + 6đ(u) vđ(r) + 3đ^2(u)đ(v)r+ 3đ^2(u)vđ(r) + 3đ(u)đ^2(v)r + 3đ(u)đ(v)đ(r)+ 3đ(u)đ(v)đ((r) + 3đ(u)vđ^2(r) = 0 , \forall u, v, r \in R \quad (18)$$

From (17) and (18):

$$6đ(u)vđ(r)+ 3đ^2(u)vđ(r) + 3đ(u) đ(v) đ(r) + 3đ(u) đ(v)đ(r) +3đ(u)vđ^2(r) = 0, \forall u, v, r \in R \quad (19)$$

Putting đ(v) instead of v in (19):

$$6đ(u)đ(v)đ(r)+ 3đ^2(u)đ(v)đ(r) + 3đ(u)đ^2(v) đ(r) + 3đ(u) đ^2(v)đ(r) + 3đ(u)đ(v)đ^2(r) = 0 , \forall u, v, r \in R \quad (20)$$

From (17) and (20):

$$3đ(u)đ^2(v)đ(r)+3đ(u)đ(v)đ^2(r)=0,\forall u,v,r\in R \quad (21)$$

That is:

$$3đ(u)[đ^2(v)đ(r)+đ(v)đ^2(r)]=0,\forall u,v,r\in R \quad (22)$$

Putting us instead of u in (22):

$$3[đ(u)s+uđ(s)][đ^2(v)đ(r)+đ(v)đ^2(r)]=0,\forall u,v,r,s\in R \quad (23)$$

From (22) and (23):

$$3đ(u) s [đ^2(v)đ(r) + đ(v)đ^2(r)] = 0$$

$$đ(u) s [3đ^2(v)đ(r) +3 đ(v) đ^2(r)] = 0, \forall u, v, r, s \in R$$

Since R is prime, either đ = 0, or,

$$3đ^2(v)đ(r)+3đ(v)đ^2(r)=0,\forall v,r\in R \quad (24)$$

From (17) and (24):

$-6đ(u)đ(v)=0, \forall u,v \in R$ (25)

Since char (R) ≠ 6:

$đ(u)đ(v)=0, \forall u,v \in R$ (26)

Putting vr instead of v in (26):

$đ(u)đ(v)r+đ(u)vđ(r)=0, \forall u,v,r \in R$ (27)

From (26) and (27):

$đ(u)vđ(r)=0, \forall u,v,r \in R$ (28)

Since R is prime, (28) gives đ=0.

Theorem 2-3:

Let R be prime ring with char (R) ≠ 6, and let δ be right generalized derivation on R with associated derivation đ. If δ is with period 3 on R , then đ(C) = {0}.

Proof:

$uv = \delta^3(uv)\,, \forall u,v \in R$

That is:

$$uv = \delta^2\big(\delta(uv)\big)$$
$$= \delta^2\big(\delta(u)v + uđ(v)\big)$$
$$= \delta(\delta^2(u)v + \delta(u)đ(v) + \delta(u)đ(v) + uđ^2(v))$$
$$= \delta^3(u)v + \delta^2(u)đ(v) + \delta^2(u)đ(v) + \delta(u)đ^2(v) + \delta^2(u)đ(v) + \delta(u)đ^2(v) + \delta(u)đ^2(v) + u\,đ^3(v)$$
$$= \delta^3(u)v + 3\delta^2(u)đ(v) + 3\delta(u)đ^2(v) + uđ^3(v), \forall u,v \in R$$

Since δ is period 3:

$$uv = uv + 3\delta^2(u)đ(v) + 3\delta(u)đ^2(v) + uđ^3(v), \forall u,v \in R$$

That is:

$$3\delta^2(u)đ(v) + 3\delta(u)đ^2(v) + uđ^3(v) = 0, \forall u,v \in R \quad (29)$$

Putting $\delta^2(u)$ instead of u in (29):

$$3\delta(u)đ(v) + 3uđ^2(v) + \delta^2(u)đ^3(v) = 0, \forall u,v \in R \quad (30)$$

Letting $c \in C$, $u \in R$ and Putting uc instead of u in (30):

$$3\delta(uc)đ(v) + 3ucđ^2(v) + \delta^2(uc)đ^3(v) = 0$$
$$3\delta(u)cđ(v) + 3uđ(c)đ(v) + 3ucđ^2(v) + \delta^2(u)cđ^3(v) + \delta(u)đ(c)đ^3(v) + \delta(u)đ(c)đ^3(v) + uđ^2(c)đ^3(v) = 0,$$
$$\forall u,v \in R, c \in C$$

That is:

$3\delta(u)cđ(v) + 3uđ(c)đ(v) + 3ucđ^2(v) + \delta^2(u)cđ^3(v) + 2\delta(u)đ(c)đ^3(v) + uđ^2(c)đ^3(v) = 0,$
$\forall u, v \in R, c \in C$ (31)

From (30) and (31):

$3uđ(c)đ(v) + 2\delta(u)đ(c)đ^3(v) + uđ^2(c)đ^3(v) = 0$
$\forall u, v \in R, c \in C$ (32)

Again letting c∈ C , u ∈ R and Putting uc instead of u in (32):

$3ucđ(c)đ(v) + 2\delta(uc)đ(c)đ^3(v) + ucđ^2(c)đ^3(v) = 0$
$, \forall u, v \in R, c \in C$ (33)

Left multiplication of (32) by c, c∈C:

$3cuđ(c)đ(v) + 2c\delta(u)đ(c)đ^3(v) + cuđ^2(c)đ^3(v) = 0$
$, \forall u, v \in R, c \in C$ (34)

Comparing (33) and (34):

$2\delta(uc)đ(c)đ^3(v) = 2c\delta(u)đ(c)đ^3(v) \ , \forall u, v \in R, c \in C$ (35)

That is:

$2\delta(u)cđ(c)đ^3(v) + 2uđ(c)đ(c)đ^3(v) = 2c\delta(u)đ(c)đ^3(v)$
$, \forall u, v \in R, c \in C$ (36)

This implies:

$2uđ^2(c)đ^3(v) = 0, \forall u, v \in R, c \in C$ (37)

Since char (R) ≠ 6:

$u\ đ^2(c)đ^3(v) = 0, \forall\ u, v \in R, c \in C$ (38)

Since R is prime:

$đ^2(c)đ^3(v) = 0, \forall\ v \in R, c \in C$ (39)

From (32) and (39):

$3uđ(c)đ(v) + 2\delta(u)đ(c)đ^3(v) = 0, \forall u, v \in R, c \in C$ (40)

Letting c∈ R , and Putting uc instead of u in (40):

$3ucđ(c)đ(v) + 2\delta(u)cđ(c)đ^3(v) + 2uđ(c)đ^3(v) = 0$
$, \forall u, v \in R, c \in C$ (41)

From (40) and (41):

$3uđ(c)đ^3(v) = 0, \forall u, v \in R, c \in C$ (42)

Since char (R) ≠ 6:

$uđ(c)đ^3(v) = 0, \forall u, v \in R, c \in C$ (43)

Putting δ(u) instead of u by in (43):

$\delta(u)đ(c)đ^3(v) = 0\ , \forall u, v \in R, c \in C$ (44)

From (40) and (44):

$3uđ(c)đ(v) = 0, \forall u, v \in R, c \in C$ (45)
Since char (R) ≠6 :
$uđ(c)đ(v) = 0, \forall u, v \in R, c \in C$ (46)
Since R is prime :
$đ(c)đ(v) = 0, \forall v \in R, c \in C$ (47)
Putting vr instead of v in (47) , we get
$đ(c)đ(v)r + đ(c)vđ(r) = 0 , \forall r, v \in R, c \in C$ (48)
From (47) and (48):
$đ(c)vđ(r) = 0 , \forall r, v \in R, c \in C$ (49)
Putting c instead of r in (49):
$đ(c)vđ(c) = 0 , \forall v \in R, c \in C$ (50)
Since R is prime , hence đ(C)={0}

Theorem 2-4:

Let R be a domain with 1, with char (R) ≠ 6. If δ is a right generalized derivation on R with period 3, then δ is the identity map

Proof: Note:
$\delta(u) = \delta(1.u) = \delta(1)u + đ(u), \forall u \in R$ (51)
Taking u=1 in (29) And letting K = δ(1):
$3\delta^2(1)đ(v) + 3\delta(1)đ^2(v) + đ^3(v) = 0$
$3K^2đ(v) + 3Kđ^2(v) + đ^3(v) = 0 , \forall v \in R$ (52)
Putting $\delta(u)$ instead of u in (29):
$3\delta^3(u)đ(v) + 3\delta^2(u)đ^2(v) + \delta(u)đ^3(v) = 0$
That is:
$3uđ(v) + 3\delta^2(u)đ^2(v) + \delta(u)đ^3(v) = 0, \ \forall u, v \in R$ (53)
Putting u=1 in (53) and Putting K =δ(1):
$3đ(v) + 3K^2đ^2(v) + Kđ^3(v) = 0, \ \forall v \in R$ (54)
Left multiplication of (52) by K:
$3K^2đ(v) + 3K^2đ^2(v) + Kđ^3(v) = 0, \ \forall v \in R$ (55)
From (54) and (55):
$3K^3đ(v) = 3đ(v)$
$3(K^3 - 1)đ(v) = 0 , \forall v \in R$ (56)
Since R is 6-torsion free:
$(K^3 - 1)đ(v) = 0 \quad \forall v \in R$ (57)
Since R is domain , and if $đ \neq 0$, we get $K^3 = 1$

That is K =1, so that:

$\delta(u) = u + đ(u) \quad$ for all $u \in R$

and by Theorm $2-2$, δ is the identity map

3. Other Results

Lemma 3-1,[3]

Let Ň be prime. Let Ī ≠ 0, Ī is semigroup right ideal (respe, semigroup left ideal). If u is an element in Ň s.t. Īu = {0} (respe. uĪ = {0}), then u = 0.

Definition 3-2:

A map δ: Ň→Ň is with period 3 on Ň, if $\delta^3(u) = u$, ∀u∈Ň, and if Ī is non empty subset of Ň, we call a map δ with period 3 on Ī if $\delta^3(u) = u$, ∀ u∈ Ī.

Example 3-3:

Let S be a zero symmetric left near-ring, and let

$$\check{N} = \left\{ \begin{bmatrix} 0 & h & k \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, 0, h, k \in S \right\}$$

Define a map f: Ň→Ň as follows:

$$f\left(\begin{bmatrix} 0 & h & k \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & h & k \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We show that Ň is a zero symmetric left near-ring, and we show f is with period 3

Theorm 3.4:

Let LŇ be prime. Let Ī ≠0, Ī be semigroup ideal of LŇ. Suppose δ is an anti-homomorphism in LŇ. If δ is with period 3 on Ī, then LŇ is commutative Ring.

Proof:

$$\begin{aligned} uv &= \delta^3(uv) \\ &= \delta^2(\delta(uv)) \\ &= \delta^2(\delta(v)\delta(u)) \\ &= \delta(\delta(\delta(v).\delta(u))) \end{aligned}$$

$= \delta\big(\delta\big(\delta(u)\big).\delta(\delta(v))\big)$
$= \delta\big(\delta^2(u).\delta^2(v)\big)$
$= \delta\big(\delta^2(v)\big).\delta\big(\delta^2(u)\big)$
$= \delta^3(v).\delta^3(u)$
$= vu, \forall\ u, v, \in \bar{I}$ (58)

$\forall u, v \in \bar{I}$ and $b, s \in L\check{N}$:

$uv(bs - sb) = uvbs - uvsb,$ (59)

Since $(\bar{I}L\check{N} \subset \bar{I})$, and from (58):

$uv(bs - sb) = vubs - vsub, \forall u, v \in \bar{I}, \forall b, s \in L\check{N}$ (60)

Since $\bar{I}$ is semigroup ideal of $L\check{N}$, and from(58):

$uv(bs - sb) = ubvs - ubvs, \forall u, v \in \bar{I}\ \ \forall b, s \in L\check{N}$ (61)

That give:

$\bar{I}v(bs - sb) = \{0\}, \forall v \in \bar{I}\ \ \forall b, s \in L\check{N}$ (62)

Lemma 3-1 give that:

$v(bs - sb) = 0, \quad \forall v \in \bar{I}\ \ \forall b, s \in L\check{N}$ (63)

That is:

$\bar{I}(bs - sb) = 0, \quad \forall b, s \in L\check{N}$ (64)

Lemma 3-1 give that:

$bs = sb, \ \ \forall b, s \in L\check{N}$ (65)

By using (65):

$(u + v)h = h(u + v), \forall\ u, v, h \in L\check{N}$ (66)

That give:

$(u + v)h = hu + hv\ , \forall\ u, v, h \in L\check{N}$ (67)

Use (65) and (67) give:

$(u + v)h = uh + vh\ \ , \forall\ u, v, h \in L\check{N}$ (68)

Now, let $u, v, h \in L\check{N}$, applying (68):

$(h + h)(u + v) = h(u + v) + h(u + v), \forall\ u, v, h \in L\check{N}$ (69)

That is:

$(h + h)(u + v) = hu + hv + hu + hv, \forall\ u, v, h \in L\check{N}$ (70)

On the other hand:

$(h + h)(u + v) = (h + h)u + (h + h)v\ , \forall\ u, v, h \in L\check{N}$ (71)

That is:

$(h + h)(u + v) = hu + hu + hv + hv\ \ , \ \ \forall\ u, v, h \in L\check{N}$ (72)

Comparing (70) and (72):

$$hu + hv = hv + hu, \quad \forall\, u, v, h \in L\check{N} \tag{73}$$
give that:
$$h(u + v - u - v) = 0 \;, \quad \forall\, u, v, h \in L\check{N} \tag{74}$$
$L\check{N}$ is left near ring, give that:
$$(u + v - u - v)L\check{N}(u + v - u - v) = \{0\}, \forall\, u, v \in L\check{N} \tag{75}$$
$L\check{N}$ is semiprime:
$$u + v - u - v = 0, \quad \forall\, u, v \in L\check{N} \tag{76}$$
Thus (65) , (68) , and (76) give $L\check{N}$ is commutative ring.

Corollary 3-5:

Let $L\check{N}$ be prime, δ is an anti-homomorphism in $L\check{N}$, such that δ is with period 3 on $L\check{N}$. Then $L\check{N}$ is commutative ring.

Theorm 3-6:

Let $L\check{N}$ be prime. Let $\bar{I} \neq 0$, $\bar{I}$ be semigroup ideal of $L\check{N}$. Suppose δ is an anti-homomrphism in $L\check{N}$. If δ is with period 3 on $\bar{I}$, then $(t + b)y = ty + by, \quad \forall\; t, b, y \in L\check{N}$

Proof:

$$\begin{aligned}
(u + v)h &= \delta^3(u + v).\, \delta^3(h) \\
&= \delta(\delta^2(u + v)).\, \delta(\delta^2(h)) \\
&= \delta(\delta^2(h).\, \delta^2(u + v)) \\
&= \delta\big(\delta\big(\delta(h)\big).\, \delta(\delta(u + v))\big) \\
&= \delta(\delta(\delta(u + v).\, \delta(h))) \\
&= \delta^2(\delta(u + v).\, \delta(h)) \\
&= \delta^2(\delta\big(h(u + v)\big)) \\
&= \delta^3(h(u + v)) \\
&= h(u + v), \quad \forall u, v, h \in \bar{I}
\end{aligned} \tag{77}$$

That is:

$$\begin{aligned}
h(u + v) &= \delta^3(h).\, \delta^3(u + v) \\
&= \delta(\delta^2(h)).\, \delta(\delta^2(u + v)) \\
&= \delta(\delta^2(u + v).\, \delta^2(h)) \\
&= \delta(\delta(\delta(u + v)).\, \delta(\delta(h))) \\
&= \delta(\delta(\delta(h).\, \delta(u + v))) \\
&= \delta^2(\delta(h).\, \delta(u + v)) \\
&= \delta^2(\delta(h)(\delta(u) + (\delta(v))) \\
&= \delta^2(\delta(h)\delta(u) + \delta(h).\, \delta(v))
\end{aligned}$$

$= \delta^2(\delta(uh) + \delta(vh))$
$= \delta^2(\delta(uh) + \delta^2(\delta(vh))$
$= \delta^3(uh) + \delta^3(vh)$
$= uh + vh\,, \;\forall\, u, v, h \in \bar{I}$ (78)

From (77) , (78):

$(u + v)h = uh + vh\,, \;\forall u, v, h \in \bar{I}$ (79)

Putting u = ua, and v = ub, in (60), we obtain:

$(ua + ub)h = uah + ubh, \;\forall\, u, h \in \bar{I}, \quad a, b \in L\check{N}$ (80)

From (78) , (80):

$u(a + b)h = u(ah + bh), \;\forall\, u, h \in \bar{I}, \quad a, b \in L\check{N}$ (81)

That is mean:

$u((a + b)h - (ah + bh)) = 0, \;\forall\, u, h \in \bar{I}, a, b \in L\check{N}$ (82)

Hence:

$\bar{I}((a + b)h - (ah + bh)) = 0, \;\forall\, h \in \bar{I}, \quad a, b \in L\check{N}$ (83)

Use Lemma 3-1:

$(a + b)h = ah + bh, \;\forall\, h \in \bar{I}, \quad a, b \in L\check{N}$ (84)

Now Putting yh instesd of h, y ∈ $L\check{N}$, in (84):

$(a + b)yh = (ayh + byh), \;\forall\, h \in \bar{I}, \quad a, b, y \in L\check{N}$ (85)

From (84), we have:

$(ayh + byh) = (ay + by)h, \;\forall\, h \in \bar{I}, \quad a, b, y \in L\check{N}$ (86)

Using (85) in (86):

$(a + b)yh = (ay + by)h, \;\forall\, h \in \bar{I}, \quad a, b, y \in L\check{N}$ (87)

Relation (79) , give $\forall\, h \in \bar{I}, \; a, b, y \in L\check{N}$:

$\big((a + b)y - (ay + by)\big)h = (a + b)yh + \big(-(ay + by)\big)h,$
$\forall\, h \in \bar{I}, a, b, y \in L\check{N}$ (88)

Relations (87) and (88), give:

$\big((a + b)y - (ay + by)\big)h = (ay + by)h + \big(-(ay + by)\big)h$
$= ((ay + by) + (-(ay + by)))h$
$= 0.\,h$
$= 0\,, \;\forall\, h \in \bar{I}, \; a, b, y \in L\check{N}$ (89)

Implies:

$\big((a + b)y - (ay + by)\big)\bar{I} = 0\,, \forall\, a, b, y \in L\check{N}$ (90)

Lemma 3-1 give:

$(a + b)y = ay + by, \;\forall\; a, b, y \in L\check{N}$ (91)

Corollary 3-7:

Let LŇ be prime. Suppose δ is an anti-homomrphism in LŇ. If δ is with period 3 on LŇ, then:

$$(a+b)y = ay + by, \quad \forall\ a, b, y \in \text{LŇ}$$

Theorm 3-8:

Let LŇ be prime. Let Ī ≠0, Ī be semigroup ideal of LŇ. Suppose δ is an endomorphism in LŇ. If δ is with period 3 on Ī, and δ(Ī) ⊆ Ī, and δ(uv)=δ(vu) , ∀u, v ∈ Ī , then LŇ is commutative ring.

Proof:

$$\begin{aligned} uv &= \delta^3(uv) \\ &= \delta^2(\delta(uv)) \\ &= \delta^2(\delta(vu)) \\ &= \delta^2(\delta(v).\delta(u)) \\ &= \delta(\delta^2(v).\delta^2(u)) \\ &= \delta^3(v).\delta^3(u) \\ &= vu \quad \forall\ u, v \in \bar{I} \qquad (92) \end{aligned}$$

From (59) to the end of (76) we obtain LŇ is commutative.

Corollary 3-9:

Let LŇ be prime. Suppose δ is an endomorphism in LŇ. If δ is with period 3, and δ(uv)=δ(vu) for all u,v ∈ LŇ, then LŇ is commutative ring.

Theorm 3-10:

Let LŇ be prime, Ī is a non-zero semigroup ideal of LŇ , δ is a map on Ī , such that δ is with period 3 and δ(uv)=δ(vu) , ∀ u, v ∈ Ī, then LŇ is commutative ring.

Proof:

$$\begin{aligned} uv &= \delta^3(uv) \\ &= \delta^2(\delta(uv)) \\ &= \delta^2(\delta(vu)) \\ &= \delta^3(vu) \\ &= vu \quad \forall\ u, v \in \bar{I} \qquad (93) \end{aligned}$$

Relations (59) , (76) obtain LŇ is commutative ring.

Corollary 3-11 :
Let LŇ be prime, δ is a map in LŇ, such that δ is with period 3 and δ(uv)=δ(vu) , ∀ u, v ∈ LŇ, then LŇ is commutative ring.

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الدالة الجمعية ذات الدورة 3 في الحلقات والحلقات المقتربة

شـيماء بدر ياس
shai.bader@yahoo.com
الجامعة العراقية

المستخلص

في هذا البحث قدمنا تعريف الدالة ذات الدورة 3 في الحلقة R وفي المثالي الايمن (او الايسر) Ī في R, وبرهنا عندما R حلقة اولية مع char (R) ≠ 2 و Ī مثالي ايمن غير صفري , اذا كان đ اشتقاق ذات الدورة 3 في R فان اما đ=0 او $u^2=0$ لكل u في R. كذلك برهنا عندما R ساحة مع وجود العنصر المحايد 1 و 6 ≠ char (R) , و δ تعميم اشتقاق ذات الدورة 3 في R , فان δ دالة محايدة . أخيرا, قدمنا تعريف الدالة ذات الدورة 3 في الحلقات المقتربة وبرهنا نظريات للحلقات المقتربة اليسرى ذوات الدورة 3 مع الدوال ذات التشاكلات ضد و(ذات التشاكلات), ذات الدورة 3, للحصول على الحلقات المتبادلة.

الكلمات الرئيسية: الحلقة الاولية, الاشتقاق, تعميم الاشتقاق الايمن, حلقة مقتربة اولية, حلقة مقتربة شبه اولية, الدوال ذات الدورة 2, ذات التشاكلات وذات التشاكلات ضد.