4- Serge Lang , " Linear Algebra ", Addition Lasley Publishing Co. ,1983.

الحلول المبسطة للمعادلات الجبرية الخطية

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المستخلص :

إن حل المعادلات الخطية المتجانسة باستخدام قاعدة كرا يمر في متناول اليد بشكل مبسط إذا كانت عدد العناصر الصفرية أكبر ما يمكن. وقد تضمن البحث دراسة الحلول وبشكل أكثر عمومية وبساطة .

إن ضرورة تقديم هكذا حلول مبسطة يكون إستخدامها بشكل خاص في التطبيقات الكيميائية .

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Here according to Theorem 5, {al,...........aq}constitutes a simplex,where $q\text{-}1 \leq r$ (see (6)), So we may complete the linearly independent vectors al,......... .aq-l with, r-(q-l) ≥ 0 vectors: ajl,.............ajr-q+1,the new vector System {al,.........aq-l,ajl,..........ajr-q+l }becoming hereby a basis. Consider now that of the base solutions belonging to this basis which is determine uniquely by the equation (see Definition 6 and Theorem 6):

X1al +......+xq-l aq-l+xjl ajl +......+xjr-q+1. ajr-q+l + Xq aq = 0.

This solution is asserted to be S. Namely,on account of the construction, xj l......... xjr-q-+1 are identically zero, Consequently the equation (15) is left over, whose unique solution is indeed the simple solution S.So S is a base solution.

Appendix

The system of homogeneous linear equations (l) has a solution in which the unknown xjl (jl =1,2,...........,..n) is uniquely (identically) zero ifand only if the rank of the matrix of(l) is by one greater than that of the matrix in which the jl-th column is dropped:

rank [aj2,..........ajn] = rank [al..........an]-l =r- l.

References

L- Bernard Kolman , " Introductory To Linear Algebra With Applications ", Macmillan Publishing Co.Mc.,New York, 1976.

2- Frank M. Stewart, " Introduction To Linear Algebra ", D,Van Nostrand Com. ,New York,1963.

3- Hoffman And Kunze , " Linear Algebra ", Prentice Hall , Inc., Newjersey ,1971.

We can now formulate our following fundamental:

Theorem 8:

The simple solutions are identical with the base solutions.

Proof:

At first we show that the base solutions are simple ones.

Consider a base solution, it is, due to Definition 6 , a solution of an equation of type (11). With out loss of generality we may assume (11) to be of the following form:

X1al + ... +xrar+xkak =0. ------------------------ (13),

Where { al,...............ar } is a basis . Here, according to Theorem 6 ,Xk can not be zero ; if, however, any of the unknowns Xl,........,,,,,Xr is zero .then it must be uniquely zero.

Thus, let the unknowns:

Xjl,............Xjq-l ($2 \leq q \leq r+1$) be different from zero,

then (13) becomes:

Xjlajl +..........+Xjq-l ajq-l + Xkak = 0 . --------------(14),

Where none of the unknowns can be zero any more .

Thus owing to Theorem 4 ,the solution of (14) , i.e. the base solution considered, will be a simple one over :

C={J1,..........Jq-l,k}.

We prove now that a simple solution is a base solution .

Consider a simple solution S , without loss of generality

Assuming it to be of the form [sl,............sq,0,.............0]. It is,because of Definition 3 ,the unique solution of the equation:

Xial +.................. + xqaq =0. ---------------------(15).

Xj1,Aj1, +..........+ XjrAjr,+ Xjk+Ajk = 0 . ------ (11).

Where { aj1,........., ajr }is abase and k = r+ 1 , ..., n .

Theorem6:

(11) Determines one one solution S,for which also Sj ≠ 0 .

Proof:

Let us solve (11) . The number of its unknowns being equal to r + 1 and the rank of its matrix [aj1 , ... , ajr , ajk] equal to r, its solution is unique ,say S (by virtue of Definition 1) Here , moreover , Sj ≠ 0 ,othenvise Sj1= .... = Sjr =0 Would have to hold because { aj1 ,............ ajr } is a basis:thus (11) would have no solution though S was one .

Q.E.D.

The base solution of equation (3) are obtained when solving it by Gramer's nile . More exactly there holds the following:

Theorem 7:

Let us solve (3) according to Gramer' s rule. As known, choosing a basis { aj1 ,............ ajr } , the general solution becomes:

$$S = \sum_{k=r+1}^{n} Xjk\,Sjk . \text{ ------------------------------ (12) .}$$

Where Xjk are so-called free variables. Consider now all the general solutions of type (12) belonging to the possible bases among a1,a2,............,an and consider the set of the different Sjk in these solutions . As a trivial consequence of Definition 6 we may assert that by these Sjk all the base solutions of (3) are represented.

theorem 4 holds consequently if and only if { aj1 ,....,...,ajq} form a simplex.

Proof:

At first we show that ,if {aj1 , ...,ajq } forms a simplex if, the solution of (4) is unique and (8) is also fufilled. The solution of (4) is unique , the number of the

unknowns , q,beng by one greater than rank [aj1,......,ajq] = q-l (see (9)). F

Consider now the (unique) solution of (4) :

S = [s1,............ sn].Were any Sjt (l ≤ t < q)zero here, (4)

without the term corresponding to a would have no solution (see (10) and Definition l), not with Standing that S was the solution of (4) . Thus (8) must be true.

Now we show that ,if the solution of (4) is unique and (8) also holds, the vectors aj1 ....ajq form a simplex.

Owing to the uniqueness (9) holds. Here, omitting any vector a (l ≤ t ≤ q) , the remaining ones linearly independent ; otherwise

Xj would namely be uniquely in (4) , which contradicts (8). Thus (10) holds too.

Q.E.D.

2- Construction of the simple solutions :

In the foregoing it has not yet been mentioned how the simple solutions of (3) can be found . A few theorems with respect to this question will now be proved.

Definition 6:

A solution S of (3) is called abase solution if it is a solution of an equation of the form :

that if a solution is not simple ,one can always find another just as good one.

Let S be a not simple solution over C={ji,,,,,....jq},then(4)

has also another solution ,say S . Let us now form the solution

S + ε S, where $\varepsilon > 0$ is a real number . If ε is small enough , the non-zero elements of S vary hereby only a little, that is ,do not become zero, Therefore S + ε S will be a solution just as good as S.

Q.E.D.

Theorem 4:

Let acombination C = { j1,............Jq } be given.

A simple solution over C exist if and only if the solution of (4) ,say S , is unique, moreover:

$$\prod_{t=1}^{q} Sjt \neq 0 \text{ . ---------------------------------- (8)}$$

This theorem is a trivial consequence of Definitions 2 and 3.

Theorem 5:

The Statement of the previous theorem holds if and only if;

Rank [aj1 .... ajq] = q-l ------ (9). ,

Rank [aj1,..........,ajt+1,..........,ajq] = q -l . -----(10).

t =l ,2,.................... ,q

A System of linearly dependent vectors should be called a simplex if ,by omitting any of them ,the remaining vectors become linearly independent .The Statement of

Proof:

The condition is trivially necessary on the basis of Definition (3). To show that the condition is sufficient we will prove that ,if a solution is not simple, one can always find a better solution.

Let S be a not simple solution over C = {J1,.......Jq },then according to Definition (3) :

Rank [aj,,........ ajq] $\leq$ q-2 .

Setting e.g. Xj in (4) equal to zero ,the new equation :

$$\sum_{t=1}^{q-1} Xjt,Ajt, = 0. \quad \text{---------------------------- (7).}$$

Becomes such that unchanged ,rank [aj1, ,...ajq -1] $\leq$ q - 2.

Therefore,(7) will still have a solution S^1 over some C^1

suchthat $C^1 \subset C^2$

So S' is a better solution. Q.E.D.

Corollary:

The number of the non-zero elements in a simple solution is at least 2 according to (6). Thus, a solution with 2 non-zero elements if existing is certainly simple because of the former theorem.

Theorem 3:

A solution is simple if and only if there does not exist any Other just as good one.

Proof:

The condition is trivially necessary, on the basis of Definition (3) . The sufficiency will be proved in the form

Theorem I:

For the number of the non-zero Elements in a simple solution the inequality holds :

$2 \leq q \leq r+1$, r = rank A. ----------------------- (6) .

Proof:

Sinse the trivial solution of (3) has been disregarded due to Definition 1, no solution with q = 0 exist.Nor does a solution exist with q = 1, A having no column with only zero elements .

Thus, for every solution of (3) , consequently for the simple ones as well, $2 \leq q$ holds.

On the other hand, the inequality rank [aj1,.........ajq] $\leq$ r is always true , hense, owing to (5) :

q = rank[aj1,.........ajq] + 1 $\leq$ r + 1 .

Q.E.D.

Definition 4:

Let S^1 be a solution over C^1 and S^2 be a solution over C^2.

The solution S^1 is said better than S^2 if C ' is a proper sub- set of C^2 ($C^1 \subset C^2$) .

Definition 5:

The solution S^1 is said just as good as S^2 if they are solutions over the same C .

Theorem 2:

A solution is simple if and only if there does no exist any better one.

over **C.**

Remark:

Consider now a solution S=[sl,............sn] of the set of equations:

$$\sum_{j=1}^{n} XjAj = 0\ ,\ Xjq+1 = ... = Xjn = 0.$$

where {jq+1 .... ,Jn} is the complementary set of **C** in

Definition 2. Let **it** agreed that in this case one says ,for the sake

of shortness ,S to be a solution of equation :

$$\sum_{t=1}^{q} Xjt\ Ajt = 0. \quad \text{-----------------------------} \quad (4)$$

Consequently, if S is a solution over C ={jl,.........jq }, S is a *solution of* (4).

Definition 3:

Let S be **a** solution over C= {jl,......... jq}, **S** will be said simple if it is the only solution of **(4)** . *under* consideration of

Definition l, **S is** simple if and only if:

Rank [a1 .... **a** q] = q-l. ----------------------------(5).

$$\sum_{j=1}^{n} XjAij=0, \; I= 1,2,...........m. \quad ------------------ (1).$$

Introducing the column vectors Aj =[Alj,,......,Am)],
(j=l,,....,n), instead of(I):

$$\sum_{j=1}^{n} XjAj=o. \quad .------------------ --- (2).$$

can be written . Defining the matrix **A** =[al,..........an] and
the column vector **X**= [xl,........xn], (I) resp. (2) have the
form:

AX = O ---------------------------------- (3).

We will assume **A** to have no column and no row consisting of
pure zero elements.

Definition L: In the set of the solutions S= [s l ,......,sn]
of (3):

(a)- The trivial solution should be disregarded, and

(b) -Tow solutions S and **a, a** $\neq$ **0** being a real number
,should be considered as a single solution .

So the number of the linearly independent solutions of **(3)**
is n-r, where **r** = rank **A.**

Definition 2: Let the *non-zero elements* of the solutions
S =[sl,........ sn]be Sjl,...................... Sjq **where**
C= {Jl,........... Jq}is a combination of the numbers
1,2,..........n taken **q** $\leq$ **n** at a time.Then **S** is said **a** solution

On The Simple Solutions Of Algebraic Homogeneous Linear Equations

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Abstract

On solving algebraic homogeneous linear equation by Gramers rule, solutions can automatically be obtained in which the number of zero elements is maximal..In the present Communication, those so-called "simple" solutions are defined more simply ,in a combinatorial manner,and their properties are formulated more generally. The necessity of introducing simple solutions emerged originally in connection with a chemical problem.

1-Definitions of simple solutions and several criteria for their existence:

Let us consider the set of homogeneous Linear equations:

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