Semi- Minimax Estimations on the Exponential Distribution Under Symmetric and Asymmetric Loss Functions

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Abstract: *In this paper the semi-minimax estimators of the scale parameter of the exponential distribution are presented by applying the theorem of Lehmann under symmetric (quadratic) loss function and asymmetric (entropy, mlinex , precautionary) loss functions .The results of comparison between these estimators are compared empirically using Monte-Carlo simulation study with respect to the mean square error(MSE) and the mean percentage error(MPE). In general, the results showed that the semi-minimax estimator under quadratic loss function is the best estimator by MSE and MPE for all sample sizes. We can notice that, when the values of the parameters β ,θ increasing the semi-minimax estimator under quadratic loss function is the best estimator by MSE while comparison by MPE showed that the semi-minimax estimator under mlinex loss function when the value of c positive is the best, but they both get worse as α ,θ increases. Also the results showed that when α, β together increase the semi-minimax* *estimator under entropy loss function is the best by MSE while by MPE the semi-minimax estimator under precautionary loss function is the best estimator.*

Keywords: Semi-minimax estimator, Exponential distribution, Bayes estimator, Monte- Carlo simulation

1. Introduction

The minimax estimation is an upgraded non classical approach in the estimation area of statistical inference. The most important element in the minimax approach is the specification of the distribution function on the parameter space, which is called prior distribution, and the minimax estimators depends on the loss functions assumed in this paper quadratic, entropy, precautionary and mlinex loss functions have been used to obtain the minimax estimators of the scale parameter of the exponential distribution. Al-kutubi and Ibrahim (2009) [1] compared Jeffrey prior and the extension of Jeffrey prior information for estimating the parameter of the exponential distribution. Asgharzadeh (2009) [2] bayes estimates of the unknown parameter and the reliability function for the generalized exponential model are derived under various loss functions such as the squared error, the absolute error, the squared log error, and the entropy loss function . Dey (2008)[3] obtained Bayesian predictive intervals of the parameter of Rayleigh distribution. Amrollah (2011)[4] finding the semi-minimax estimators of the scale parameter of the weibull distribution under quadratic and mlinex loss functions. Podder(2004) [8] studied the minimax estimator of the Pareto distribution under quadratic and mlinex loss functions. Masoud and Hassan(2010)[6]studied the minimax estimators of the shape parameter for the Burr type XII distribution under the squared log error, precautionary and weighted balanced squared error loss functions. Nassiri, Sajad and Hassan (2011) [7] studied the semi- minimax estimators of the parameter of the Rayleigh distribution for the well known quadratic and modified linear exponential and the efficiency of the estimator salso been studied. Also shadrokh and pazira (2010)[9] studied the minimax estimator for the minimax distribution under several loss functions.

The probability density function of the exponential distribution with the scale parameter θ is given by:

$$f(x|\theta) = \frac{1}{\theta}\, e^{-\frac{x}{\theta}} \quad ; x > 0 \ \ , \theta > 0 \qquad \text{....... (1-1)}$$

In this paper, we have derived the minimax estimators of θ for the exponential distribution under quadratic, entropy, precautionary, and modified linear exponential (mlinex) loss functions and the derivation depends on the Lehmann's theorem (1950)[5], which can be stated as follows :

2. Lehmann's Theorem[5]

Let $\tau = \{F_\theta : \theta \in \Omega\}$ be a family of distribution functions and D be a class of estimators of θ, suppose that $d^* \in D$ is a Bayes estimator against a prior distribution on the parameter spaceΩ, and the risk function $R(d^*, \theta)$ is constant on Ω , then d^* is a minimax estimator of θ for this theorem, if we can show that the risk function is constant then the theorem above will be followed, first we have must find the Bayes estimator of θ.

Let us assume that θ has a prior distribution function[4] are defined by:

$$g(\theta) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}}\, \theta^{-(\alpha+1)}\, e^{-\frac{1}{\beta\theta}} \qquad \theta > 0\,, \alpha, \beta > 0 \qquad \text{……(2-1)}$$

Then the posterior distribution of θ for the given random sample X=(x1 , x2 ,…., xn) is:

$$\mathrm{h}(\theta|\underline{x}) = \frac{\pi_{i=1}^{n} f(x_i|\theta)\, g(\theta)}{\int_0^{\infty} \pi_{i=1}^{n} f(x_i|\theta)\, g(\theta)\, d\theta} \qquad \text{……(2-2)}$$

The likelihood function of the distribution of f(x |θ) is given by:

$$\pi_{i=1}^{n} f(x_i|\theta) = \frac{1}{\theta^n} \; e^{-\frac{\sum_{i=1}^{n} x_i}{\theta}} \qquad \text{…… (2-3)}$$

Substituting (2-1) and (2-3) in (2-2) , we get :

$$\text{h}(\theta|\underline{x}) = \frac{e^{-\frac{1}{\theta}\left(\sum_{i=1}^{n} x_i+\frac{1}{\beta}\right)} \; \theta^{-(\alpha+1)-n}}{\int_0^\infty e^{-\frac{1}{\theta}\left(\sum_{i=1}^{n} x_i+\frac{1}{\beta}\right)} \; \theta^{-(\alpha+1)-n} \; d\theta}$$

by letting $y = \frac{1}{\theta}\left(\sum_{i=1}^{n} x_i + \frac{1}{\beta}\right)$

$$\text{h}(\theta|\underline{x}) = \frac{e^{-y}\, y^{\alpha+n+1}}{-\left(\sum_{i=1}^{n} x_i+\frac{1}{\beta}\right) \int_0^\infty e^{-y}\, y^{\alpha+n-1}\, dy}$$

$$\text{h}(\theta|\underline{x}) = \frac{-\left(\sum_{i=1}^{n} x_i+\frac{1}{\beta}\right)^{\alpha+n} \; e^{-\frac{1}{\theta}\left(\sum_{i=1}^{n} x_i+\frac{1}{\beta}\right)}}{\theta^{\alpha+n+1}\, \Gamma(\alpha+n)} \qquad \text{……….(2-4)}$$

3. Bayes estimator under quadratic loss function

We consider the quadratic loss function[3] of the form:

$$\text{L}\,(\theta, \hat{\theta}_1) = \left(\frac{\theta-\hat{\theta}_1}{\theta}\right)^2 \qquad \text{…..(3-1)}$$

Then the Bayes estimator of θ for the above loss function is given by:

$$\hat{\theta}_1 = \frac{E\left(\frac{1}{\theta}\middle|\underline{x}\right)}{E\left(\frac{1}{\theta^2}\middle|\underline{x}\right)}$$

Where $E\left(\frac{1}{\theta}\middle|\underline{x}\right) = \int_0^\infty \frac{1}{\theta}\; h(\theta|\underline{x})\, d\theta$

$$= \frac{-\left(\sum_{i=1}^{n} x_i+\frac{1}{\beta}\right)^{\alpha+n}}{\Gamma(\alpha+n)} \int_0^\infty e^{-\frac{1}{\theta}\left(\sum_{i=1}^{n} x_i+\frac{1}{\beta}\right)} \left(\frac{1}{\theta}\right)^{\alpha+n+2} d\theta$$

On simplification, we get:

$$E\left(\frac{1}{\theta}\middle|\underline{x}\right) = \frac{-\left(\sum_{i=1}^{n} x_i+\frac{1}{\beta}\right)^{\alpha+n}}{\Gamma(\alpha+n)} \int_0^\infty e^{-y} \left(\frac{y}{\sum_{i=1}^{n} x_i+\frac{1}{\beta}}\right)^{\alpha+n+2} \frac{-(\sum_{i=1}^{n} x_i+\frac{1}{\beta})}{y^2} \; dy$$

$$= \frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+n)\;\left(\sum_{i=1}^{n} x_i+\frac{1}{\beta}\right)} \qquad \text{……… (3-2)}$$

, and $E\left(\frac{1}{\theta^2}\Big|\underline{x}\right) = \int_0^\infty \frac{1}{\theta^2}\ h\left(\theta\big|\underline{x}\right) d\theta$

$$= \frac{-\left(\sum_{i=1}^{n} x_i+\frac{1}{\beta}\right)^{\alpha+n}}{\Gamma(\alpha+n)} \int_0^\infty e^{-\frac{1}{\theta}\left(\sum_{i=1}^{n} x_i+\frac{1}{\beta}\right)} \left(\frac{1}{\theta}\right)^{\alpha+n+3} d\theta$$

$$= \frac{-\left(\sum_{i=1}^{n} x_i+\frac{1}{\beta}\right)^{\alpha+n}}{\Gamma(\alpha+n)} \int_0^\infty e^{-y} \left(\frac{y}{\sum_{i=1}^{n} x_i+\frac{1}{\beta}}\right)^{\alpha+n+3} \frac{-\left(\sum_{i=1}^{n} x_i+\frac{1}{\beta}\right)}{y^2}\ dy$$

$$= \frac{\Gamma(\alpha+n+2)}{\Gamma(\alpha+n)\ \left(\sum_{i=1}^{n} x_i+\frac{1}{\beta}\right)^2} \qquad \text{……….} (3\text{-}3)$$

Hence , from (3-2) , (3-3) we get:

$$\hat{\theta}_1 = \frac{1}{(\alpha+n+1)} \left(\sum_{i=1}^{n} x_i + \frac{1}{\beta}\right) \qquad \text{………} (3\text{-}4)$$

The risk function of the estimator $\hat{\theta}_1$ is :-

$$R\left(\hat{\theta}_1\right) = E[L(\theta,\hat{\theta}_1) = \frac{1}{\theta^2}\left[\theta^2 - 2\theta\ E\left(\hat{\theta}_1\right) + E\left(\hat{\theta}_1\right)^2\right]$$

Let T= $(\sum_{i=1}^{n} x_i\ )$, such that the statistic T is distributed as gamma distribution with the parameters n, θ with the density:

$$g(t) = \frac{1}{\theta^n \Gamma(n)}\ t^{n-1}\ e^{-\frac{t}{\theta}} \qquad t \geq 0\ , \theta > 0$$

$$\therefore R\left(\hat{\theta}_1\right) = \frac{1}{\theta^2}\left[\theta^2 - \frac{2\theta}{(n+\alpha+1)} E\left(T+\frac{1}{\beta}\right) + \frac{1}{(n+\alpha+1)^2} E\left(T+\frac{1}{\beta}\right)^2\right]$$

By using the relation y= $\frac{t}{\theta}$,t= θ y , dt = θ dy we have:

$$E\left(T+\frac{1}{\beta}\right) = n\theta + \frac{1}{\beta}$$

, and $E\left(T+\frac{1}{\beta}\right)^2 = \theta^2 n(n+1) + \frac{2}{\beta} n\theta + \frac{1}{\beta^2}$

So, $R\left(\hat{\theta}_1\right)$ will be:

$$R\left(\hat{\theta}_1\right) = \{1 - \frac{2n}{(n+\alpha+1)} - \frac{2}{\theta\beta(n+\alpha+1)} + \frac{n^2}{(n+\alpha+1)^2} + \frac{n}{(n+\alpha+1)^2} + \frac{2n}{\theta\beta(n+\alpha+1)^2} + \frac{1}{\beta^2\theta^2(n+\alpha+1)^2}\}$$

We see that $R(\hat{\theta}_1)$ is not constant, there for $\hat{\theta}_1$ is not minimax estimator exactly, so if $\beta \to \infty$ then:

$$R(\hat{\theta}_1) = 1 - \frac{2n}{(n+\alpha+1)} + \frac{n(n+1)}{(n+\alpha+1)^2}$$

Which is independent on θ , hence according to the Lehmann's theorem it follows that $\hat{\theta}_1$ is semi-minimax estimator for the parameter θ.

4. Bayes estimator under entropy loss function

We consider the entropy loss function [2] of the form:

$$L(\theta, \hat{\theta}_2) = \text{b}\left[\left(\frac{\hat{\theta}_2}{\theta}\right) - \ln\left(\frac{\hat{\theta}_2}{\theta}\right) - 1 \right] \quad ,\text{b} > 0 \qquad \text{…..(4-1)}$$

Then the Bayes estimator of θ for the above loss function is given by:

$$\hat{\theta}_2 = \left[E\left(\frac{1}{\theta}\middle|\underline{x}\right)\right]^{-1}$$

From (3-2), we find that:

$$E\left(\frac{1}{\theta}\middle|\underline{x}\right) = \frac{(\alpha+n)}{\left(\sum_{i=1}^{n} x_i + \frac{1}{\beta}\right)}$$

Hence: $$\hat{\theta}_2 = \frac{1}{(\alpha+n)}\left(\sum_{i=1}^{n} x_i + \frac{1}{\beta}\right) \qquad \text{……… (4-2)}$$

The risk function of the estimator $\hat{\theta}_2$ is :-

$$R(\hat{\theta}_2) = E[L(\theta, \hat{\theta}_2)] = \text{b}\left[\frac{1}{\theta}E(\hat{\theta}_2) - E\ln(\hat{\theta}_2) + \ln\theta - 1\right]$$

Where $E(\hat{\theta}_2) = \frac{E(T+\frac{1}{\beta})}{(\alpha+n)} = \frac{(n\theta+\frac{1}{\beta})}{(\alpha+n)}$

And $E[\ln(\hat{\theta}_2)] = E(\ln T) + \frac{1}{\beta} - \ln(\alpha + n)$

Such that $E(\ln T) = \int_0^\infty \ln t \; g(t)\, dt$

$$= \frac{1}{\theta^n \Gamma(n)} \int_0^\infty \ln t \;\; t^{n-1} e^{-\frac{t}{\theta}} dt$$

Let $y = \frac{t}{\theta}$

$$= \frac{1}{\theta^n \Gamma(n)} \int_0^\infty \ln (\theta y) \; (\theta y)^{n-1} \, e^{-y} \, \theta \, dy.$$

$$= \frac{\ln \theta}{\Gamma(n)} \int_0^\infty y^{n-1} \, e^{-y} \; dy + \frac{1}{\Gamma(n)} \int_0^\infty Iny \, y^{n-1} \, e^{-y} \; dy$$

$$\therefore E(\ln \mathrm{T}) = \ln \theta + \frac{\Gamma'(n)}{\Gamma(n)} \qquad \text{…….(4-3)}$$

Where $\Gamma'(n) = \int_0^\infty \mathrm{Iny} \, \mathrm{y}^{\mathrm{n}-1} \, \mathrm{e}^{-\mathrm{y}} \, \mathrm{dy}$ is the first derivative of $\Gamma(\mathrm{n})$ with respect to n.

$$E[\ln(\hat{\theta}_2)] = \ln \theta + \frac{\Gamma'(n)}{\Gamma(n)} + \frac{1}{\beta} - \ln (\alpha + \mathrm{n})$$

So, $R(\hat{\theta}_2)$ will be:

$$R(\hat{\theta}_2) = \mathrm{b} \left[\frac{(n\theta + \frac{1}{\beta})}{\theta(\alpha+n)} - \ln \theta - \frac{\Gamma'(n)}{\Gamma(n)} - \frac{1}{\beta} + \ln (\alpha + \mathrm{n}) + \ln \theta - 1 \right]$$

We see that $\mathrm{R}(\hat{\theta}_2)$ is not constant, there for $\hat{\theta}_2$ is not minimax estimator exactly, so if $\beta \to \infty$ then :

$$R(\hat{\theta}_2) = b \left[\frac{n}{(\alpha+n)} - \frac{\Gamma'(n)}{\Gamma(n)} + \ln (\alpha + \mathrm{n}) - 1 \right]$$

Which is independent on θ, hence according to the Lehmann's theorem it follows that $\hat{\theta}_2$ is the semi-minimax estimator for the parameter θ.

5. Bayes estimator under precautionary loss function

We consider the precautionary loss function [6] of the form:

$$L(\theta, \hat{\theta}_3) = \frac{(\hat{\theta}_3 - \theta)^2}{\hat{\theta}_3 \theta} \qquad \text{….. (5-1)}$$

Then the Bayes estimator of θ for the above loss function is given by:

$$\hat{\theta}_3 = \sqrt{\frac{E(\theta|\underline{x})}{E\left(\frac{1}{\theta}\Big|\underline{x}\right)}}$$

Where $E(\theta|\underline{x}) = \int_0^\infty \theta \;\; h(\theta|\underline{x})\, d\theta$

$$= \frac{-\left(\sum_{i=1}^{n} x_i + \frac{1}{\beta}\right)^{\alpha+n}}{\Gamma(\alpha+n)} \quad \int_0^\infty e^{-\frac{1}{\theta}\left(\sum_{i=1}^{n} x_i+\frac{1}{\beta}\right)} \quad \left(\frac{1}{\theta}\right)^{\alpha+n} \; d\theta$$

And by making similar substitution as above we get:

$$E(\theta|\underline{x}) \;= \frac{-\left(\sum_{i=1}^{n} x_i + \frac{1}{\beta}\right)^{\alpha+n}}{\Gamma(\alpha+n)} \;\; \int_0^\infty e^{-y} \left(\frac{y}{\sum_{i=1}^{n} x_i+\frac{1}{\beta}}\right)^{\alpha+n} \frac{-(\sum_{i=1}^{n} x_i+\frac{1}{\beta})}{y^2} \;\; dy$$

$$= \frac{\Gamma(\alpha+n-1)}{\Gamma(\alpha+n)} \left(\sum_{i=1}^{n} x_i + \frac{1}{\beta}\right) \qquad \text{……. (5-2)}$$

From (3-2) and (5-2) , we find that:

$$\hat{\theta}_3 \;= \frac{\left(\sum_{i=1}^{n} x_i+\frac{1}{\beta}\right)}{\sqrt{(\alpha+n)\;(\alpha+n-1)}} \qquad \text{……… (5-3)}$$

The risk function of the estimator $\hat{\theta}_3$ is:-

$$R(\hat{\theta}_3) = E[L(\theta,\hat{\theta}_3)] = \left[\frac{1}{\theta}E(\hat{\theta}_3) + \theta E\left(\frac{1}{\hat{\theta}_3}\right) - 2\right]$$

$$R(\hat{\theta}_3) = \frac{1}{\theta} \;\; E\left(\frac{(T+\frac{1}{\beta})}{\sqrt{(\alpha+n)\;(\alpha+n-1)}}\right) + \theta\; E\left(\frac{\sqrt{(\alpha+n)\;(\alpha+n-1)}}{(T+\frac{1}{\beta})}\right) - 2$$

$$R(\hat{\theta}_3) = \frac{1}{\theta\;\sqrt{(\alpha+n)\;(\alpha+n-1)}} \;\; \mathrm{E}(\mathrm{T}+\frac{1}{\beta}) + \;\; \theta\;\sqrt{(\alpha+n)\;(\alpha+n-1)} \;\;\; \mathrm{E}\left(\frac{1}{(\mathrm{T}+\frac{1}{\beta})}\right) \;\; -2$$

Where $\mathrm{E}(\,\mathrm{T}+\frac{1}{\beta}) = \mathrm{n}\,\theta + \frac{1}{\beta}$

and $\mathrm{E}(\frac{1}{\mathrm{T}}) = \int_0^\infty \frac{1}{\mathrm{t}}\; g(t)\;\mathrm{dt} \;\; = \;\; \frac{1}{\theta(n-1)}$

Hence: $R(\hat{\theta}_3) = \frac{(\mathrm{n}\,\theta+\frac{1}{\beta})}{\theta\,\sqrt{(\alpha+n)\;(\alpha+n-1)}} + \;\; \frac{\theta\sqrt{(\alpha+n)\;(\alpha+n-1)}}{\theta\;\;(n-1)} + \frac{\theta\sqrt{(\alpha+n)\;(\alpha+n-1)}}{\beta} -2$

We see that $R(\hat{\theta}_3)$ is not constant, there for $\hat{\theta}_3$ is not minimax estimator exactly, so if $\beta\to\infty$ then:

$$R(\hat{\theta}_3) = \frac{n}{\sqrt{(\alpha+n)\ (\alpha+n-1)}} + \frac{\sqrt{(\alpha+n)\ (\alpha+n-1)}}{(n-1)} -2$$

Which is independent on θ, according to the Lehmann's theorem it follows that $\hat{\theta}_3$ is the semi-minimax estimator for the parameter θ.

6. Bayes estimator under mlinex loss function

We consider the modified linear exponential (mlinex) loss function[4] of the form:

$$L(\theta\,,\hat{\theta}_4) = K\left[\left(\frac{\hat{\theta}_4}{\theta}\right)^c - c\ln\left(\frac{\hat{\theta}_4}{\theta}\right) - 1\right] \quad ; k>0\,,\ c\neq 0 \qquad \ldots\ldots\ (6\text{-}1)$$

Then the Bayes estimator of θ for the above loss function is given by:

$$\hat{\theta}_4 = \frac{1}{\left[E\left(\frac{1}{\theta^c}\middle|\underline{x}\right)\right]^{\frac{1}{c}}}$$

Where $E\left(\frac{1}{\theta^c}\middle|\underline{x}\right) = \int_0^\infty \frac{1}{\theta^c}\, h(\theta|\underline{x})\, d\theta$

$$= \frac{-\left(\sum_{i=1}^n x_i+\frac{1}{\beta}\right)^{\alpha+n}}{\Gamma(\alpha+n)} \int_0^\infty e^{-\frac{1}{\theta}\left(\sum_{i=1}^n x_i+\frac{1}{\beta}\right)} \left(\frac{1}{\theta}\right)^{\alpha+n+c+1} d\theta$$

$$= \frac{-\left(\sum_{i=1}^n x_i+\frac{1}{\beta}\right)^{\alpha+n}}{\Gamma(\alpha+n)} \int_0^\infty e^{-y} \left(\frac{y}{\sum_{i=1}^n x_i+\frac{1}{\beta}}\right)^{\alpha+n+c+1} \frac{-\left(\sum_{i=1}^n x_i+\frac{1}{\beta}\right)}{y^2}\, dy$$

$$E\left(\frac{1}{\theta^c}\middle|\underline{x}\right) = \frac{\Gamma(\alpha+n+c)}{\Gamma(\alpha+n)\ \left(\sum_{i=1}^n x_i+\frac{1}{\beta}\right)^c}$$

Hence, $\hat{\theta}_4 = \left[\frac{\Gamma(\alpha+n)}{\Gamma(\alpha+n+c)}\right]^{\frac{1}{c}} \left(\sum_{i=1}^n x_i + \frac{1}{\beta}\right)$ …….. (6-2)

The risk function of the estimator $\hat{\theta}_4$ is:

$$R(\hat{\theta}_4) = E[L(\theta, \hat{\theta}_4)] = k\left[\frac{1}{\theta^c} E(\hat{\theta}_4)^C - c\,\mathrm{E}\ln(\hat{\theta}_4) + c\ \ln\ \theta - 1\right]$$

$$\mathrm{E}(\hat{\theta}_4)^c = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha+n+c)}\ E\left(T + \frac{1}{\beta}\right)^c$$

Where $E(T)^c = \int_0^\infty t^c\ g(t)\,dt = \frac{\Gamma(n+c)}{\Gamma(n)}\ \theta^c$

$$\therefore E(\hat{\theta}_4)^c = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha+n+c)}\ \frac{\Gamma(n+c)}{\Gamma(n)}\ \theta^c\ + \frac{1}{\beta^c}$$

And $E[\ln(\hat{\theta}_4)] = \ln\ \left[\frac{\Gamma(\alpha+n)}{\Gamma(\alpha+n+c)}\right]^{\frac{1}{c}} + E(\ln \mathrm{T} + \frac{1}{\beta})$

From (4-3) ,we get:

$$E\ [\ln(\hat{\theta}_4)] = \ln\ \left[\frac{\Gamma(\alpha+n)}{\Gamma(\alpha+n+c)}\right]^{\frac{1}{c}} + \ln\ \theta + \frac{\Gamma'(n)}{\Gamma(n)} + \frac{1}{\beta}$$

$$R\ (\hat{\theta}_4) = k\left[\frac{\Gamma(\alpha+n)\Gamma(n+c)}{\Gamma(\alpha+n+c)\Gamma(n)} + \frac{1}{\beta^c} - c\ \ln\ \left[\frac{\Gamma(\alpha+n)}{\Gamma(\alpha+n+c)}\right]^{\frac{1}{c}} - c\ln\theta - \frac{c\Gamma'(n)}{\Gamma(n)} - \frac{c}{\beta} + c\ln\theta - 1\right]$$

We see that $R(\hat{\theta}_4)$ is not constant, there for $\hat{\theta}_4$ is not minimax estimator exactly, so if $\beta\rightarrow\infty$ then :

$$R(\hat{\theta}_4) = k\left[\frac{\Gamma(\alpha+n)\Gamma(n+c)}{\Gamma(\alpha+n+c)\Gamma(n)} - \ln\frac{\Gamma(\alpha+n)}{\Gamma(\alpha+n+c)} - \frac{c\Gamma'(n)}{\Gamma(n)} - 1\right]$$

Which is independent on θ, according to the Lehmann's theorem it follows that $\hat{\theta}_4$ is the semi-minimax estimator for the parameter θ.

Empirical study

The estimated values, MSE and MPE of the different estimators of the parameter θ are computed by the Monte-Carlo simulation method using the exponential distribution, where:

$$\mathrm{MES}(\hat{\theta}) = \frac{\sum_{i=1}^{R}(\hat{\theta}_i - \theta)^2}{R}, \qquad \mathrm{MPE}(\hat{\theta}) = \frac{\sum_{i=1}^{R}\frac{|\hat{\theta}_i - \theta|}{\theta}}{R}$$

In the simulation study, we have chosen n = 5, 15, 25 and 50 for several values of c = -1, 2 also for θ =1, 3, $\alpha = 0.2, 1.5$ and

$\beta = 0.3$, 2.5 all results are based on replications (R=2000).The results were summarized and tabulated in the following tables for each estimator and for all sample sizes.

Table (1): Estimated values, MSE and MPE of the different estimators for θ of the exponential distribution for different values of n when θ=1, $\alpha = 0.2$, and $\beta = 0.3$

n	Criteria	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{\theta}_4$	
					c=-1	c=2
5	estimated value	1.3480	1.6073	1.7884	1.9899	1.4719
	MSE	0.2493	0.5509	0.8472	1.2593	0.3756
	MPE	0.3835	0.6131	0.7897	0.9901	0.4883
15	estimated value	1.1333	1.2078	1.2496	1.2929	1.1699
	MSE	0.0746	0.1078	0.1314	0.1598	0.0895
	MPE	0.2108	0.2562	0.2863	0.3204	0.2317
25	estimated value	1.0796	1. 1224	1.1454	1. 1688	1.1008
	MSE	0.0434	0.0550	0.0628	0.0719	0.0487
	MPE	0.1621	0.1831	0.1965	0.2116	0.1718
50	estimated value	1.0404	1.0611	1.0719	1.0827	1.0507
	MSE	0.0206	0.0234	0.0252	0.0273	0.0218
	MPE	0.1134	0.1209	0.1256	0.1309	0.1169

Table (2): estimated values, MSE and MPE of the different estimators for θ of the exponential distribution for different values of n when θ=1, $\alpha = 0.2$ and $\beta = 2.5$

n	criteria	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{\theta}_4$	
					c=-1	c=2
5	estimated value	0.8749	1.0432	1.1607	1.2915	0.9553
	MSE	0.1438	0.1841	0.2514	0.3643	0.1548
	MPE	0.3092	0.3266	0.3715	0.4447	0.3098
15	estimated value	0.9522	1.0148	1.0499	1.0863	0.9830
	MSE	0.0591	0.0648	0.0716	0.0814	0.0609
	MPE	0.1969	0.2022	0.2101	0.2215	0.1981
25	estimated value	0.9676	1.0060	1.0266	1.0476	0.9866
	MSE	0.0381	0.0401	0.0424	0.0457	0.0387
	MPE	0.1565	0.1587	0.1623	0.1675	0.1568
50	estimated value	0.9831	1.0027	1.0128	1.0231	0.9929
	MSE	0.0192	0.0197	0.0202	0.0210	0.0193
	MPE	0.1112	0.1119	0.1132	0.1151	0.1113

Table (3): estimated values, MSE and MPE of the different estimators for θ of the exponential distribution for different values of n when θ=1 , $\alpha = 1.5$ and $\beta = 0.3$

n	criteria	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{\theta}_4$	
					c=-1	c=2
5	estimated value	1.1144	1.2858	1.3978	1.5196	1.1970
	MSE	0.1007	0.1983	0.2961	0.4328	0.1399
	MPE	0.2344	0.3355	0.4243	0.5312	0.2759
15	estimated value	1.0491	1.1127	1.1479	1.1844	1.0804
	MSE	0.0511	0.0675	0.0802	0.0961	0.0581
	MPE	0.1769	0.2005	0.2188	0.2408	0.1867
25	estimated value	1.0285	1.0674	1.0881	1.1092	1.0478
	MSE	0.0344	0.0407	0.0454	0.0510	0.0372
	MPE	0.1460	0.1574	0.1658	0.1760	0.1509
50	estimated value	1.0147	1.0344	1.0445	1.0548	1.0245
	MSE	0.0182	0.0199	0.0211	0.0225	0.0189
	MPE	0.1073	0.1117	0.1118	0.1184	0.1092

Table (4): estimated values, MSE and MPE of the different estimators for θ of the exponential distribution for different values of n when θ=1 , $\alpha = 1.5$ and $\beta = 2.5$

n	criteria	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{\theta}_4$	
					c=-1	c=2
5	estimated value	0.7233	0.8345	0.9072	0.9863	0.7769
	MSE	0.1642	0.1439	0.1464	0.1631	0.1508
	MPE	0.3492	0.3147	0.3077	0.3139	0.3488
15	estimated value	0.8815	0.9349	0.9646	0.9952	0.9078
	MSE	0.0628	0.0590	0.0596	0.0621	0.0602
	MPE	0. 2071	0.1978	0.1969	0.1994	0.2012
25	estimated value	0.9219	0.9567	0.9752	0.9942	0.9391
	MSE	0.0397	0.0381	0.0382	0.0391	0.0386
	MPE	0.1616	0.1569	0.1564	0.1574	0.1586
50	estimated value	0.9588	0.9774	0.9870	0.9968	0.9680
	MSE	0.0197	0.0192	0.0193	0.0195	0.0194
	MPE	0.1133	0.1114	0.1111	0.1115	0.1121

Table (5): estimated values, MSE and MPE of the different estimators for θ of the exponential distribution for different values of n when θ=3, $\alpha = 0.2$ and $\beta = 0.3$

n	criteria	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{\theta}_4$	
					c=-1	c=2
5	estimated value	2.9688	3.5397	3.9386	4.3825	3.2417
	MSE	1.1545	1.9312	2.9114	2.4251	1.5503
	MPE	0.2783	0.3417	0.4205	0.5309	0.2995
15	estimated value	2.9883	3.1848	3.2951	3.4091	3.0849
	MSE	0.5117	0.6153	0.7091	0.8332	0.5525
	MPE	0.1907	0.2042	0.2171	0.2348	0.1958
25	estimated value	2.9843	3.1027	3.1662	3.2309	3.0429
	MSE	0.3336	0.3709	0.4028	0.4440	0.3484
	MPE	0.1531	0.1596	0.1655	0.1731	0.1556
50	estimated value	2.9910	3.0506	3.0815	3.1126	3.0207
	MSE	0.1704	0.1797	0.1874	0.1971	0.1741
	MPE	0.1099	0.1123	0.1145	0.1172	0.1109

Table (6): estimated values, MSE and MPE of the different estimators for θ of the exponential distribution for different values of n when θ=3, $\alpha = 0.2$ and $\beta = 2.5$

n	criteria	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{\theta}_4$	
					c=-1	c=2
5	estimated value	2.4957	2.9756	3.3109	3.6841	2.7251
	MSE	1.4079	1.6405	2.1270	2.9817	1.4509
	MPE	0.3274	0.3311	0.3643	0.4245	0.3222
15	estimated value	2.8072	2.9918	3.0954	3.2026	2.8981
	MSE	0.5488	0.5812	0.6311	0.7067	0.5556
	MPE	0.2009	0.2031	0.2092	0.2188	0.2004
25	estimated value	2.8723	2.9863	3.0474	3.1097	2.9288
	MSE	0.3496	0.3605	0.3775	0.4028	0.3516
	MPE	0.1585	0.1591	0.1619	0.1663	0.1580
50	estimated value	2.9337	2.9922	3.0224	3.0530	2.9628
	MSE	0.1746	0.1772	0.1813	0.1872	0.1751
	MPE	0.1119	0.1121	0.1131	0.1147	0.1117

Table (7): estimated values, MSE and MPE of the different estimators for θ of the exponential distribution for different values of n when θ=3 , $\alpha = 1.5$ and $\beta = 0.3$

n	criteria	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{\theta}_4$	
					c=-1	c=2
5	estimated value	2.4542	2.8317	3.0785	3.3467	2.6362
	MSE	1.0762	1.0778	1.2465	1.5860	1.0419
	MPE	0.2746	0.2842	0.2919	0.3124	0.2786
15	estimated value	2.7663	2.9339	3.0271	3.1232	2.8489
	MSE	0.4930	0.4975	0.5257	0.5740	0.4878
	MPE	0.1887	0.1914	0.1924	0.1987	0.1891
25	estimated value	2.8432	2.9505	3.0078	3.0662	2.8964
	MSE	0.3272	0.3283	0.3387	0.3563	0.3247
	MPE	0.1524	0.1537	0.1539	0.1570	0.1535
50	estimated value	2.9169	2.9736	3.0029	3.0325	2.9452
	MSE	0.1689	0.1690	0.1716	0.1761	0.1681
	MPE	0.1097	0.1102	0.1103	0.1114	0.1099

Table (8): estimated values, MSE and MPE of the different estimators for θ of the exponential distribution for different values of n when θ=3, $\alpha = 1.5$ and $\beta = 2.5$

n	criteria	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{\theta}_4$	
					c=-1	c=2
5	estimated value	2.0631	2.3805	2.5878	2.8133	2.2161
	MSE	1.6661	1.4333	1.4102	1.5240	1.5007
	MPE	0.3741	0.3350	0.3237	0.3250	0.3517
15	estimated value	2.5986	2.7561	2.8437	2.9339	2.6762
	MSE	0.5995	0.5526	0.5494	0.5698	0.5632
	MPE	0.2142	0.2026	0.2002	0.2035	0.2072
25	estimated value	2.7365	2.8398	2.8949	2.9512	2.7877
	MSE	0.3719	0.3515	0.3496	0.3591	0.3543
	MPE	0.1654	0.1593	0.1581	0.1601	0.1616
50	estimated value	2.8611	2.9167	2.9454	2.9744	2.8887
	MSE	0.1813	0.1753	0.1746	0.1775	0.1757
	MPE	0.1147	0.1123	0.1118	0.1119	0.1132

Results and Discussions

In the most cases, the results in tables (1,2,3,5) showing that the semi-minimax estimators under quadratic loss function have the smallest MSE and MPE this is mean these estimators are better than others with all sample sizes. From table (4) we can see clearly that the semi-minimax estimators under entropy loss function became the best with respect to MSE ,but with respect to MPE we see that the semi-minimax estimators under precautionary loss function was better. Also to be noted that from table (6)when β, θ increases the comparison by MSE showed that the semi-minimax estimators under quadratic loss function was the best while by MPE the semi-minimax estimators under mlinex loss function when c positive gives better results, but from table (7) they both get worse as α , θ increases. From table (8) we see that the semi-minimax estimators under precautionary loss function became the best when α, β θ increases. In general the results show for all sample sizes that the semi-minimax estimators under mlinex loss function when c positive gives better results than mlinex loss function when c negative with respect to MSE and MPE.

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Appendix (1)

برنامج محاكاة يوضح مقدرات بيز $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\theta}_3$, $\hat{\theta}_4$ عندما قيمة c موجبة

```
INPUT "INPUT THE NUMBER OF ITERATIONS"; NM

INPUT "INPUT THE NUMBER OF OBSERVATIONS"; N

INPUT "INPUT TH"; TH

INPUT "INPUT THE A"; A

INPUT "INPUT B"; B

INPUT "INPUT C"; C

DIM X(N), S(N), D(N), M(NM), B1(NM), B2(NM), B3(NM), B4(NM)

DIM PX(N), TY(N), SR(N), DR(N), AV(NM)

DIM N1(N), N2(N), XR(N), YR(N), XJ(N ^ 2), YJ(N ^ 2), P(N), T(N),
RX(N), RY(N)

DIM ML(NM)

SM = 0: sb1 = 0

sb2 = 0: sb3 = 0: SB4 = 0

F1 = 1

NA = N + A: NC = N + A + C - 1

FOR I = NA TO NC

F1 = F1 * I

PRINT I, F1

NEXT I

SX = 0

FOR RP = 1 TO NM
```

```
SX = 0
FOR I = 1 TO N
U = RND
X(I) = (-1) * TH * LOG(RND)
SX = SX + X(I)
NEXT I
ML(RP) = SX / N
B1(RP) = (SX + (1 / B)) / (N + A + 1)
B2(RP) = ((1 / F1) ^ (1 / C)) * (SX + (1 / B))
B3(RP) = (SX + (1 / B)) / (N + A)
B4(RP) = (SX + (1 / B)) / (SQR((N + A) * (N + A - 1)))
PRINT RP, ML(RP), B2(RP), B3(RP)
s1 = s1 + ((B1(RP)) - TH) ^ 2
s2 = s2 + ((B2(RP)) - TH) ^ 2
S3 = S3 + ((B3(RP)) - TH) ^ 2
S4 = S4 + ((B4(RP)) - TH) ^ 2
SAM = SAM + (ABS(ML(RP) - TH)) / TH
sb1 = sb1 + (ABS(B1(RP) - TH)) / TH
sb2 = sb2 + (ABS(B2(RP) - TH)) / TH
sb3 = sb3 + (ABS(B3(RP) - TH)) / TH
SB4 = SB4 + (ABS(B4(RP) - TH)) / TH
SUM = SUM + ML(RP)
su1 = su1 + B1(RP)
```

```
SU2 = SU2 + B2(RP)

SU3 = SU3 + B3(RP)

SU4 = SU4 + B4(RP)

5104 NEXT RP

PRINT "N="; N; ; "ITERA. = "; NM, "TH = "; TH; "A  = "; A; "B="; B;
"C="; C

PRINT "EXPECTED."; su1 / NM; SU2 / NM, SU3 / NM, SU4 / NM

PRINT

PRINT "MSE"; "B1"; s1 / NM; s2 / NM; S3 / NM; S4 / NM

PRINT

PRINT "MPE"; sb1 / NM; sb2 / NM; sb3 / NM; SB4 / NM

PRINT

END
```

Appendix (2)

برنامج محاكاة يوضح مقدربيز $\hat{\theta}_4$ عندما قيمة **c** سالبة

```
INPUT "INPUT THE NUMBER OF ITERATIONS"; NM

INPUT "INPUT THE NUMBER OF OBSERVATIONS"; N

INPUT "INPUT TH"; TH

INPUT "INPUT THE A"; A

INPUT "INPUT B"; B

INPUT "INPUT C"; c

DIM X(N), S(N), D(N), M(NM), B1(NM), B2(NM), B3(NM), B4(NM)

DIM PX(N), TY(N), SR(N), DR(N), AV(NM)

DIM N1(N), N2(N), XR(N), YR(N), XJ(N ^ 2), YJ(N ^ 2), P(N), T(N),
RX(N), RY(N)

DIM ML(NM)

SM = 0: SB1 = 0

sb2 = 0: sb3 = 0

F1 = 1

NA = N + A + c: NC = N + A - 1

FOR I = NA TO NC

F1 = F1 * I

PRINT I, F1

NEXT I

SX = 0

FOR RP = 1 TO NM
```

```
SX = 0
FOR I = 1 TO N
U = RND
X(I) = (-1) * TH * LOG(RND)
SX = SX + X(I)
NEXT I
B3(RP) = ((F1) ^ (1 / c)) * (SX + (1 / B))
PRINT  B3(RP)
s3 = s3 + ((B3(RP)) - TH) ^ 2
sb3 = sb3 + (ABS(B3(RP) - TH)) / TH
su3 = su3 + B3(RP)
5104 NEXT RP
PRINT "N="; N; ; "ITERA. = "; NM, "TH = "; TH; "A  = "; A; "B="; B;
"C="; c
PRINT "EXPECTED."; su3 / NM
PRINT
PRINT "MSE" ; s3 / NM
PRINT
PRINT "MPE";   sb3 / NM
PRINT
PRINT PAR
END
```

تقديرات Semi-minimax للتوزيع الأسي تحت دوال خسارة متناظرة وغير متناظرة

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المستخلص

في هذا البحث تم إيجاد مقدرات semi-minimax لمعلمة القياس للتوزيع الآسي بتطبيق مبرهنة Lehmann باستخدام دوال خسارة متناظرة وغير متناظرة . وان نتائج المقارنة بين هذه المقدرات وجدت تجريبيا وباستخدام دراسة المحاكاة وبالاعتماد على متوسط مربعات الخطأ ومتوسط الخطأ النسبي. أظهرت النتائج وبصورة عامة إن مقدر semi-minimax تحت دالة الخسارة المتناظرة quadratic كان الأفضل تبعا إلى MSE وMPE ولكافة إحجام العينة وتبين انه في حالة زيادة قيم المعالم θ, β كان مقدرsemi-minimax تحت دالة الخسارة المتناظرة quadratic هو الأفضل وفقا إلى MSE إما بالنسبة إلى MPE فان المقدر تحت دالة الخسارة mlinex عندما قيمة c موجبة هو الأفضل بينما يحصل العكس في حالة زيادة قيم المعالم α,θ كما أظهرت النتائج في حالة زيادة قيم المعالم α,β سوية كان المقدر تحت دالة الخسارة الغير متناظرة entropy هو الأفضل وفقا إلى MSE، إما وفقا إلى MPE فان المقدر تحت دالة الخسارة precautionary كان هو الأفضل.

الكلمات الرئيسية: مقدر semi-minimax، التوزيع الآسي، مقدر بيز، محاكاة مونت كارلو