On Threshold Relation & Their Application

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Abstract

In this paper, The notion of. Threshold relations by using resemblance relation are introduced. To get a similarity relation from a resemblance relation R, we must build the transitive closure $\hat{R}$ of the relation R..

Introduction

A fuzzy set μ on a universe X is a mapping from X to the unit interval [0,1], with the value μ(x) of μ in x of X the degree of membership of x in μ. μ(x) = 1 means full membership, μ(x) =0 means non–membership, and all values μ (x) in (0,1) denote partial membership.[1]

In order to analyze a fuzzy set μ in X at a particular membership degree $\alpha \in [0,1]$,we can" cut" the fuzzy set at the degree, and consider only the set of elements x of X that have a membership degree μ(x) of at least . α The crisp set constructed in this way is called the α –cut of the fuzzy set μ ,and denoted as. μ_α Just as a fuzzy set extends a classical or crisp set, a fuzzy relation then extends the concept of a crisp relation from X to Y.

The product $X \times X$ is defined by the membership function. $\mu_{X \times X}(x,y) = \mu(x) \wedge \mu(y)$ and a fuzzy relation R on X is a subset Of $X \times X$i,e.; $\mu_R(x,y) \le \mu_{X\times X}(x,y)\ \forall x,y \in X$.

In this paper, we introduce a unifying point of view based on the theory of fuzzy sets [2] and, more particularly, fuzzy relations. We will then discover some new and very interesting properties.

We introduce and study threshold relation by using resemblance relation. We illustrate other programs, by using some results.

Similarity Relation

The notion of "similarity" as defined in this paper is essentially a generalization of the notion of equivalence. The notion of a distance, $d(x, y)$, between objects x and y has long been used in many contexts as a measure of similarity or dissimilarity between elements of a set.

Definition 1 [3],[4],[5]

A similarity relation R, is a fuzzy relation in X that is reflexive, Symmetric, and Transitive. Thus let x_i, x_j be elements of X, and let $\mu_R(x_i, x_j)$ denoted the grade-membership of the order pair (x_i, x_j) in R. Then R is a similarity relation in X if and only if $\forall\ x, y, z \in X$:

1. $\mu_R(x,x) = 1$ (reflexive)
2. $\mu_R(x,y) = \mu_R(y,x)$ (symmetric)
3. $\mu_R(x,z) \ge \bigvee_{y \in X} (\mu_R(x,y) \wedge \mu_R(y,z))$ (transitive)

Now, we introduce the following definition

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Definition 2 [4]

Let $\mu_R(x,y)$ be a resemblance relation. Then $\mu_R^n(x,y)$ is called the n-purlieus relation, defined as

$$\mu_R^n(x,y) = \bigvee_{x_1,x_2,x_3,\dots x_{n-1}\in X} [\mu_R(x,x_1) \wedge \mu_R(x_1,x_2) \wedge \mu_R(x_2,x_3) \wedge \dots \wedge \mu_R(x_{n-1},y)]$$

for $n = 2,3,4$

...since $\mu_R^1(x,y) = \mu_R(x,y)$.

Clearly we have

$$0 \le \mu_R(x,y) \le \mu_R^2(x,y) \le \dots \le \mu_R^n(x,y) \le \mu_R^{n+1}(x,y) \le \dots \le 1$$

since

$$\mu_R^{n+1}(x,y) = \bigvee_{x_1,x_2,\dots,x_n\in X} [\mu_R(x,x_1) \wedge \mu_R(x_1,x_2) \wedge \dots \wedge \mu_R(x_{n-1},x_n) \wedge \mu_R(x_n,y)]$$

$$\ge \bigvee_{x_1,x_2,\dots,x_{n-1}\in X} [\mu_R(x,x_1) \wedge \mu_R(x_1,x_2) \wedge \dots \wedge \mu_R(x_{n-1},y) \wedge \mu_R(y,y)]$$

$$= \bigvee_{x_1x_2\dots x_{n-1}\in X} [\mu_R(x,x_1) \wedge \mu_R(x_1,x_2) \wedge \dots \wedge \mu_R(x_{n-1},y) \wedge 1]$$

$$= \mu_R^n(x,y).$$

Clearly, the n-purlieus relation defined as:

$$\mu_R^n(x,y) = \bigvee_{\vec{x}\in\hat{X}} [\mu_R(x,x_1) \wedge \mu_R(x_1,x_2) \wedge \dots \wedge \mu_R(x_{n-1},y)]$$

Where

$\hat{X}$ $\vec{x} = (x_1,x_2,\dots,x_{n-1}) \in \hat{X}$

is the (n-1)-fold Cartesian product of X with itself where $x, y \in X$) implies that for all $x, y \in X$ and all $n \ge 1$.

$$0 \le \mu_R^n(x,y) \le \mu_R^{n+1}(x,y) \le 1$$

Consequently

$$\lim_{n\to\infty} \mu_R^n(x,y) = \bar{\mu}_R(x,y)$$

exists by the monotone convergence principle, namely, for every $\in > 0$, there is an integer N such that:

$$\left|\mu_R^n(x,y) - \bar{\mu}_R(x,y)\right| < \in \text{ for } n > N.$$

Since the sequence is non-decreasing and is bounded from above and below we can conclude that the limit exists.

Now we introduce the following definition which we help to define threshold relation.

Definition 3

Let x and y be two elements of X, and Let $\mu_R^n(x,y)$ be the n-purlieus relation as defined above. Then we define the propinquity $\bar{\mu}(x,y)$ in [0,1] such that

$\bar{\mu}_R(x,y) = \lim_{n\to\infty} \mu_R^n(x,y)$ or

$\bar{\mu}(x,y) = \lim_{n\to\infty} \mu_R^n(x,y)$

Definition 4

Let $x, y \in X$. Then x and y are said to have a threshold relation xR_Ty if and only if $\bar{\mu}_R(x,y) \ge T$ .It is easy to show that $\forall x,y,z \in X$ we have

$$\bar{\mu}_R(x,z) \ge \bar{\mu}_R(x,y) \wedge \bar{\mu}_R(y,z)$$

Now we introduce the following theorems about threshold relation property

Theorem 1

The threshold relation is a similarity relation in X.

Proof

i [0,1] xR_Tx $\forall T \in$ -

since $1 = \mu_R(x,x) \le \bar{\mu}_R(x,x) \le 1$ $\forall x \in X$

ii xR_Ty if and only if yR_Tx -

since

$\lim_{n\to\infty} \mu_R^n(x,y) = \overline{\mu}_R(x,y) = \overline{\mu}_R(y,x)$

$= \lim_{n\to\infty} \mu_R^n(y,x)$

iii $x\, R_T\, y \;\; \wedge y\, R_T\, z \rightarrow x\, R_T\, z$ -

Since $\overline{\mu}_R(x,z) \ge \overline{\mu}_R(x,y) \wedge \overline{\mu}_R(y,z)$

Clearly we can associate with every relation an appropriate matrix to represent the relation, and hence we can classify the set using the partition induced by the threshold relation .

The fuzzy matrix approach is a pictorial representation of a graph theoretic principle involving the selection of a threshold distance. Once the threshold distance d_0 is selected, two elements are said to be in the same set if the distance between them is less than d_0.

Suppose that we pick a threshold value d_0 and say that α is similar to β if $R(\alpha,\beta) > d_0$

$$R_{ij} = \begin{cases} 1 & \text{if } R(\alpha_i,\beta_j) > d_0 \quad i,j = 1,2,..,n \\ 0 & \text{otherwise} \end{cases}$$

This matrix defines a similarity graph in which nodes correspond to points and an edge joins node i and node j if and only if $R_{ij} = 1$.

It is clear that if R_T is a threshold relation induced by $\mu_R(x,y)$ and R'_T is a threshold relation induced by $\mu'_R(x,y)$ and

$\mu_R(x,y) \le \mu'_R(x,y)$ for all $x,y \in X$.

Then R_T refines R'_T.

It is our assumption that if $x \ne y$ then $(0,1]\, \overline{\mu}_R(x,y) \in$ and thus the function $\overline{\eta}_R(x,y) = 1 - \overline{\mu}_R(x,y)$ acts as a distance function. This is obvious, since

i. $\overline{\eta}_R(x,y) > 0$ for $x \ne y$

and $\overline{\eta}_R(x,x) = 0$

ii. $\overline{\eta}_R(x,y) = \overline{\eta}_R(y,x)$

iii. $\overline{\eta}_R(x,z) \le \overline{\eta}_R(x,y) + \overline{\eta}_R(y,z)$

as

$\overline{\mu}_R(x,z) \ge \overline{\mu}_R(x,y) \wedge \overline{\mu}_R(y,z) \ge$

$\overline{\mu}_R(x,y) + \overline{\mu}_R(y,z) - 1$

We shall assume for simplicity of the analysis that we deal with only a finite number of sets, and hence we shall consider our threshold relation on finite sets only.

Theorem 2

Let $x_1, x_2, x_3, \ldots, x_n \in X$ where n is a finite number .

$R_T = \bigcup_j (R_T^*)^j = (R_T^*)^{n-1}$. Then

$x\, R_T^*\, y$ iff $\mu_R(x,y) \ge T$ and

$\bigcup_j (R_T^*)^j = \sup_j (R_T^*)^j = adj(R_T^*)$

Proof

i - clearly $\bigcup_j (R_T^*)^j$ refines R_T since

$\bigcup_j (R_T^*)^j = \sup_j (R_T^*)^j$ and if

$x[\bigcup_j (R_T^*)^j]y$ then

$\exists\, x_1, x_2, x_3, \ldots, x_{n-1} \in X$ .such that

$\mu_R(x,x_1) \ge T$, $\mu_R(x_1,x_2) \ge T$

$,\ldots\ldots, \mu_R(x_{n-1},y) \ge T$

and hence

$\mu_R^n(x,y) \ge \mu_R(x,x_1) \wedge$

$\mu_R(x_1,x_2) \wedge \ldots\ldots \wedge \mu_R(x_{n-1},y) \ge T$

which implies that

$\overline{\mu}_R(x,y) \ge \mu_R^n(x,y) \ge T$

and thus $x\ R_T\, y$

ii- Assume $x\, R_T\, y$ then

$\mu_R(x,y) = \mu_R^{n-1}(x,y)$

$= \vee_{x_1,x_2,\ldots,x_{n-2}\in X}[(\mu_R(x,x_1) \wedge \mu_R(x_1,x_2)$

$\wedge \ldots\ldots \wedge \mu_R(x_{n-2},y)] \geq T$

Therefore $\exists\, x_1, x_2, x_3, \ldots, x_{n-2} \in X$. such that

$\mu_R, \mu_R(x,x_1) \geq T$ $(x_1, x_2) \geq T, \ldots,$

$\mu_R(x_{n-2},y) \geq T$

and thus we have $x R_T^* x_1, x_1 R_T^* x_2, \ldots\ldots,$

$x_{n-2} R_T^*$ y

$\rightarrow x\ (R_T^*)^{n-1} y$ exists

$\rightarrow x[\bigcup_j (R_T^*)^j]y$ exists

iii– Let $- R_T^*$ $[P_{ij}]$. The ij entry of

$(R_T^*)^2$ is $\sum_{k=1}^{n} P_{ik}P_{kj}$ and this term has a grade membership of

$\vee_k [P_{ik} \wedge P_{kj}]$

If there is a direct path between vertices i and j or there is a path from i to j through one intermediate vertex. Extending this argument to $(R_T^*)^i$ it is clear that no path requires more then n-2 intermediate vertices, since there are only n vertices and internal loops are excluded. Hence the ij entry of $(R_T^*)^{n-1}$ has a grade membership of

$\vee_{subterm} \{ij \text{ terms of } (R_T^*)^{n-1}\}$.

If and only if i and j are connected, namely $(R_T^*)^{n-1} = \bigcup_j (R_T^*)^j$.

Corollary 3

$R_T^* = \bigcup_j (R_T^*)^j$ if and only if $(R_T^*)^2 = R_T^*$.

Proof

$R_T^* = \bigcup_j (R_T^*)^j = (R_T^*)^{n-1}$ by above theorem

Now, if

$R_T^* = \bigcup_j (R_T^*)^j \Rightarrow R_T^* = \bigcup_j (R_T^*)^j = (R_T^*)^{n-1}$

$\Rightarrow n-1 = 1 \Rightarrow n = 2 \Rightarrow \bigcup_j (R_T^*)^j = R_T^* \Rightarrow (R_T^*)^2 = R_T^*$

Conversely if

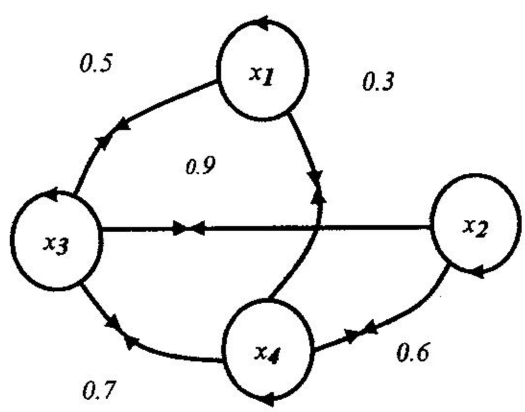
$(R_T^*)^2 = R_T^* \Rightarrow \bigcup_j (R_t^*)^j = R_T^*$

Now we introduce the following examples which clear above definitions and theorems :

Example 1

Let $X = \{x_1, x_2, x_3, x_4\}$ and $\mu_R(x_i, x_j)$ i, j = 1,2,3,4 be as follows

$$R_T^* = \begin{array}{c|cccc} & x_1 & x_2 & x_3 & x_4 \\ \hline x_1 & 1 & 0 & 0.5 & 0.3 \\ x_2 & 0 & 1 & 0.9 & 0.6 \\ x_3 & 0.5 & 0.9 & 1 & 0.7 \\ x_4 & 0.3 & 0.6 & 0.7 & 1 \end{array}$$



Then

$$\bigcup_j (R_T^*)^j = \begin{array}{c|cccc} & x_1 & x_2 & x_3 & x_4 \\ \hline x_1 & 1 & 0.5 & 0.5 & 0.5 \\ x_2 & 0.5 & 1 & 0.9 & 0.7 \\ x_3 & 0.5 & 0.9 & 1 & 0.7 \\ x_4 & 0.5 & 0.7 & 0.7 & 1 \end{array} = (R_T^*)^2 = (R_T^*)^3$$

and we have the partitions

$R_{T=1} = \{[x_1], [x_2], [x_3], [x_4]\}$

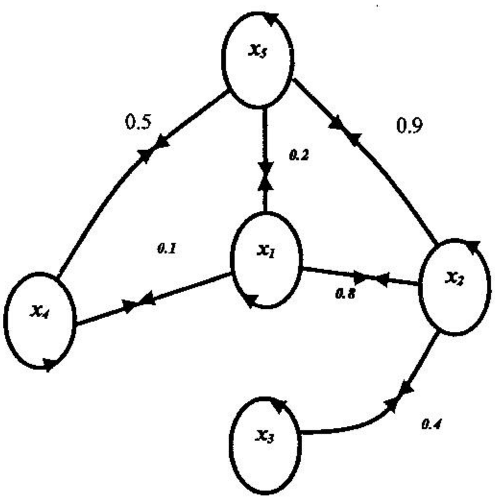
$R_{0.7<T\leq 0.9} = \{[x_1], [x_2, x_3], [x_4]\}$

$R_{0.5<T\leq 0.7} = \{[x_2, x_3, x_4], [x_1]\}$

$R_{0<T\leq 0.5} = \{[x_1, x_2, x_3, x_4]\}$

Example 2

$$R_T^* = \begin{matrix} & x_1 & x_2 & x_3 & x_4 & x_5 \\ x_1 & 1 & 0.8 & 0 & 0.1 & 0.2 \\ x_2 & 0.8 & 1 & 0.4 & 0 & 0.9 \\ x_3 & 0 & 0.4 & 1 & 0 & 0 \\ x_4 & 0.1 & 0 & 0 & 1 & 0.5 \\ x_5 & 0.2 & 0.9 & 0 & 0.5 & 1 \end{matrix}$$



$$\bigcup_j (R_T^*)^j = \begin{matrix} & x_1 & x_2 & x & x_4 & x_5 \\ x_1 & 1 & 0.8 & 0.4 & 0.5 & 0.8 \\ x_2 & 0.8 & 1 & 0.4 & 0.5 & 0.9 \\ x_3 & 0.4 & 0.4 & 1 & 0.4 & 0.4 \\ x_4 & 0.5 & 0.5 & 0.4 & 1 & 0.5 \\ x_5 & 0.8 & 0.9 & 0.4 & 0.5 & 1 \end{matrix} = (R_T^*)^4$$

Thus we have the partitions

$R_{T=1} = \{[x_1], [x_2], [x_3], [x_4], [x_5]\}$

$R_{1>T>0.8} = \{[x_1], [x_2, x_5], [x_3], [x_4]\}$

$R_{0.8\geq T>0.5} = \{[x_1, x_2, x_5], [x_3], [x_4]\}$

$R_{0.5\geq T>0.4} = \{[x_1, x_2, x_4, x_5], [x_3]\}$

$R_{0.4\geq T\geq 0} = \{[x_1, x_2, x_3, x_4, x_5]\}$

Now, we introduce the following definitions and theorems which represent some application about threshold relation .

Definition 5

A vertex x is said to be $\in$-reachable form another vertex y, for some $0 < \in \leq 1$ if and only if $\mu_{\bigcup_j (R_T^*)^j}(x,y) \geq \in$

The reachability matrix of R_T^* denoted . $\bigcup_j (R_T^*)^j$ The $\in$ -reachability matrix of

R_T^* denoted by $\left(\bigcup_j (R_T^*)^j\right)^{\in}$ is obtained from $\bigcup_j (R_T^*)^j$ such that

$\mu_{\left(\bigcup_j (R_T^*)^j\right)^{\in}}(x,y) = 1$ if and only if $\geq \in \mu_{\bigcup_j (R_T^*)^j}(x,y)$.

Definition 6

A relation R_T^* is called strongly $\in$ -connected if and only if every pair of nodes are mutually $\in$-reachable. R_T^* is said to be initial $\in$-connected if and only if there exists $x \in X$ such that every node y in R_T^* is $\in$-reachable from x.

A maximal strongly $\in$-connected sub relation (MS-$\in$CS) of R_T^* is a strongly $\in$-connected sub graph not property contained in any other Ms-$\in$CS. It is easily seen that strongly $\in$ -connectedness implies initial $\in$ -connectedness .

Also, the following result is straight for ward.

Theorem 4

A fuzzy relation R_T^* is strongly $\in$ -connected if and only if there exists a node x such that for any other node y in $\bigcup_j (R_T^*)^j (x,y) \geq \in$ ، R_T^*.

Example 3

Let R_T^* given in Figure (1 . a),and $\left(\bigcup_j (R_T^*)^j\right)^{0.8}$ is given in Figure (1.b)

$$R_T^* = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0.5 & 0.3 \\ 0 & 1 & 0.9 & 0.6 \\ 0.5 & 0.9 & 1 & 0.7 \\ 0.3 & 0.6 & 0.7 & 1 \end{bmatrix} \end{array},$$

$$\bigcup_j (R_T^*)^j = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} 1 & 0.5 & 0.5 & 0.5 \\ 0.5 & 1 & 0.9 & 0.7 \\ 0.5 & 0.9 & 1 & 0.7 \\ 0.5 & 0.7 & 0.7 & 1 \end{bmatrix} \end{array}$$

(Figure 1.a)

$$\left(\bigcup_j (R_T^*)^j\right)^{0.8} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Figure (1. b)

We see that the MS.8–CS'S of R_T^* contain the following nodes sets $\{x_1\},\{x_2, x_3\},\{x_4\}$, respectively.

The previous result is now applied to clustering analysis we assume that a data is given, where X is a set of data and ($\mu_{R_T^*}$ x, y) is a quantitive measure of the similarity of the two data items x and y. For 0<1 ،$\in\leq$ an $\in$ -cluster in X is a maximal subset Y of X such that each pair of elements in Y is $\in$ -cluster of X is tantamount of finding all maximal strongly $\in$ -connected sub relation of R_T^*

Example 4

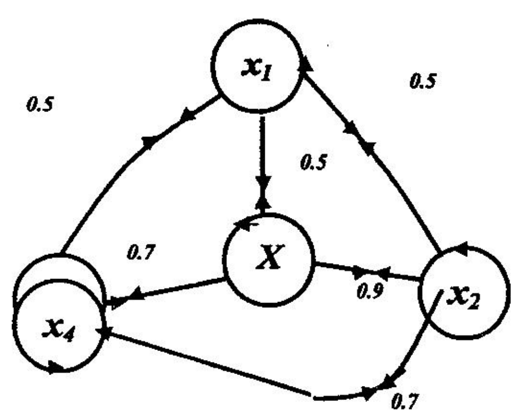
Let

$$R_T^* = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0.5 & 0.3 \\ 0 & 1 & 0.9 & 0.6 \\ 0.5 & 0.9 & 1 & 0.7 \\ 0.3 & 0.6 & 0.7 & 1 \end{bmatrix} \end{array}, (R^*_T)^2 =$$

$$(R_T^*)^2 = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} 1 & 0.5 & 0.5 & 0.5 \\ 0.5 & 1 & 0.9 & 0.7 \\ 0.5 & 0.9 & 1 & 0.7 \\ 0.5 & 0.7 & 0.7 & 1 \end{bmatrix} \end{array} = (R_T^*)^3 = (R_T^*)^4$$

Let

$$S = \sum_{i=1}^{2} (R_T^*)^i = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} 1 & 0.5 & 0.5 & 0.5 \\ 0.5 & 1 & 0.9 & 0.7 \\ 0.5 & 0.9 & 1 & 0.7 \\ 0.5 & 0.7 & 0.7 & 1 \end{bmatrix} \end{array} = R_T^* \vee (R_T^*)^2$$



clear that S is reflexive and symmetric and transitive since $(S)^2 = S$, $V = \{x_1, x_2, x_3, x_4\}$

$$F_{0.8}^{s} = \{k \subseteq V \mid (\forall x \in V)[x \in K \leftrightarrow (\forall y \in k)(\mu_S(x,y) \geq 0.8)]\}$$

$$= [\{x_1\},\{x_2,x_3\},\{x_4\}]$$

Then, each element in $F_{\in}^{S}$ is an $\in$ -cluster we may also define an $\in$ -cluster in X as a maximal subset Y of X such that every element of Y is $\in$ -reachable from a special element y in Y. In this case, the construction of $\in$ -clusters is equivalent to finding all maximal initial $\in$ -connected.

Another application is the use of R_T^* ، $\bigcup_j (R_T^*)^j$ to model information net works.[6], [7].

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حـــول علاقـــة العتبـــة وتطبيقـــاتهـــا

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المستخلص

في هذا البحث عرفنـا علاقـة العتبـة باستخدام العلاقـة المتماثلـة وللحـصول علـى علاقـة متجانسة من علاقـة متماثلـة R يجب أن ننشـأ انغلاق التعدي للعلاقـة R.