Fuzzy Linear Transformations
التحويلات الخطية الضبابية

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ABSTRACT

In this paper , the concept of fuzzy linear transformation have been investigated , this lead us to study and give some properties concerning with it .

Moreover , we give some types of fuzzy as a fuzzy Kernel and its relationships with fuzzy linear transformation and a characterizations of fuzzy linear transformation is presented .

المستخلص

في هذا البحث قدمنا التحويلات الخطية الضبابية كتعميم للتحويلات الخطية الاعتيادية , والذي قادنا الى دراسة واعطاء العديد من الخواص المتعلقة بهذا المفهوم .
كذلك برهنا النتائج الاساسية المناظرة للمفهوم الاعتيادي , بالاضافة الى ذلك اعطينا بعض المفاهيم الضبابية الاخرى مثل النواة وغيرها لبيان مدى علاقتها بالتحويلات الخطية الضبابية .

INTRODUCTION

The present paper introduces and studies fuzzy linear transformation . In fact , some basic definitions and results which will be needed later are recalled.

In section one , we applies the concept of fuzzy set on a vector space and we give some of properties , a binary operations addition and scalar multiplication . Finally , we studied and debated some properties that are necessary in this work .

In section two , we studied and discusses the concept of fuzzy linear transformation on vector space , a binary operations addition and scalar multiplication and some theorems we studied and discusses the concept of a fuzzy kernel on a vector space .

In section three , we shall give definition and some properties of fuzzy coset and we shall give definition and some properties of quotient fuzzy ring . Moreover , we studied the concept of a fuzzy isomorphism .

Throughout this paper (R,+, .) be a commutative ring with identity .

PRELIMINARY CONCEPTS

In this section some basic definitions and results which we will be used in the next section are considered .

Let X be a nonempty set , **A fuzzy subset of X** is a function from X into [0,1] , ([7] , [2]) .

Let A and B be fuzzy subset of X . If for all $x \in X$, $A(x) \le B(x)$ then we write $\mathbf{A \subseteq B}$. If $A \subseteq B$ and there exists $x \in X$ such that $A(x) < B(x)$, then we write $\mathbf{A \subset B}$ and we say that A is a proper fuzzy subset of B, [7] . Note that **A = B** if and only if A(x) = B(x) , for all $x \in X$, ([7] , [8]) .

Let λ_X denote the characteristic function of X defined by $\lambda_X(x) = 1$ if $x \in X$ and $\lambda_X(x) = 0$ if $x \notin X$, ([8] , [1]) .

Let (R,+,·) be a commutative ring with identity , for each $t \in [0,1]$, the set $\mathbf{A_t = \{ x \in R \mid A(x) \ge t \}}$ is called **a level subset of R** and A = B if and only if $A_t = B_t$ the set $\mathbf{A_* = \{ x \in R \mid A(x) > 0 \}}$ is called **the support of R** , ([2] , [1]) .

Let $x \in X$ and $t \in [0,1]$, let x_t denote the fuzzy subset of X defined by $x_t(y) = 0$ if $x \neq y$ and $x_t(y) = t$ if $x = y$ for all $y \in R$. x_t is called **a fuzzy singleton** , ([1],[8]) . If x_t and y_s are fuzzy singletons, then $x_t + y_s = (x + y)_\lambda$ and $x_t \circ y_s = (x . y)_\lambda$, where $\lambda = \min \{ t ,s \}$, ([7],[4],[8]).
Let $I^R = \{A_i \mid i \in \Lambda\}$ be a collection of fuzzy subset of R. Define the fuzzy subset of R (**intersection**) by ($\bigcap_{i \in \Lambda} A_i$) (x) = inf { $A_i(x) \mid i \in \Lambda$ } for all $x \in R$, ([2] ,[4]). Define the fuzzy subset of R **(union)** by ($\bigcup_{i \in \wedge} A_i$) (x) = sup { $A_i(x) \mid i \in \Lambda$ } for all $x \in R$, ([3] ,[4]).

The empty fuzzy subset of R denote by ϕ is definition by : $\phi (x) = 0$ for all $x \in R$, ([7] , [4]) .

Let A and B be fuzzy subsets of R, **the product $A \circ B$** define by :$A \circ B (x) = \sup \{A(y), B(z) \} \mid x = y \cdot z \}$ y, z $\in$ R , for all $x \in R$, ([2],[3]) . And **the addition A+B** define by : $A+B (x) = \sup \{A(y), B(z) \} \mid x = y +z \}$ y, z $\in$ R, for all $x \in R$, ([4],[2]) .

The complement of A denoted by $E = A^c$ and define by $E(x) = A^c(x) =1- A(x)$, for all $x\in R$,[7].

When we say fuzzy subset we mean a non empty fuzzy subset. We let Im (A) denotes **the image of A**. We say that A is a **finite –valued** if Im(A) is finite and **|Im(A)|** denotes the cardinality of Im(A) ,([2],[8]).

Let $f : X \to Y$, A and B are two nonempty fuzzy subsets of nonempty sets X and Y respectively , the fuzzy subset f (A) of Y defined by : f (A) (y) = sup A(x) if $x \in f^{-1} (y) = \varnothing$, $y \in Y$ and f(A) (y) = 0 , otherwise , where $f^{-1} (y) = \{ x: f(x) = y \}$. It is called **the image of A under f** and denoted by f (A) . The fuzzy subset f^{-1} (B) of R defined by : $f^{-1} (B) (x) = B(f(x))$, for $x \in X$.(i.e. $f^{-1} (B) = (B \circ f)$. Is called **the inverse image of B** and denoted by f^{-1} (B) , [2] . A fuzzy subset A of X is called **f-invariant** if f(x) = f(y) implies A(x) = A(y) , where x ,y $\in$ X , [8] . A is called **the sup property ,** if every set of Im (A) , the image of A has a maximal element , ([8],[4]) .

Let X be a nonempty set and a fuzzy set A in X can be represented by the set of pairs : A = {(x , A(x) : $x \in X$ }. **The family of all fuzzy sets in X** is denoted by $\mathbf{I^X}$ ([7],[2]) .

Let A be a non empty fuzzy subset of a group G, A is called **a fuzzy subgroup of G** if for all x, $y \in G$, $A(x + y) \geq \min \{A(x), A(y)\}$ and $A(x) = A(-x)$, ([8],[8],[4]) .

A is a non empty fuzzy subset of R, A is called **a fuzzy ring of R** if and only if for all x, $y \in R$, then $A(x - y) \geq \min \{A(x), A(y)\}$ and $A(x \cdot y) \geq \min \{A(x), A(y)\}$, ([8] , [2]) .

A non empty fuzzy subset A of R is called **a fuzzy ideal of R** if and only if for all x, $y \in R$, then $A(x - y) \geq \min \{A(x), A(y)\}$ and $A(x \cdot y) \geq \max \{A(x), A(y)\}$, ([4] , [2]) . It is clear that every fuzzy ideal of R is a fuzzy ring of R, but the converse is not true.

SECTION ONE

Fuzzy Vector Space

In this section, we applies the concept of fuzzy set on vector space and we give some of properties , a binary operations addition and scalar multiplication . Finally , we studied and debated some properties that are necessary in this work .

DEFINITION 1.1 [3]:

A vector space over a field F is a set X , whose elements are called vectors which two operations , addition ($+ : X\times X \to X$) and scalar multiplication($. : F\times X \to X$) with conditions is satisfies :-

1. $x + y \in X$, for all x , $y \in X$;
2. $x + y = y + x$, for all x , $y \in X$;
3. $x + (y + z) = (x + y) + z$, for all x , y , $z \in X$;
4. There exists $0 \in X$ such that $0 + x = x$, for all $x \in X$ and 0 is the zero vector or the origin ;
5. For all $x \in X$, there is a unique element $(-x) \in X$ such that $x + (-x) = 0$;
6. $\lambda x \in X$, for $\lambda \in F$ and for all $x \in X$;
7. $\lambda (x + y) = \lambda x + \lambda y$, for $\lambda \in F$ and x , $y \in X$;

8. $(\lambda + \alpha) x = \lambda x + \alpha x$, for $\lambda , \alpha \in F$ and $x \in X$;
9. $(\lambda \alpha) x = \lambda (\alpha x)$, for $\lambda , \alpha \in F$ and $x \in X$;
10. $I. x = x .I = x$, for all $x \in X$ and I is the unity element of the field F .

DEFINITION 1.2 [2]:

If X be a vector space over F and $A , B \subseteq X$, $G \subseteq F$, the following notations will be used : $A + B = \{ x = a + b : a \in A , b \in B\}$ and $G A = \{ x = \lambda a : a \in A , \lambda \in G \}$.

DEFINITION 1.3 [15]:

If A , B are fuzzy sets in vector space X over F and let $\lambda \in X$, $G \subseteq F$. We define A + B and λ A by :-

1. $A + B = f (A \times B)$, where $A\times B\ (x , y) = \min \{ A(x) , B(y) \}$ and $f\ X \times X \to X$ is a function defined by $f (x ,y) = (x + y)$, for all $x , y \in X$.
2. $\lambda A = g (A)$ where $g : X \to X$ is a function defined by $g (x) = \lambda x$, for all $x \in X$.

DEFINITION 1.4 [11] :

A fuzzy subset A of a field F is **a fuzzy field of F** if

1. $A(1)=1$.
2. $A(x-y) \geq \min\{A(x) , A(y)\}$, for each $x , y \in F$.
3. $A (x y^{-1}) \geq \min\{A(x) , A(y)\}$, for each $x , y \in F ,y \neq 0$.

Let A be a fuzzy field of F . If $x \in F$, $x \neq 0$, then $A(0) = A(1) \geq A(x) = A(-x) = A(x^{-1})$.

- **DEFINITION 1.5 [15] :**

Let X be a vector space over F . A fuzzy set A in X is called **a fuzzy subspace over F** if :

1. $A + A \subseteq A$;
2. $\lambda A \subseteq A$, for all $\lambda \in F$.

- **DEFINITION 1.6 [11] :**

A is a fuzzy set of a vector space V over a field F . A is **a fuzzy subspace of V over a fuzzy subfield K of F** if :

1. $A(0) > 0$;
2. $A(x-y) \geq \min \{A(x) ,A(y) \}$, for all $x , y \in V$;
3. $A(cx) \geq \min \{ K(c) , A(x) \}$, for all $c\in F$, for all $x \in V$.

THEROEM 1.7 [8] :

Let A be a fuzzy set in a vector space X over F , then the following statements are equivalent :

1. A is a fuzzy subspace of X .
2. For all $\alpha , \beta \in F$, we have $\alpha A + \beta A \subseteq A$.
3. For all $\alpha , \beta \in F$ and for all $x , y \in X$, we have $A (\alpha x + \beta y) \geq \min \{A(x) ,A(y) \}$.

PROPOSITION 1.8 [16] :

1. If A is a fuzzy subspace of vector space X over F . Then $A(0) > A(x)$, for all $x \in X$.
2. If A is a fuzzy set in a vector space X over F . Then A is a fuzzy subspace of X if and only if A_t is a subspace of X , for all $0 \leq t \leq A(0)$.

PROPOSITION 1.9 [15] :

1. If A , B are fuzzy subspaces of vector space X over F and $\lambda \in F$. Then λA , $A + B$, $A \cap B$ are fuzzy subspaces in X .
2. Let $x , y \in X$ and A be a fuzzy set of a vector space X over F such that $A(x) > A(y)$, then $A(x + y) = A (y)$.
3. If A is a fuzzy subspace of vector space X over F and $x , y \in X$ with $A(x) \neq A(y)$, then $A(x + y) = \min \{A(x) , A (y) \}$.

PROPOSITION 1.9 [15] :

Let X , Y be two vector spaces over F and let $f : X \to Y$, be a linear function . Then

1. If A is a fuzzy subspace in X , then f(A) is a fuzzy subspace in Y .
2. If B is a fuzzy subspace in Y , then $f^{-1} (B)$ is a fuzzy subspace in X .

SECTION TWO

Fuzzy Linear Transformations

In this section , we studied and discusses the concept of fuzzy linear transformation on vector space , a binary operations addition and scalar multiplication and citation some theorems . Finally , we studied and debated some properties that are necessary in this work .

DEFINITION 2.1 [3] :

Let X , Y be two vector spaces over F and let $f : X \to Y$ is called a linear transformation on a vector space if :-

1. $f(x + y) = f(x) + f(y)$, for all $x , y \in X$;
2. $f(\lambda x) = \lambda f(x)$, for all $x \in X$ and $\lambda \in F$.

Or $f(\lambda x + \alpha y) = \lambda f(x) + \alpha f(y)$, for all $x , y \in X$ and $\lambda , \alpha \in F$.

The linear transformation $f : X \to Y$ is called a linear functional on X .

DEFINITION 2.2 :

Let A , B be two fuzzy subspaces of vector spaces X , Y over F respectively . $K : A \to B$ is called a fuzzy linear transformation on a fuzzy subspace if :-

$K(\lambda x_t + \alpha y_h) \geq \min\{ K(x_t) , K(y_h) \}$, for all $x_t , y_h \in A$, $\lambda , \alpha \in F$ and $t , h \in [0,1]$.

The fuzzy linear transformation $K : A \to B$ is called a fuzzy linear functional on A .

EXAMPLES 2.3 :

1. Let A be a fuzzy subset of R^3 such that $A(a , b , c) = 1$ for all $(a ,b ,c) \in R^3$ and B be a fuzzy subset of R^2 such that $B(a , b) = 1/2$ for all $(a ,b) \in R^2$. $K : A \to B$ such that $K(a ,b ,c) = (a ,b)$, for all $(a ,b ,c) \in R^3$. Is K a fuzzy linear transformation on a fuzzy subspace A .

Solution :

To prove A and B are two fuzzy subspaces of vector spaces R^3 and R^2 respectively .

Let $x , y \in R^3$, $\alpha , \beta \in F$(F is a field) , then $A(\alpha x + \beta y) \geq \min\{A(x) ,A(y)\}$, where $x = (a_1 ,b_1 ,c_1)$, $y = (a_2 ,b_2 ,c_2)$.

$A(\alpha x + \beta y) = \sup\{ \min\{ A(u) ,A(w)\} \mid u + w = \alpha x + \beta y \}, u ,w \in R^3$.

$= \sup\{ \min\{ A(\alpha x) ,A(\beta y)\} \}$,

$= \sup\{ \min\{ \sup\{ \min\{A(\alpha) , A(x)\}\} , \sup\{ \min\{ A(\beta) ,A(y)\}\} \} \}$,

$= \sup\{ \min\{ A(\alpha) , A(x) , A(\beta) ,A(y) \} \}$, [4] ,

$\geq \sup\{ \min\{ A(x) ,A(y)\} \}$,

$\geq \min\{ A(x) ,A(y)\}$.

A and B are two fuzzy subspaces of vector spaces R^3 and R^2 respectively .

$K : A \to B$ such that $K(a ,b ,c) = (a ,b)$, for all $(a ,b ,c) \in R^3$.

$K(\alpha x+ \beta y) = \sup\{ \min\{ K(\alpha x) , K(\beta y) \}\}$, for all $x, y \in R^3$.

$= \sup\{ \min\{ K(s) , K(d) \} \mid s= \alpha x , d = \beta y \}$, for all $s, d \in R^3$.

$= \sup\{ \min\{ K(s) , K(d) \} \mid s= (\alpha a_1, \alpha b_1, \alpha c_1) , d = (\beta a_2 , \beta b_2 , \beta c_2)\}$,

$\geq \sup\{ \min\{ K(x) , K(y) \}\}$,

$\geq \min\{ K(a) , K(b) \}$,

Hence K is a fuzzy linear transformation on a fuzzy subspace A .

2. Let A be a fuzzy subset of R^2 such that $A(a , b) = 1/3$ for all $(a ,b) \in R^2$ and B be a fuzzy subset of R such that $B(a) = 1/4$ for all $a \in R$. $K : A \to B$ such that $K(a ,b) = a$, for all $(a ,b) \in R^2$. Is K a fuzzy linear transformation on a fuzzy subspace A .

Solution :

To prove A and B are two fuzzy subspaces of vector spaces R^2 and R respectively .

Let $x , y \in R^2$, $\alpha , \beta \in F$(F is a field) , then $A(\alpha x + \beta y) \geq \min\{A(x) ,A(y)\}$, where $x = (a_1 ,b_1)$, $y = (a_2 ,b_2)$.

$A(\alpha x + \beta y) = \sup\{ \min\{ A(u) ,A(w)\} \mid u + w = \alpha x + \beta y \}, u ,w \in R^2$.

$= \sup\{ \min\{ A(\alpha x) ,A(\beta y)\} \}$,

$= \sup\{ \min\{ \sup\{ \min\{A(\alpha) , A(x)\}\} , \sup\{ \min\{ A(\beta) ,A(y)\}\} \} \}$,

$= \sup \{ \min \{ A(\alpha), A(x), A(\beta), A(y) \} \}$, [4] ,
$\geq \sup \{ \min \{ A(x), A(y) \} \}$,
$\geq \min \{ A(x), A(y) \}$.

A and B are two fuzzy subspaces of vector spaces R^2 and R respectively .
$K : A \rightarrow B$ such that $K(a,b) = a$, for all $(a,b) \in R^2$.
$K(\alpha x + \beta y) = \sup\{ \min \{ K(\alpha x), K(\beta y) \}\}$, for all $x, y \in R^2$.
$= \sup\{ \min \{ K(s), K(d) \} \mid s = \alpha x, d = \beta y \}$, for all $s, d \in R^2$.
$= \sup\{ \min \{ K(s), K(d) \} \mid s = (\alpha a_1, \alpha b_1), d = (\beta a_2, \beta b_2)\}$,
$\geq \sup\{ \min \{ K(x), K(y) \}\}$,
$\geq \min \{ K(a), K(b) \}$,

Hence K is a fuzzy linear transformation on a fuzzy subspace A .

REMARK 2.4 :

1. Let A , B be fuzzy subspaces of vector spaces X , Y over F respectively . $K : A \rightarrow B$ is called **a fuzzy Zero transformation** on a vector space if $K(x_t) = 0_t$, for all $x_t \in A$ and $t \in [0,1]$.
2. Let A , B be fuzzy subspaces of vector spaces X , Y over F respectively . $K : A \rightarrow B$ is called **a fuzzy Identity transformation** on a vector space if $K(x_t) = x_t$, for all $x_t \in A$ and $t \in [0,1]$.

THEROEM 2.5 :

Let A , B be fuzzy subspaces of vector spaces X , Y over F respectively . $K : A \rightarrow B$ is a fuzzy linear transformation . Then , for all $t \in [0,1]$,

1. $K(0_t) = 0_t$;
2. $K(-x_t) = -K(x_t)$, for all $x_t \in A$;
3. $K(x_t - y_h) = K(x_t) - K(y_h)$, for all $x_t , y_h \in A$ and $t , h \in [0,1]$;
4. $K(\sum_{i=1}^{n} \lambda_i x_{ti}) \geq \sum_{i=1}^{n} \lambda_i K(x_{ti})$, for all $x_{ti} \in A$, $\lambda_i \in F$ and $t_i \in [0,1]$, i = 1, 2, … , n .

PROOF:

1. Since $0_t \circ 0_t = 0_t$, then $K(0_t) = K(0_t \circ 0_t) = K(0_t) \circ K(0_t) = 0_t$.
2. $K(-x_t) = K[(-1)(x_t)] = -1\ K(x_t) = -K(x_t)$, for all $x_t \in A$.
3. $K(x_t - y_h) = K[(x_t) - (y_h)] = K(x_t) + K(-y_h) = K(x_t) - K(y_h)$, for all $x_t , y_h \in A$.
4. Since $K(\lambda_1 x_{t1}) = \lambda_1 K(x_{t1}) = \lambda_1 x_{t1}$, let $K(\sum_{i=1}^{k} \lambda_i x_{ti}) \geq \sum_{i=1}^{k} \lambda_i K(x_{ti})$, for all $x_{ti} \in A$, $\lambda_i \in F$ and $t_i \in [0,1]$, i = 1, 2, … , k .

To prove $K(\sum_{i=1}^{k+1} \lambda_i x_{ti}) \geq \sum_{i=1}^{k+1} \lambda_i K(x_{ti})$, for all $x_{ti} \in A$, $\lambda_i \in F$ and $t_i \in [0,1]$, i = 1, 2, … , k+1 .

$$K(\sum_{i=1}^{k+1} \lambda_i x_{ti}) = K(\sum_{i=1}^{k} \lambda_i x_{ti} + \lambda_{k+1} x_{t\,k+1})$$
$$\geq K(\sum_{i=1}^{k} \lambda_i x_{ti}) + K(\lambda_{k+1} x_{t\,k+1})$$
$$\geq \sum_{i=1}^{k} \lambda_i K(x_{ti}) + \lambda_{k+1} K(x_{t\,k+1})$$
$$\geq \sum_{i=1}^{k+1} \lambda_i K(x_{ti})$$

Hence $K(\sum_{i=1}^{n} \lambda_i x_{ti}) \geq \sum_{i=1}^{n} \lambda_i K(x_{ti})$, for all $x_{ti} \in A$, $\lambda_i \in F$ and $t_i \in [0,1]$, i = 1, 2, … , n .

PROPOSITION 2.6 :

Let A be a fuzzy subspace of a vector space X over F and B be a fuzzy subspace of a vector space Y over F . Let K : A→ B, be an epimorphism fuzzy linear transformation , then :

1. K(A) is a fuzzy subspace of B .
2. K^{-1} (B) is a fuzzy subspace of A .

PROOF:

1. Let x_{t1} , $y_{t2} \in$ B such that $K(a_{t3}) = x_{t1}$, $K(b_{t4}) = y_{t2}$, where a_{t3} , $b_{t4} \in$ A , since t_1 , t_2 , t_3 , $t_4 \in [0,1]$ and $a_{t3} = K^{-1}(x_{t1})$, $b_{t4} = K^{-1}(y_{t2})$, where a_{t3} , $b_{t4} \in$ A

$K(A)(\lambda x_{t1} + \alpha y_{t2}) = \sup\{ \min \{K(A)(a_{t3}), K(A)(b_{t4})\} \mid a_{t3}=K^{-1}(\lambda x_{t1}), b_{t4}=K^{-1}(\alpha y_{t2}) ;\ \lambda x_{t1} + \alpha y_{t2} = a_{t3} + b_{t4} \}$,

$\geq \sup\{ \min \{K(A)(a_{t3}), K(A)(b_{t4})\} \mid a_{t3}=K^{-1}(x_{t1}), b_{t4}=K^{-1}(y_{t2}) \}$,

$= \sup\{ \min \{K(A)(a_{t3}), K(A)(b_{t4})\} \mid K(a_{t3}) = x_{t1} , K(b_{t4}) = y_{t2}\}$,

$\geq \min \{ K(A)(x_{t1}) , K(A)(y_{t2}) \};\ K(a_{t3}) = x_{t1} , K(b_{t4}) = y_{t2}$ [4] .

$K(A)(\lambda x_{t1} + \alpha y_{t2}) \geq \min \{ K(A)(x_{t1}) , K(A)(y_{t2})$. Then K(A) is a fuzzy subspace of B .

2. Let a_{t3} , $b_{t4} \in$ A such that $a_{t3} = K^{-1}(x_{t1})$, $b_{t4} = K^{-1}(y_{t2})$, where x_{t1} , $y_{t2} \in$ B , since t_1 , t_2 , t_3 , $t_4 \in [0,1]$ and $K(a_{t3}) = x_{t1}$, $K(b_{t4}) = y_{t2}$, where a_{t3} , $b_{t4} \in$ A

$K^{-1}(B)(\lambda a_{t3}+ \alpha b_{t4}) = \sup\{\min \{ B(\lambda x_{t1}) , B(\alpha y_{t2}) \} \mid x_{t1} =K(a_{t3}), y_{t2}= K(b_{t4}) \}$,

$\geq \sup\{\min \{ B(x_{t1}) , B(y_{t2}) \} \mid x_{t1} =K(a_{t3}), y_{t2}= K(b_{t4}) \}$

$\geq \sup \{ \min \{ K^{-1}(B)(a_{t3}) , K^{-1}(B)(b_{t4})\} \mid a_{t3} = K^{-1}(x_{t1}) , b_{t4} = K^{-1}(y_{t2}) \}$,

$\geq \min \{ K^{-1}(B)(a_{t3}) , K^{-1}(B)(b_{t4})\}; a_{t3} = K^{-1}(x_{t1}) , b_{t4} = K^{-1}(y_{t2})$,[4] .

$K^{-1}(B)(\lambda a_{t3}+ \alpha b_{t4}) \geq \min \{ K^{-1}(B)(a_{t3}) , K^{-1}(B)(b_{t4})\}$. Then $K^{-1}(B)$is a fuzzy subspace of A .

PROPOSITION 2.7 :

Let A be a fuzzy subspace of a vector space X over F and B be a fuzzy subspace of a vector space Y over F . Let K : A→ B be a fuzzy linear transformation if and only if f : X → Y is a linear transformation on vector space .

PROOF:

(→) Since K : A→ B is fuzzy linear transformation , that mean :

$K(\lambda x_{t1} + \alpha y_{t2}) \geq \min \{ K(x_{t1}) , K(y_{t2}) \}$, for all x_{t1} , $y_{t2} \in$ A and $\lambda , \alpha \in F$, t_1 , $t_2 \in [0,1]$.

To prove f : X → Y is a linear transformation on vector space ,(i.e.) $f(\lambda x +\alpha y) = \lambda f(x)+\alpha f(y)$, for all x , y ∈ X and λ , α ∈ F .

Since x_{t1} , $y_{t2} \in$ A , t_1 , $t_2 \in [0,1]$, then there exists x , y ∈ X such that $K(x_{t1}) = f(x)$, $K(y_{t2}) = f(y)$ implies that $x = f^{-1}(K(x_{t1}))$ and $y = f^{-1}(K(y_{t2}))$.

$f(\lambda x +\alpha y) = f(\lambda x) + f(\alpha y)$

$= f(\lambda f^{-1}(K(x_{t1}))) + f(\alpha f^{-1}(K(y_{t2})))$

$= \lambda f(f^{-1}(K(x_{t1}))) + \alpha f(f^{-1}(K(y_{t2})))$

$= \lambda K(x_{t1}) + \alpha K(y_{t2})$

$= \lambda f(x) +\alpha f(y)$, for all x , y ∈ X .

Then $f(\lambda x +\alpha y) = \lambda f(x) +\alpha f(y)$, for all x , y ∈ X and λ , α ∈ F .

Hence f : X → Y is a linear transformation on vector space .

(←) Since f : X → Y is a linear transformation on vector space , that mean :

$f(\lambda x +\alpha y) = \lambda f(x) +\alpha f(y)$, for all x , y ∈ X .

To prove K : A→ B is fuzzy linear transformation , (i.e.)

$K(\lambda x_{t1} + \alpha by_{t2}) \geq \min \{ K(x_{t1}) , K(y_{t2}) \}$, for all x_{t1} , $y_{t2} \in$ A and $\lambda , \alpha \in F$, t_1 , $t_2 \in [0,1]$.

Since λ , α ∈ F and x , y ∈ X , then there exists x_{t1} , $y_{t2} \in$ A , t_1 , $t_2 \in [0,1]$ such that $K(x_{t1}) = f(x)$, $K(y_{t2}) = f(y)$ implies that $x_{t1} = K^{-1}(f(x))$ and $y_{t2} = K^{-1}(f(y))$.

$K(\lambda x_{t1} + \alpha y_{t2}) = K(\lambda x_{t1}) + K(\alpha y_{t2})$

$= K(\lambda K^{-1}(f(x))) + K(\alpha K^{-1}(f(y)))$

$= \lambda K(K^{-1}(f(x))) + \alpha K(K^{-1}(f(y)))$

$= \lambda f(x) + \alpha f(y)$

$= \lambda K(x_{t1})+\alpha K(y_{t2})$, for all x , y $\in$ X , $\lambda , \alpha \in F$.

$\geq K(x_{t1})+ K(y_{t2})$, for all x , y $\in$ X , $\lambda , \alpha \in F$.

$K(\lambda x_{t1} + \alpha y_{t2}) \geq K(x_{t1})$ and $K(\lambda x_{t1} + \alpha y_{t2}) \geq K(y_{t2})$, [4] .

$K(\lambda x_{t1} + \alpha y_{t2}) \geq \min \{ K(x_{t1}) , K(y_{t2}) \}$, for all $x_{t1} , y_{t2} \in A$ and $\lambda , \alpha \in F$, $t_1 , t_2 \in [0,1]$.

Hence K : A$\rightarrow$ B is a fuzzy linear transformation .

PROPOSITION 2.8 :

Let A be a fuzzy subspace of a finite vector space $X = \{ x_{t1} , x_{t2} , \ldots , x_{tn} \}$ over F and B be a fuzzy subspace of a finite vector space $Y= \{ y_{t1} , y_{t2} , \ldots , y_{tn} \}$ over F . Then K : A$\rightarrow$ B such that $K(x_{ti}) = y_{ti}$, for all i = 1 ,2 ,… , n , t$\in$ [0,1] is a fuzzy linear transformation .

PROOF:

Since a finite vector space $X = \{ x_{t1} , x_{t2} , \ldots , x_{tn} \}$ over F and a finite vector space $Y= \{ y_{t1} , y_{t2} , \ldots , y_{tn} \}$ over F , then $f : X\rightarrow Y$ such that $f(x_i) = y_i$, for all i = 1 ,2 ,… , n , by [3] .

Then A_t is a subspace of X , for all $0 \leq t \leq A(0)$, and proposition (1.8 (2)) , A is a fuzzy subspace of X .

Hence $K \approx f$ by proposition (2.7) , and $K(x_{ti}) = y_{ti}$, for all i = 1 ,2 ,… , n , t$\in$ [0,1] .

To prove K is a fuzzy linear transformation .Let $x_{t1} , x_{t2} \in A$ such that :

$x_{t1} = \sum_{i=1}^{n} \lambda_i y_{ti}$, $x_{t2} = \sum_{i=1}^{n} \alpha_i y_{ti}$, then $(\lambda x_{t1}+ \alpha x_{t2}) = (\sum_{i=1}^{n} (\beta\lambda_i + \mu\alpha_i) x_{ti})$, $\beta , \mu \in F$.

$$K (\lambda x_{t1}+ \alpha x_{t2}) = K (\sum_{i=1}^{n} (\beta\lambda_i + \mu\alpha_i) y_{ti})$$

$$= \beta K (\sum_{i=1}^{n} \lambda_i y_{ti}) + \mu K (\sum_{i=1}^{n} \alpha_i y_{ti})$$

$$= \beta K (x_{t1}) + \mu K (x_{t2})$$

$K (\lambda x_{t1}+ \alpha x_{t2}) \geq \beta K (x_{t1})$ and $K (\lambda x_{t1}+ \alpha x_{t2}) \geq \mu K (x_{t2})$.

$K (\lambda x_{t1}+ \alpha x_{t2}) \geq K (x_{t1})$ and $K (\lambda x_{t1}+ \alpha x_{t2}) \geq K (x_{t2})$.

$K (\lambda x_{t1}+ \alpha x_{t2}) \geq \min \{K (x_{t1}) , K (x_{t2}) \}$.

Hence K is a fuzzy linear transformation .

DEFINITION 2.9 [7] :

A linear transformation f from a ring (R ,+,.) to a ring $(R' ,+' ,.')$ is called **ring homomorphism** if it satisfies the following properties : for all a, b $\in$ R ,

1. $f(a + b) = f(a) +' f(b)$

2. $f(a \cdot b) = f(a) \cdot' f(b)$.

PROPOSITION 2.10 [7] :

If $f : R \rightarrow R'$ and $g : R' \rightarrow R''$ are homomorphism between the fuzzy subsets A , B and C, then $f \circ g$ is a homomorphism between A and C.

REMARK 2.11 [7] :

1. If f and g are isomorphism, then $g \circ f$ is an isomorphism since f and g are one – to- one and onto implies that $g \circ f$ is one – to - one and onto.

2. If f and g are homomorphism , one – to - one and onto, then $g \circ f$ is an isomorphism .

THEROEM 2.12 :

Let A , B , C be fuzzy subspaces of vector space X , Y , Z over F respectively and let K : A$\rightarrow$ B , G : B $\rightarrow$ C be fuzzy linear transformations . Then $G \circ K$: A $\rightarrow$ C be a fuzzy linear transformation .

PROOF:

Since K and G are fuzzy linear transformations , then :

$K (\lambda x_{t1} + \alpha y_{t2}) = \sup \{ \inf \{ \lambda, K(x_{t1}), \alpha , K (y_{t2})\}$, for all $x_{t1} , y_{t2} \in A$ and $\lambda , \alpha \in F$, $t_1 , t_2 \in [0,1]$.

$G (\lambda z_{t3} + \alpha u_{t4}) = \sup \{ \inf \{ \lambda, G(z_{t3}), \alpha , G(u_{t4})\}$, for all $z_{t3} , u_{t4} \in B$ and $\lambda , \alpha \in F$, $t_3 , t_4 \in [0,1]$ and $K(x_{t1}) = z_{t3}$, $K(y_{t2}) = u_{t4}$, $K(z_{t3}) = a_{t5}$, $K(u_{t4}) = b_{t6}$, $a_{t5} , b_{t6} \in A$, $t_5 , t_6 \in [0,1]$.

To prove $G \circ K : A \to C$ be a fuzzy linear transformation .

Let a_{t5} , $b_{t6} \in A$ (that mean a , b $\in$ X and t_5 , $t_6 \in [0,1]$) , for all λ , $\alpha \in F$, then :

$G \circ K (\lambda\, a_{t5} + \alpha\, b_{t6}) = G(K (\lambda\, a_{t5} + \alpha\, b_{t6}))$

$\geq G(\min \{K(a_{t5}) , K (b_{t6})\})$,

$= G(\min \{ z_{t3}, u_{t4} \})$,

$= \min \{G(z_{t3}) , G(u_{t4}) \}$, [4] .

$= \min \{G(K(x_{t1})) , G(K (b_{t6}))\}$,

$= \min \{ G \circ K (a_{t5}), G \circ K (b_{t6})\}$.

$G \circ K (\lambda\, a_{t5} + \alpha\, b_{t6}) \geq \min \{ G \circ K (a_{t5}), G \circ K (b_{t6})\}$.

Then $G \circ K : A \to C$ be a fuzzy linear transformation .

DEFINITION 2.13 ([8], [4]):

Let $X : R \to [0,1]$, $Y : R' \to [0,1]$ are fuzzy sets . $f : R \to R'$ be homomorphism between them. We define **the fuzzy kernel of f**, ker $f_{zz}f : R \to [0,1]$ by :

$$\ker f_{zz}\, f(x) = \begin{cases} X(0) & x \in \ker f \\ 0 & x \notin \ker f \end{cases}$$

DEFINITION 2.14 :

Let A be a fuzzy subspace of a vector space X over R and B be a fuzzy subspace of a vector space Y over R′ . Let $K : A \to B$ be a fuzzy linear transformation and $f : X \to Y$ be homomorphism between them .We define **the fuzzy kernel of K** , ker f_{zz} K : $A \to [0,1]$ by :

$$\ker f_{zz}\, K(x) = \begin{cases} A(0) & x \in \ker f \\ 0 & x \notin \ker f \end{cases}.$$

PROPOSITION 2.15 :

ker f_{zz} K : $A \to [0,1]$ is a fuzzy subspace of X .

PROOF:

Let a , b $\in$ X , for all λ , $\alpha \in F$, since ker f_{zz} K(0) = X(0) , if x $\in$ ker f , then :

$$\ker f_{zz}\, K(\lambda a + \beta b) = \begin{cases} A(0) & \lambda a + \alpha b \in \ker f \\ 0 & \lambda a + \alpha b \notin \ker f \end{cases}.$$

$$\ker f_{zz}\, K(a) = \begin{cases} A(0) & a \in \ker f \\ 0 & a \notin \ker f \end{cases}.$$

$$\ker f_{zz}\, K(b) = \begin{cases} A(0) & b \in \ker f \\ 0 & b \notin \ker f \end{cases}.$$

Then ker f_{zz} K ($\lambda a + \alpha b$) = sup { inf { λ , ker f_{zz} K (a) , α , ker f_{zz} K (b) } .

Hence ker f_{zz} K is a fuzzy subspace of X .

PROPOSITION 2.16 :

Let A be a fuzzy subspace of a vector space X over F and B be a fuzzy subspace of a vector space Y over F and $K : A \to B$ be a fuzzy linear transformation . Then ker f_{zz} K = ϕ if and only if K is one – to - one.

PROOF:

Since ker f_{zz} K = ϕ , then ker f = {0} , by theorem (2.1.10) in [3] , K is a one – to - one.

SECTION THREE

Fuzzy Coset and Quotient Fuzzy Rings

In this section, two definitions about fuzzy coset and quotient fuzzy ring are given , some properties concerning with this definitions are given and we studied the concept of a fuzzy isomorphism.

DEFINITION 3.1 [3]:

Let A and B be fuzzy subsets of vector space X over F such that $B \subseteq A$ and $x_t \subseteq A$, $t \in [0,A(0)]$. Then $x_t + B$ ($B + x_t$) is called **a fuzzy left (right) coset of B in A with representative x_t** .

REMARK 3.2 [3]:

Let A and B be fuzzy subsets of vector space X over F such that $B \subseteq A$ and $x_t \subseteq A$, $t \in [0,A(0)]$. For all $z \in X$, $(x_t + B)(z) = \inf \{ t , B(z-x) \}$ and $(A/B) = \{ x_t + B : x_t \subseteq A, x \in B \}$ is commutative group under + .

PROPOSITION 3.3 [3]:

Let A and B be fuzzy subsets of vector space X over F such that $B \subseteq A$ and $x_t , y_s \subseteq A$,t, $s \in [0,A(0)]$. Then :

1. For all $z \in G$, $(x_t + B)(z) = \inf \{ t , B(z-x) \}$ and $(B+ x_t)(z) = \inf \{ t , B(x + (-z)) \}$.
2. (a) $x_t + B = y_s + B$ iff $\inf \{ t, B(e) \} = \inf \{s , B((-y)+x) \}$ and $\inf \{ s, B(e) \} = \inf \{ t, B(x + (-y)) \}$.
 (b) $x_t + B = y_s + B$ iff $\inf \{ t, B(e) \} = \inf \{s , B(x+ (-y)) \}$ and $\inf \{ s, B(e) \} = \inf \{ t, B(y + (-x)) \}$.
3. If $B((-y) +x) = B(e)$, then $x_t + B = y_t + B$.

DEFINITION 3.4 [18]:

Let A and B be fuzzy subsets of vector space X over F such that $B \subseteq A$ and $x_t \subseteq A$,$t \in [0,A(0)]$. $B(e) = A(e)$ and B is a fuzzy normal in A . Then $(A/B)_t = \{ x_t + B : x_t \subseteq A, x \in G \}$, for all $t \in [0,1]$ is a group under "+" . $(A/B)_t$ is called **a quotient group of fuzzy subgroup .**

$(A/B) = \{ x_t + B : x_t \subseteq A, x \in G, t \in [0,1]\}$. Then ((A/B) , +) is a semigroup with identity and (A/B) is completely regular ((A/B) is a union of disjoint groups) i.e. , $(A/B) = \bigcup_{t \in [0,A(0)]} (A/B)_{(t)}$) .

PROPOSITION 3.5 [3]:

Let A and B be fuzzy subsets of vector space X over F such that $B \subseteq A$ and $x_t \subseteq A$, $t \in [0,A(0)]$. Then $(A/B)_t = A_t / B_t$.

PROPOSITION 3.6 :

Let A and B be two fuzzy subspaces of a vector space X over F such that $B \subseteq A$ and $x_t , y_t \subseteq A$, $t \in [0,A(0)]$. Then (A/B) is a fuzzy subspace over F on (+ and .) such that :

1. $(x_t + B) + (y_t + B) = (x_t + y_t) + B$.
2. $\lambda (x_t + B) = (\lambda x_t) + B$, for all $\lambda \in F$.

PROOF:

Let $x_t , y_t \subseteq A$,$t \in [0,A(0)]$ and $B \subseteq A$, then $(x_t + y_t) \subseteq A$ and $\lambda x_t \subseteq A$. Thus $(x_t + y_t) + A \subseteq (A/B)_t$, then (A/B , +) and (A/B , .) are closure on (+ and .) .

Let $z_t , u_t \subseteq A$,$t \in [0,A(0)]$ and $B \subseteq A$, then $(x_t - z_t) \subseteq A$ and $(y_t - u_t) \subseteq A$, since A is a vector subspace , $(x_t - z_t) + (y_t - u_t) \subseteq A$ implies that $(x_t+ y_t) - (z_t + u_t) \subseteq A$ implies that $(x_t+ y_t) + A = (z_t + u_t) + A \subseteq A$ implies that $(A/B)_t$ is a well defined of (+) .

And $(x_t - z_t) \subseteq A$ and $(\lambda (x_t - z_t)) \subseteq A$ implies that $(\lambda x_t – \lambda z_t) \subseteq A$ implies that $(\lambda x_t) + A = (\lambda y_t) + A \subseteq A$ implies that $(A/B)_t$ is a well defined of (.) .

Since A(0) > 0 , A(x – y) ≥ min{ A(x) , A(y) } , for all x , y ∈ X and A(cx) ≥ min { F(c) , A(x) } , for all x ∈X and c ∈ F , then (A/B) is a fuzzy subspace over F on (+ and .)

THEOREM 3.7 :

Let A and B be fuzzy subspaces of vector space X over F such that B ⊆ A and x_t , y_t ⊆ A , t ∈ [0,A(0)] . Then K : X → X / A define by: f (x) = x_t + A . Then K is an epimorphism fuzzy linear transformation and ker fzz K = A .

PROOF:

Let x_t , y_t ⊆ X ,t ∈ [0,A(0)] and α , β ∈ F , then :

1. K (α x_t + β y_t) = (α x_t + β y_t) + A
 = α (x_t+ A) + β (y_t + A)
 ≥ min { (x_t+ A) , (y_t + A)}}
 = min {K(x_t) , K (y_t)}} .

Then K is a fuzzy linear transformation .

2. Let z_t ⊆ X / A , then there exists x ∈ X such that z_t = (x_t + A) = K (x_t) , then K is a onto .
3. Since the fuzzy kernel of K is ker f_{zz} K : A → [0,1] by :

$$\ker f_{zz}\,K(x)=\begin{cases} A(0) & x\in \ker f \\ 0 & x\notin \ker f \end{cases}$$. Then ker f_{zz} K = A .

REMARK 3.8 :

The function K is called **fuzzy Canonical function** . In general , K is one – to- one since x_t , y_t ⊆ A , t ∈ [0,A(0)] . Then (x_t - y_t) ⊆ A implies that (x_t+ A) = (y_t + A) , then K(x) = K(y) .

DEFINITION 3. 9 :

Let X and Y be fuzzy subsets over F. Then we define **fuzzy linear isomorphism** , if there exists K : X → Y is a fuzzy linear transformation , one – to – one and onto . We denoted by X ≈ Y .

THEOREM 3.10 :

Let A , B , C be fuzzy subspaces of vector spaces X , Y and Z over F respectively such that A = B ⊕ C . Then B ≈ A / C or C ≈ A / B .

PROOF:

Define K : B → A / C such that K(x) = x_t + C , x_t ⊆ B .

Let x_t , y_t ⊆ B , t ∈ [0, 1] and α , β ∈ F , then :

1. K (α x_t + β y_t) = (α x_t + β y_t) + C
 = α (x_t+ C) + β (y_t + C)
 ≥ min { (x_t+ C) , (y_t + C)}}
 = min {K(x_t) , K (y_t)}} .

Then K is a fuzzy linear transformation .

2. Let z_t ⊆ A / C , then there exists x_t ∈ B such that z_t = (x_t + C) = K (x_t) , but A = B ⊕ C , then x_t = (u_t + w_t) , u_t ⊆ B and w_t ⊆ C)) implies that u_t = (x_t - w_t) , thus x_t + C = w_t + C , then z_t = K(u_t) . Hence K is a onto .
3. Since x_t , y_t ⊆ B such that K(x_t) = K (y_t) , then x_t + C = y_t + C, t ∈ [0, 1] and (x_t - w_t) ⊆ C , K is a one – to – one .

Then B ≈ A / C , by this style C ≈ A / B .

(First Fuzzy Isomorphism Theorem For Fuzzy Subspaces)

THEOREM 3.11 :

Let X and Y are fuzzy subspaces of a vector subspace over F and K be onto homomorphism between them. Then X / ker f_{zz} K ≈ K (X).

PROOF:

Define G : X/ ker K → K (X)such that: G (a_t + ker K)= K (a_t), for each a_t + ker K∈ X /ker K .

By definition, G is a non empty function of X / ker K since g (0_t + ker K) = K (0_t)

Let a_t + ker K, b_t + ker K $\in$ X/ ker K , a_t + ker K = b_t + ker K implies that $a_t - b_t \in$ ker K, therefore K $(a_t - b_t)=0_t$ and K is homomorphism, then K (a_t) – K $(b_t) = 0_t$ implies K (a_t) = K (b_t). Thus G $(a_t$ + ker K) = G $(b_t$ + ker K). Hence G is well – define.

Now, we must prove G is an isomorphism

First, if G $(a_t$ + ker K) = G $(b_t$ + ker K), then K (a_t) = K (b_t) and K (a_t) – K$(b_t) = 0_t$ implies that K $(a_t - b_t) = 0_t$. Thus $a_t - b_t \in$ ker K therefore a_t + ker K = b_t + ker K , G is one – to – one.

Second, for any $b_t \in$ K (X) there exists $a_t \subseteq$ X such that: K $(a_t) = b_t$ since K is onto then K (a_t) = G $(a_t$ + ker K) = b_t , G is onto .

Finally, Let a_t + ker K, b_t + ker K $\in$ X / ker K] and α , β ∈ F , then :

$$\begin{aligned} G[\alpha(a_t + \ker K) \oplus \beta(b_t + \ker K)] &= G[(\alpha a_t + \beta b_t) + \ker K] \\ &= K(\alpha a_t + \beta b_t)] \\ &= \alpha K(a_t) +' \beta K(b_t) \\ &= G(\alpha(a_t + \ker K)) +' G(\beta(b_t + \ker K)) \\ &\geq \min\{G((a_t + \ker K)), G((b_t + \ker K)\}\} \end{aligned}$$

Then G is a fuzzy linear transformation .

Hence X/ ker f_{zz}K ≈ K (X).

(Second Fuzzy Isomorphism Theorem For Fuzzy Subspaces)

THEOREM 3.12 :

Let A and B be fuzzy subspaces of a fuzzy subspace X over F ,with A ⊆ B such that. B(x) = B(0) , whenever A(x) = A(0) . Then (X / A) / (B / A) ≈ (X / B).

PROOF:

Define G : (X / A) / (B / A) → (X / B) such that : G ($(x_t$ + A) + (B / A)) = x_t + B is an isomorphism by [7] .

By definition, G is a non empty function of (X / A) / (B / A) since g $(0_t$ + (B / A)) = K $(0_t$+A)

Let (a_t +A + (B / A)), $(b_t$ +A + (B / A)) ∈ (X / A) / (B / A), $(a_t$ +A+ (B / A)) =(b_t +A + (B / A)) implies that $((a_t - b_t)$ +A) ∈ (B / A) , therefore K$((a_t - b_t)$+A) =0_t +A and K is homomorphism, then K $(a_t$+A) – K $(b_t$+ A) = 0_t +A implies K $(a_t$+A) = K $(b_t$+A). Thus G $(a_t$+A+ (B / A)) = G $(b_t$ +A + (B / A)). Hence G is well – define.

Now, we must prove G is an isomorphism

First, if G $(a_t$ +A + (B / A)) = G $(b_t$ +A + (B / A)), then K $(a_t$+A) = K $(b_t$+A) and K $(a_t$+A) – K$(b_t$+A) = 0_t+A implies that K $((a_t - b_t)$+A) = 0_t +A . Thus $(a_t - b_t)$+ A∈ (B / A) therefore $(a_t$ + A) + (B / A)) =(b_t +A+ (B / A)) , G is one – to – one.

Second, for any $(b_t$ +A) ∈ K ((X / A)) there exists a_t+A ⊆ (X / A) such that: K $(a_t$+A) = b_t +a , since K is onto then K $(a_t$+A) = G($(a_t$ +A+ (B / A)) = $(b_t$ +A) , G is onto .

Finally, Let $(a_t$ +A+ (B / A)) , $(b_t$ +A+ (B / A)) ∈ (X / A) / (B / A)] and α , β ∈ F , then :

$$\begin{aligned} G[\alpha(a_t + A + (B/A)) \oplus \beta(b_t + A + (B/A))] &= G[(\alpha(a_t + A)) + (\beta(b_t + A)) + (B/A)] \\ &= K(\alpha(a_t + A) + \beta(b_t + A)] \\ &= \alpha K(a_t + A) +' \beta K(b_t + A) \\ &= G(\alpha(a_t + A + (B/A))) +' G(\beta(b_t + A + (B/A))) \\ &\geq \min\{G(a_t + A + (B/A)), G(b_t + A + (B/A))\}. \end{aligned}$$

Then G is a fuzzy linear transformation .

Hence (X / A) / (B / A) ≈ K (X / A) .

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