i-Generalized Homeomorphisms and Generalized i-Homeomorphisms in Topological Spaces

Sabih W. Askandar
Department of Mathematics/College of Education For Pure Science
University of Mosul
Mosul-Iraq
sabeehqaqus@yahoo.com

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التشاكلات المعممة من النوع-i والتشاكلات من النوع-i المعممة في الفضاءات التبولوجية

م.م. صبيح وديع اسكندر
قسم الرياضيات/كلية التربية للعلوم الصرفة/جامعة الموصل
الموصل/العراق

الملخص:

في هذا البحث، تم تقديم التشاكلات المعممة من النوع-i والتشاكلات من النوع-i المعممة في الفضاءات التبولوجية. اكثر من ذلك تم ايجاد العلاقة بين هذين الصنفين وبعض الاصناف الاخرى من التشاكلات التبولوجية.

الكلمات المفتاحية: التشاكل التبولوجي.

ABSTRACT:

In this paper, topological ig-Homeomorphisms and topological gi-Homeomorphisms are introduced. Further, the relations between these two classes and some other classes of Homeomorphisms are investigated.

Keywords: Topological Homeomorphism

INTRODUCTION:

In 1970, Levine [7], has generalized the concept of closed sets to generalized closed sets [1]. In 1987, Bhattacharyya and Lahiri [3], generalized the concept of closed sets to semi-generalized closed sets with the help of semi-open sets and they obtaining various topological properties. In 1990, Arya and Nour [1], defined generalized semi-open sets with the help of semi-openness and used them to obtain some characterizations of s-normal spaces. In 1995, Devi, Balachandran and maki [٥], defined two classes of maps that are called semi-generalized homeomorphisms and generalized semi-homeomorphisms. In 2006, Rajesh, N., Ekici, E. and Thivagar, M. L.[9], introduced a new class of homeomorphisms called, gs*-homeomorphisms. In 2009, Caldas, M., Jafari, S. and Rajesh, N. [4], introduced $\tilde{g}$ -homeomorphisms. In 2012, Mohammed, A.A. and Askandar, S.W., [8][2], introduced the concept of i-open sets. In this paper we generalize the concept of closed sets to i-generalized closed sets with the help of i-open sets and obtained various topological properties. We introduce two classes of maps are called ig-homeomorphisms and gi-homeomorphisms and we study their properties. Throughout this paper (X,τ) and (Y,δ) are always topological spaces, *f* is always a mapping from (X,τ) into (Y,δ) and *Cl(A*) denotes the closure of a set *A*.

1. Definitions and Examples.

In this Section, we introduce important new concepts of sets and maps and use them to prove the main results. $A \subseteq X$ is called i-open set [2][8] if $A \subseteq Cl(A \cap G)$ (for some $G \neq \phi$, X is open set in *X* . *X/A* is called i-closed set [2][8]. For example if X= {1, 2, 3}, τ= {ϕ, {1}, {1, 3}, X}, we have: Closed sets which are: *X,{2,3}, {2},* ϕ, i-open sets are : ϕ,{1},{1, 3},{3},{1, 2},{2, 3},X. i-closed sets are: *X,{2,3}, {2}, {1,2}, {3}, {1},* ϕ.

Lemma1.1. Every open set is i-open. But the converse is not necessary to be trrue [8][2].

Corollary1.2. Every closed set is i-closed. But the converse is not necessary to be trrue [8][2].

Definition1.3. Let τ^i be the set of all i-open sets of *X* and let $A \subseteq X$ then:

1. $Cl_i(A) = \bigcap_{\circ \in \Lambda} F_i \,.\, A \subseteq F_i \;\; \forall i$ (Where $F_i \subseteq X$ is i-closed set $\forall i$ and $Cl_i(A)$ is the i-closure of *A[8][2]*. $A = Cl_i(A)$ if and only if *A* is i-closed set.

2. $Int_i(A)=\bigcup_{i\in\Lambda} I_i \quad I_i \subseteq A \ \forall i$. (Where $I_i \subseteq X$ is i-open set $\forall i$ and $Int_i(A)$ is i-interior of *A[8][2]*.

Definition1.4. A subset *A* of *X* is said to be:

1. *generalized closed set (g−closed)*[1] if $Cl(A) \subseteq U$ where $A \subseteq U$ and $U \subseteq X$ is *open set*. $B \subseteq X$ is generalized open set ($g-open$)[1] if its complement is $g-closed$.

2. *generalized i−open set (gi−open)* if $F \subseteq Int_i(A)$,
Where, $F \subseteq A \subseteq X$ is *closed set*. $B \subseteq X$ is *generalized i−closed set (gi−closed)* if its complement is *gi−open*.

3. *i−generalized closed set (ig−closed)* if $Cl_i(A) \subseteq U$ where $A \subseteq U$ and $U \subseteq X$ is an *i−open set*. $B \subseteq X$ is i-generalized open set *(ig−open)* if its complement is $ig-closed$.

Definition1.5. i-Generalized Closure of *A*, denoted by $Cl_{ig}(A)$: $Cl_{ig}(A)=\bigcap_{i\in\Lambda} F_i . A \subseteq F_i \quad \forall i$ Where $F_i \subseteq X$ is ig-closed set $\forall i$, $A=Cl_{ig}(A)$ if and only if *A* is ig-closed set.

Example1.6. Let X= {4, 5, 6}, τ= {ϕ, {4}, X}.
Therefore, Closed sets are: X, {5, 6}, ϕ.
i-open sets are : ϕ,{4},{4,5},{4,6},X, i-closed sets are: X,{5, 6},{6},{5},ϕ.
gi-open sets are: : ϕ,{4},{5},{6},{4,5},{4,6},X,
gi-closed sets are: X,{5, 6},{4,6},{4,5},{6},{5},ϕ.
ig-closed sets are: ϕ,{5},{6},{5, 6},X,
ig-open sets are: X,{4, 6},{4,5},{4},ϕ.

Example1.7. Let X={7, 8, 9} , τ={ ϕ,{7},{7,8},{7,9},X}.
Therefore, Closed sets are: X, {8, 9}, {9},{8},ϕ.
i-open sets are : ϕ,{7},{8},{9},{7,8},{7,9},X,
i-closed sets are: X,{8, 9},{7,9},{7,8},{9},{8},ϕ.
gi-open sets are: : ϕ,{7},{8},{9},{7,8},{7,9},{8,9},X,
gi-closed sets are: X,{8, 9},{7,9},{7,8},{9},{8},{7},ϕ.
ig-closed sets are: ϕ,{7},{8},{9},{7,8},{7,9},{8, 9},X,
ig-open sets are: X,{8,9},{7, 9},{7,8},{9},{8},{7},ϕ.

Theorem1.8. Every i-closed set is i-generalized-closed. [6]

Proof: Let $A \subseteq X$ be i-closed. Then $A = Cl_i(A)$ It is clear that $A \subseteq Cl_i(A)$ and $Cl_i(A) \subseteq A$.
Suppose that $A \subseteq U$ and $U \subseteq X$ are i-open. Therefore, $Cl_i(A) \subseteq U$. Hence A is i-generalized closed. Also since $A \subseteq Cl_{ig}(A) \subseteq Cl_i(A) \subseteq Cl(A)$ we have, every closed and i-closed sets are i-generalized-closed (where $Cl(A)$ denotes the closure of a set A). ■
In (example 1.7) it is clear that A= {7} is i-generalized closed, A is not i-closed.

Theorem1.9. Every i-generalized closed set is generalized i-closed.[6]
Proof: suppose that $A \subseteq X$ is i-generalized closed. We have, $Cl_i(A) \subseteq U$ where $A \subseteq U$ and $U \subseteq X$ is i-open . Now $(Cl_i(A) \subseteq U)^C = U^C \subseteq Int_i(A^C)$, we have U^C is i-closed set in X, we have A^C *is* generalized i-open. Hence, A is generalized i-closed. ■
In (example 1.6) A= {4, 5} is gi-closed set but it is not ig-closed.

Corollary1.10. Every closed set (respectively i-closed, ig-closed set) is gi-closed. But the converse is not necessary to be true (see corollary (1.2.), theorem (1.8) and theorem (1.9)).

Definition1.11. A mapping f is said to be:
1. Closed mapping (resp. open mapping) if $f(F)$ is closed set (resp. open set) in Y For each closed set(resp. open set) F *in* X.
2**.** i-closed mapping (resp. i-open mapping) if the $f(F)$ is i-closed (resp. i-open) in Y[8][2] for each closed set (resp. open set) F *in* X.
3. i-generalized closed mapping (resp. i-generalized open mapping) if $f(F)$ i-generalized closed (resp. i-generalized open set) in Y for each closed set (resp. open set) F in X.
4. Generalized i-open mapping (resp. generalized i-closed mapping) if $f(F)$ is gi-open (resp. gi-closed set) in Y for each open set (resp. closed set) F in X .

Definition1.12. A mapping f is said to be:
1. Continuous if and only if the inverse of I^* is closed (resp. open set) in X for all closed (resp. open sets) I^* in Y.
2. i-continuous[8] [2] if and only if the inverse of I^* is i-closed (resp. i-open set) in X for every closed (resp. open sets) I^* in Y.
3. i-generalized continuous if and only if the inverse of I^* is ig-closed (resp. ig-open set) in X for all closed (resp. open sets) I^* in Y.

4. Generalized i-continuous if and only if the inverse of I^* is gi-closed (resp. gi-open set) in X for all closed (resp. open sets) I^* in Y.
5. i-generalized irresolute (in short, ig-irresolute) if and only if the inverse of I^* is ig-closed set in X for all ig-closed sets I^* in Y.
6. Generalized i-irresolute (in short, gi-irresolute) if and only if the inverse of I^* is gi-closed set in X for all gi-closed sets I^* in Y.

Definition1.13. A mapping f is said to be:
i) Homeomorphism, if f is (1) continuous, (2) open.
ii) i-Homeomorphism [8] [2] if f is (1) i-continuous. (2) i-open.
iii) i-generalized Homeomorphism (in short igh) if f is (1) ig-continuous, (2) ig-open.
iv) Generalized i-Homeomorphism (in short gih) if $\boldsymbol{f}$ is (1) gi-continuous, (2) gi-open.

2. The Main Results.

In this section we find the relation between the new concepts of mappings.

Theorem2.1. Every open-mapping is i- open. [8][2]

Example2.2. Let X={10,11,12}, τ ={ϕ,{10, 11},X}, Y={10,11,12}, δ={ϕ,{11},{10, 11},Y}, *f(10)=11, f(11)=12 f(12)=10,* δ^i={ϕ,{11},{10, 11},{10},{10,12},{11,12},Y }. f is not open mapping because {10,11} is open set in X , f({10,11})={11,12} is not open set in Y. Hence f is i-open mapping.

Corollary2.3. Every closed-mapping is i- closed.[8][2]
In (example 2.2) f is not closed mapping because {12} is closed set in X but f({12})={10} is not closed set in Y. Hence f is i-closed mapping.

Theorem2.4. Every i-closed mapping is i-generalized closed.
Proof: Suppose that f is i-closed mapping. Then by suppose, *f(F)* is i-closed set in Y for each closed set F in X. Then *f(F)* is ig-closed set in *Y(*theorem (1.8)*)*. Hence f is ig-closed.■

Example2.5. Let X=Y={13,14,15}, τ ={ϕ,{13},{13,14},{13,15},X }=δ,, *f(13)=14, f(14)=13, f(15)=15*, f is not i-closed because {14,15} is closed set in X , f({14,15})={13,15} is not i-closed in Y. Hence f is ig-closed.
Corollary2.6. Every i-open mapping is ig- open.
In (example 2.5) f is not i-open because {13,15} is open set in X , f ({13,15})={14,15} is not i-open set in Y. Hence f is ig-open.

Theorem2.7. Every ig-closed mapping is gi-closed.
Proof: Suppose that *f* be ig-closed mapping, F be a closed set in X, then *f(F)* is ig-closed in *Y* (by suppose). *f(F)* is gi-closed in *Y* (theorem 1.9). Hence *f* is gi-closed. ■

Example2.8. Let X=Y={p,q,r}, τ ={ϕ,{p},X }=δ, *f(p)=q, f(q)=p, f(r)=r, f* is not ig-closed mapping because {q,r} is closed in *X* , *f* ({q,r})={p,r} is not ig-closed in *Y*. Hence *f* is gi-closed.

Corollary2.9. Every ig-open mapping is gi- open.
In (example 2.8) *f* is not ig-open because {p} is open in *X* , *f* ({p}) = {q} is not ig-open in *Y*. Hence *f* is gi-open.

Theorem2.10. Every continuous mapping is i-continuous.[8][2]

Example2.11. Let X={s, t, u, v}, τ ={ ϕ,{t}, {u, v},{t, u, v},X},Y={x, y}, δ={ ϕ,{x},{y},Y} , *f(s)= f(u)= f(v)=x, f(t)=y.*
τ^i ={ ϕ,{t},{u, v},{t, u, v},{s, u, v},………,X} . *f* is not continuous because {x} is open set in *Y* but f^{-1}({x})={s, u, v} is not open in *X*. *f* is i-continuous.

Theorem2.12. Every i-continuous mapping is i-generalized continuous.
Proof: let *f* be i-continuous mapping and let G^* be closed set in *Y,* then $f^{-1}(G^*)$ is i-closed in *X*. By (theorem 1.10) we have: $f^{-1}(G^*)$ is ig-closed in *X*. Hence *f* is i-generalized continuous. ■
In (Example 2.5) *f* is not i-continuous because {14} is closed set in *Y* but f^{-1}({14}) = {13} is not i-closed in *X*. *f* is i-generalized continuous.

Theorem2.13. Every i-generalized continuous mapping is generalized i-continuous.
Proof: Let *f* be ig-continuous mapping. If G^* is closed set in *Y* then $f^{-1}(G^*)$ is ig-closed in *X*. By theorem (1.9) we have: $f^{-1}(G^*)$ is gi-closed in *X*. Hence *f* is gi-continuous. ■
In example 2.8. *f* is not ig-continuous because {q,r} is closed *Y* , f^{1} ({q,r})={p,r} is not ig-closed in *X*. *f* is gi-continuous.
Theorem2.14. Every topological Homeomorphism is i-Homeomorphism.[8][2]

Example2.15. Let $X=\{g,h,i\}=Y$, $\tau=\{\phi,\{g,h\},X\}$, $\delta=\{\phi,\{g\},Y\}$, $f(g)=g$, $f(h)=i$, $f(i)=h$, f is not continuous because {g} is open set in Y, $f^{-1}(\{g\})=\{g\}$ is not open in X. f is one-one because $\forall\ x_1\neq x_2$ and $f(x_1)\neq f(x_2)$. f is onto because $f(X)=Y$. Hence f is not open because {g, h} is open in X but $f(\{g,h\})=\{g,i\}$ is not open in Y. Hence f is not a Homeomorphism.
$\tau^i=\{\phi,\{g,h\},\{g\},\{h\},\{g,i\},\{h,i\},X\}$. $\delta^i=\{\phi,\{g\},\{g,h\},\{g,i\},Y\}$.
f is i-continuous and it is i-open mapping . Hence f is i-Homeomorphism.

Theorem2.16. Every topological i-Homeomorphism is ig-Homeomorphism.
Proof: let f be a topological i-Homeomorphism. Then f is (1) i-continuous mapping. , (2) (onto, one-one). (3) i-open mapping. By (theorem 2.12) we have, f is ig-continuous and by corollary (2.6.) we have, f *is* ig-open. Hence f *is* ig-Homeomorphism. ■

In example 2.5. We have, *1.* f is not i-open because {13,15} is open in X but $f(\{13,15\})=\{14,15\}$ is not i-open in Y, f is ig-open. 2. f is not i-continuous because {14} is closed in Y, $f^{-1}(\{14\})=\{13\}$ is not i-closed in X. f is ig-continuous. 3. f is bijective. Therefore, f is ig-homeomorphism is not i-homeomorphism.

Theorem2.17. Every topological ig-Homeomorphism is gi-Homeomorphism.
Proof: let f *be* a topological ig-Homeomorphism. We have f is (1) ig-continuous. (2)bijective (onto, one-one)., (3) ig-open. By theorem (2.13) we have, f is gi-continuous mapping and by corollary (2.9.) we have, f *is* gi-open mapping. Hence f *is* gi-Homeomorphism. ■

In Example2.8 we have *1*. f is not ig-open because {p} is open in X but $f(\{p\})=\{q\}$ is not ig-open in Y. f is gi-open mapping.*2*. f is not ig-continuous because {q,r} is closed in Y but $f^{1}(\{q,r\})=\{p,r\}$ is not ig-closed in X. f is gi-continuous . 3. f is bijective. Therefore, f is gi-homeomorphism but is not ig-homeomorphism.

Corollary2.18. Every homeomorphism (respectively i-homeomorphism, ig-homeomorphism) is gi-homeomorphism. But the converse is not necessary to be true(theorem (2.14), theorem (2.16) and theorem (2.17)).

Corollary2.19. Every continuous (respectively i-continuous and ig-continuous mapping) is gi-continuous. But the converse is not necessary to be true(theorem (2.10), theorem (2.12) and theorem (2.13)).

Corollary2.20. Every open (respectively i-open and ig-open mapping) is gi-open. But the converse is not necessary to be true(theorem (2.1), corollary (2.6) and corollary (2.9)).

Proposition2.21. For any bijection (onto, one-one) *f*, the following statements are equivalent:
1)Its inverse map $f^{-1}(Y,\sigma)\rightarrow(X,\tau)$ is ig-continuous. 2) *f* is ig-open. 3) *f* is ig-closed.

Proposition2.22. Let *f* be a bijective and ig-continuous map. Then the following statements are equivalent.
1) *f* is ig-open map.
2) *f* is ig-homeomorphism.
3) *f* is ig-closed map.

Proposition2.23. For any bijection (onto, one-one) *f*, the following statements are equivalent:
1)Its inverse map $f^{-1}(Y,\sigma)\rightarrow(X,\tau)$ is gi-continuous. 2) *f* is gi-open. 3) *f* is gi-closed.

Proposition2.24. Let *f* be a bijective and gi-continuous map. Then the following statements are equivalent.
1) *f* is gi-open map.
2) *f* is gi-homeomorphism.
3) *f* is gi-closed map.

Proposition2.25. If a map *f* is ig-irresolute then, it is ig-continuous.

Proposition2.26. If a map *f* is gi-irresolute then, it is gi-continuous.

Theorem2.27. A mapping $f:(X,\tau)\rightarrow(Y,\delta)$ is ig-irresolute if and only if the inverse of I^* is ig-open set in *X* for all ig-open sets I^* in *Y*.
Proof. Suppose that f is ig-irresolute mapping.
Let I^* be ig-open set in Y. then I^{*c} *is ig-closed set in Y*. we have, $f^{-1}((I^*)^C)$ is ig-closed set in X (by suppose). We have, $(f^{-1}(I^*))^C$ is ig-closed set in X. Hence $f^{-1}(I^*)$ is ig-open set in X.

Conversely, suppose that the inverse of every ig-open set in Y is ig-open in X. Let I^* be ig-closed set in Y. then I^{*c} *is* ig-open in Y. we have, $f^{-1}((I^*)^C)$ is ig-open set in X (by suppose). We have, $(f^{-1}(I^*))^C$ is ig-open set in X. Hence $f^{-1}(I^*)$ is ig-closed set in X. Therefore, f is ig-irresolute map. ■

Theorem2.28. A mapping $f:(X,\tau)\rightarrow(Y,\delta)$ is gi-irresolute if and only if the inverse of I^* is gi-open set in X for all gi-open sets I^* in Y.

Proof. Suppose that f is gi-irresolute mapping.

Let I^* be gi-open set in Y. then I^{*c} *is* gi-closed in Y. we have, $f^{-1}((I^*)^C)$ is gi-closed in X (by suppose). We have, $(f^{-1}(I^*))^C$ is gi-closed in X. Hence $f^{-1}(I^*)$ is gi-open set in X.

Conversely, suppose that the inverse of *every* gi-open set in Y is gi-open in X. Let I^* be gi-closed set in Y. then I^{*c} *is gi-open in Y*. we have, $f^{-1}((I^*)^C)$ is gi-open set in X (by suppose). We have, $(f^{-1}(I^*))^C$ is gi-open in X. Hence $f^{-1}(I^*)$ is gi-closed set in X. Therefore, f is gi-irresolute map. ■

Theorem2.29. Every i-generalized irresolute mapping is generalized i-irresolute.

Proof: Let f be ig-irresolute mapping. If G^* is ig-closed set in Y then $f^{-1}(G^*)$ is ig-closed in X. By theorem (1.9) we have: $f^{-1}(G^*)$ is gi-closed in X. Hence f is gi-irresolute. ■

In example 2.8. f is not ig-irresolute because {q,r} is ig-closed set in Y, f^{-1}({q,r})={p,r} is not ig-closed in X. f is not gi-irresolute because{q}is gi-closed set in Y, f^{-1}({q})={p}. is not gi-closed in X.

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