On cyclic of Steiner system (v); V=2,3,5,7,11,13

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Abstract:

A steiner system can be defined by the triple S(t,k,v), where every block Bi, (i=1,2,…,b) contains exactly K-elementes taken from aset V-elements; every two distinct block Bi and Bj have at most (t-1) elements in common of V, the set V usually called the base set.

Steiner system can be represented geometrical by letting the varieties be "points" and representing a block by "line" (not necessary straight) through the points it contains . In this paper we connected between Steiner system and geometry where we'll find all blocks of S(2, p+1,p^2 + p+1); p is prime number using cyclic Steiner system putting S(2 p+1,p^2 +p+1)in one cycle.

Introduction:

A special kind of bloke designs is called a Steiner system. The idea a rose in November (1852) by Steiner who posed a many questions concerning what so called now a Steiner triple systems . The second of these questions asked whether or not it was always possible to introduce a Steiner triple system on V-points.

Steiner system: [3]

Given three integers t, k and V ; 2≤ t< k < V a Steiner system S(t , k ,V) is a set V which contains V points (or verities) together with a family B of k-subset of V is contained in exactly one block. Equivalently an S(t ,k ,V) is a t-(V,k,I) design .

Lemma: [6]

Let S be a steiner system with the parameters (t ,k, V) and let I $\subseteq$V be an i-set ; 0≤ i ≤ t . then

$$\lambda_i = \binom{V-i}{t-i} \Big/ \binom{k-i}{t-i}$$

where λ_i is the number of blocks containing i and depends only on the parameters V,k,t and i , while it is independent on i .

Lemma: [3]

A S(t, k, V) exists only if,

$$\binom{V-i}{t-i} \Big/ \binom{k-i}{t-i}$$ is an integer for every i=0,1,…..,t-1 .

Lemma: (Fisher's Inequality): [1]

Let S be a Steiner system with the parameters (2,k,V) , then b ≥V ,where b is the number of blocks in S .

Block design:

A block design is an arrangement of V distinct objects into blocks ; each block contains exactly K distinct objects , each object occurs in exactly r different blocks . Furthermore every pair of distinct object a_i ,a_j occurs in exactly λ blocks .

Galois field: [5]

Given a positive prime integer p , let k be a set with P elements {0,1,2,…,p-1} .Define addition in k by a+b= c if c is the remainder of a+b devided by p , i.e. a+b=c if c is a+b reduced modulo P . similary multiplication in K is defined by a.b=c if c is the remainder of a.b on dividing by p. If m is a positive integer and $a \in k$ then we denoted by ma the element of K obtained by adding a to itself m times in k, i.e ma=b if b is ma reduced modulo p . Similarly a^{m}=b if b is a^{m} reduced modulo p. Then k with the two operations, addition and multiplication, defined as above is a field with zero and multiplication identity l which is usually called Galois field with characteristic p and denoted by GF(p). For every $a \in GF(p)$, pa=0 and $a^{p}=0$ in particular,

-a=(p-1)a , and if $a \neq 0$ then $a^{-1} = a^{(p-2)}$

Projective plane : [2]

A projective plan π is a triple (P,L, I) where P is a set of objects(called points), L is a certain subsets of P (called lines) , and I is a relation between the points and lines;

(1) every pair of lines intersect in exactly one point,
(2) every pair of points lies in exactly one line,
(3) there exist four distinct points, no three of them are conteined in
one line.

Theorem (1): [5]

The points (x_1, x_2, x_3) incidence on the line $[y_1, y_2, y_3]$ if and only if $x_1y_1+x_2y_2+x_3y_3=0$

Theorem (2): [5]

If$\{p_1, p_2, p_{(n+2)}\}$ and $\{p_1, p_2, \ldots, p'_{(n+2)}\}$are any two sets of n+2 points of PG(n,k); n+1 points are chosen from same set lie in a prime then there exists a unique projectivity A; $P'_i = P_i.A$ for all i in N_{n+2}

Theorem(3) : [3]

Let S be asymmetric steiner system with the parameters (2,k,V) .Then S is a projective plane with parameters $(2,p+1,p^2+p+1)$.

Cyclic steiner triple system: [2]

An automorphism of a STS(S,T) is abijection $\alpha:S \lozenge S$ such that T={x, y, z} ∈T if and only if $t\alpha=\{x\alpha,y\alpha,z\alpha\} \in T$, ASTS(V) is a cyclic if it has an automophism that is a permutation consisting of a single cycle of length V.

Group action on a set: [4]

Def: Let X be a set and G a group . An action of G on X is a map:

*: X×G◊ such that:

(1) x.e= x, for all x∈X,

(2) $x(g_1.g_2)=(xg_1)g_2$,for all x∈X and all $g_1,g_2 \in G$

Under these conditions, X is a G-set.

Theorem: [4]

Let X be a G-set. For $x_1,x_2 \in X$, Let $x_1 \sim x_2$ if and only if there exists $g \in G$ such that :

$x_1 g = x_2$. Then ~ is an equivalence relation on S.

Def: [4]

Let X be a G-set. Each cell in the partition of the equivalence relation described in the theorem above is an orbit in X under G.

If $x \in X$, the cell containing x is the orbit of x . We let this cell be G_x .

Steiner system S(2,3,7) as PG(2,2):

The projective plane K= PG(2,2) contains 7 points and lines, 3 points in every line and 3 lines through every point.

Since (k,+) is abelian group with identity O , and (k/{0},*) is abelian group with identity 1 ,and distributive law is satisfied , then (1k,+,*) is a field.

Let P_i and L_i; i=1,2,…,7 be the points and the lines of PG(2,2) respectively, and let A be a square matrix (cyclic projectivity);

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Then $P_i = P_{i-1}A$ are the 7 points of PG(2,2).Writing i for P_i , the vectors of The 7 points of PG(2,2) are given in table (1)

Table (1). The points of PG (2,2)

i	Pi
1	(1,0,0)
2	(0,1,0)
3	(0,0,1)
4	(1,1,0)
5	(0,1,1)
6	(1,1,1)
7	(1,0,1)

By theorem (1), the points(1,0,0),(0,1,0) and(1,1,0) incidence on the line L_1 =[0,0,1], L_1 is the first line which contains the points 1,2 and 4 then $L_i = L_1 A^{i-1}$, i=1,2,…,7 are the lines of PG(2,2).

Projective plane PG(2,2) can be written as S(2,3,7) and this set satisfy the condition of steiner system where,

$$\binom{7}{2} \Big/ \binom{3}{2} = 7, \text{ which is integer,}$$

$$\binom{6}{1} \Big/ \binom{2}{1} = 3, \text{ is an integer too.}$$

So, the set S(2,3,7) is realy steiner system , the points of PG(2,2) are the points of steiner system and the lines of PG(2,2) are the blocks of steiner system and the number of lines equivalent to the number of blocks below . All block of S(2,3,7) are given in table (2).

Table(2). The block of S(2,3,7)

1	2	3	4	5	6	7
1	1	0	1	0	0	0
0	1	1	0	1	0	0
0	0	1	1	0	1	0
0	0	0	1	1	0	1
1	0	0	0	1	1	0
0	1	0	0	0	1	1
1	0	1	0	0	0	1

Steiner system S(2,4,13) as PG(2,3) :

The projective plane K=PG (2,3) contains 13 points and 13 lines, 4 points in every line and 4 line through every point.
(K,+) is abelian group with identity O , and (1K/{0},*) is abelian group With identity 1, and the distributive law is satisfied , then (1K,+,*) is a field .

Given cyclic projectivity,

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$$ on PG(2,3) therefor, the 13 points on PG(2,3) are

given in table (3) below :

Table (3) points of PG (2,3)

i	P_i
1	(1,0,0)
2	(0,1,0)
3	(0,0,1)
4	(2,1,0)
5	(0,2,1)
6	(1,2,1)
7	(1,1,1)
8	(2,2,1)
9	(1,0,1)
10	(1,1,0)
11	(0,1,1)
12	(2,1,1)
13	(2,0,1)

Let L_i be the line which contains the points 1,2,4 and 10 , then $L_i = L_1 A^{i-1}$,i=1,2,…13 are the lines of PG(2,3). Projective plane PG(2,3) can be written as S(2,4,13) and satisfies the conditions of steiner system where,

$$\binom{13}{2} \Big/ \binom{4}{2} = 13$$, which is an integer,

$$\binom{12}{1} \Big/ \binom{3}{1}$$

= 4, is an integer .

So, the set S(2,4,13) is really steiner system . All blocks of S(2,4,13) in table (4) below .

Table (4). The block of S (2,4,13)

1	2	4	10
2	3	5	11
3	4	6	12
4	5	7	13
1	5	6	8
2	6	7	9
3	7	8	10
4	8	9	11
5	9	10	12
6	10	11	13
1	7	11	12
2	8	12	13
1	3	9	13

Steiner system S(2,6,31) as PG(2,5) :

By the same way we can find the points of PG(2,5) where the cyclic Projectivity

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

, and all points of PG(2,5) given in table(5) .

Table (5). The points of PG (2,5)

i	Pi	i	Pi	i	Pi
1	(1,0,0)	11	(3,2,1)	21	(3,1,0)
2	(0,1,0)	12	(2,1,1)	22	(0,3,1)
3	(0,0,1)	13	(3,1,1)	23	(4,0,1)
4	(1,0,1)	14	(3,4,1)	24	(1,4,1)
5	(1,1,1)	15	(2,1,0)	25	(1,1,0)
6	(3,3,1)	16	(0,2,1)	26	(0,1,1)
7	(4,2,1)	17	(2,0,1)	27	(3,0,1)
8	(2,3,1)	18	(1,2,1)	28	(1,3,1)
9	(4,3,1)	19	(2,2,1)	29	(4,4,1)
10	(4,1,1)	20	(2,4,1)	30	(4,1,0)
				31	(0,4,1)

And table (6) we can find all blocs of S(2,6,31)

1	2	15	21	25	30
2	3	16	22	26	31
3	4	17	23	27	1
4	5	18	24	28	2
5	6	19	25	29	3
6	7	20	26	30	4
7	8	21	27	31	5
8	9	22	28	1	6
9	10	23	29	2	7
10	11	24	30	3	8
11	12	25	31	4	9
12	13	26	1	5	10
13	14	27	2	6	11

14	15	28	3	7	12
15	16	29	4	8	13
16	17	30	5	9	14
17	18	31	6	10	15
18	19	1	7	11	16
19	20	2	8	12	17
20	21	3	9	13	18
21	22	4	10	14	19
22	23	5	11	15	20
23	24	6	12	16	21
24	25	7	13	17	22
25	26	8	14	18	23
26	27	9	15	19	24
27	28	10	16	20	25
28	29	11	17	21	26
29	30	12	18	22	27
30	31	13	19	23	28
31	1	14	20	24	29

Steiner system S(2,8,57) as PG(2,7) :

Using the same way to construct S (2,8,57) and the first block is

1 2 4 14 33 37 44 53

and the last block is

57 1 3 13 32 36 43 52

Steiner system S (2,12,133) as PG(2,11) :

By the same way we can find all blocks of S(2,12,133) and the first block is

1 2 4 13 21 35 39 82 89 95 105 110
and the last block is:
133 1 3 12 20 34 38 81 88 94 104 109

Steiner system S (2,14,183) as PG (2,13):

Using the same way to construct S (2,14,183) and we can find all blocks and the first block is:

1 2 4 17 24 29 43 77 83 87 120 138 155 176
and the last block is:
183 1 3 16 23 28 42 76 82 86 119 137 154 175

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