# On Pair-Wise s*-Compactness in Bitopological Spaces

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**Abstract:** *The main goal of this paper is to create special types of compactness on topological spaces* $(X,\tau)$ *and on bitopological spaces* $(X,\tau_1,\tau_2)$, *where we studied the concepts of s*-compactness and pair-wise s*-compactness. Also, we study the characterizations and basic properties of s*-compact spaces and pair-wise s*-compact spaces.*

***Key words: s*-compact space, pair-wise s*-open cover, pair-wise s*-clopen set, pair-wise s*-compact space, pair-wise s*-irresolute function, pair-wise s*-continuous function.***

## Introduction

Levine, N. [1] introduced the concept of semi open sets. Also, Al-Meklafi, S. [2] introduced and investigated s*-closed sets by using the concept of semi-open sets. Khan, M. and et.al.,[3] we can prove that the family of all s*-open subsets of a topological space $(X,\tau)$ form a topology on X which is finer than $\tau$. The concept of a bitopological space $(X,\tau_1,\tau_2)$ was first introduced by Kelly, J. [4], where X is a non-empty set and $\tau_1,\tau_2$ are topologies on X. Reilly, I. and Mrsevice, M. [5] introduced the concept of pair-wise compact spaces. Recall that a subset A of a topological space $(X,\tau)$ is called a semi-open set if there exists an open subset U of

X such that $U \subseteq A \subseteq cl(U)$ [1]. The complement of a semi-open set is said to be semi-closed [1]. An s*-closed set is also called s*g-closed [3], $\hat{g}$ -closed [6] and w-closed [7].

## 1. Preliminaries

**Definition(1.1)[2]:**

A subset A of a topological space $(X,\tau)$ is called an s*-closed set if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in $(X,\tau)$. The complement of an s*-closed set is said to be s*-open. The class of all s*-open subsets of $(X,\tau)$ is denoted by $S^*O(X,\tau)$.

**Remarks (1.2):**

- ***i)*** Every open (closed) set is an s*-open (s*-closed) set respectively, but the converse is not true.
- **ii)** Semi-open sets and s*-open sets are independent.

**Theorem (1.3)[8]:**

A subset A of a topological space $(X,\tau)$ is s*-open iff $F \subseteq int(A)$ whenever F is an semi-closed subset of X and $F \subseteq A$.

**Definition (1.4):**

Let $(X,\tau)$ be a topological space and $A \subseteq X$ . Then:

- **i)** The s*-closure of A, denoted by $s^*-cl(A)$ is the intersection of all s*-closed subsets of X which contains A [3].
- **ii)** The s*-interior of A, denoted by $s^*-int(A)$ is the union of all s*-open subsets of X which are contained in A [3].
- **iii)** The semi-closure of A, denoted by $scl(A)$ is the intersection of all semi-closed subsets of X which contains A [1].

**Theorem (1.5)[9]:**

A topological space $(X,\tau)$ is an semi-$T_1$-space iff every singleton subset of X is semi-closed.

**Theorem (1.6)[8]:**

A subset A of a topological space $(X,\tau)$ is an s*-closed set iff $cl(A)-A$ contains no non-empty semi-closed set.

**Proposition (1.7):**

**(i)** If A is an s*-open set in X and B is an s*-open set in Y. Then $A\times B$ is an s*-open set in $X\times Y$.

**(ii)** If $A\subseteq X$ and $B\subseteq Y$. Then $A\times B$ is an s*-closed set in $X\times Y$ iff A and B are s*-closed sets in X and Y respectevely.

**Proof:**

**(i)** Suppose that F is a semi-closed set in $X\times Y$ such that $F\subseteq A\times B$. By theorem (1.3) we will prove that $F\subseteq \mathrm{int}(A\times B)$. Let $(x,y)\in F$. Then $(x,y)\in \mathrm{scl}\{(x,y)\}=\mathrm{scl}\{x\}\times \mathrm{scl}\{y\}\subseteq \mathrm{scl}(F)=F\subseteq A\times B$ and it follows that $\mathrm{scl}\{x\}\subseteq A$ and $\mathrm{scl}\{y\}\subseteq B$. Since A and B are s*-open sets in X and Y respectivey, then by theorem (1.3), we get $\mathrm{scl}\{x\}\subseteq \mathrm{int}(A)$ and $\mathrm{scl}\{y\}\subseteq \mathrm{int}(B)$. Thus $(x,y)\in \mathrm{scl}\{x\}\times \mathrm{scl}\{y\}\subseteq \mathrm{int}(A)\times \mathrm{int}(B)=\mathrm{int}(A\times B)$. Hence $F\subseteq \mathrm{int}(A\times B)$. Thus $A\times B$ is an s*-open set in $X\times Y$.

**(ii)** Suppose that A and B are s*-closed sets in X and Y respectevely. To prove that $A\times B$ is an s*-closed set in $X\times Y$. By theorem (1.6) is sufficient to show that $Q=cl(A\times B)-A\times B$ contains no non-empty semi-closed set. Suppose on the contary that $scl(x,y)\subseteq Q$ for some $(x,y)\in X\times Y$. It follows that $scl(x)\subseteq cl(A)$ and $scl(y)\subseteq cl(B)$. Since $cl(A)-A$ and $cl(B)-B$ contains no non-empty semi-closed set, then by theorem (1.6) $scl(x)\cap A\neq\phi$ and $scl(y)\cap B\neq\phi$. Choose $x'\in scl(x)\cap A$ and $y'\in scl(y)\cap B$. Then $(x',y')\in scl(x',y')=scl\{x'\}\times scl\{y'\}\subseteq scl\{x\}\times scl\{y\}=scl(x,y)\subseteq Q$. Thus $(x',y')\notin A\times B$ contradicting the fact that $(x',y')\in A\times B$. Conversely, it is obvious.

**Corollary(1.8):**

If A and B are subsets of topological spaces $(X,\tau)$ and $(Y,\tau')$ respectevily. Then:

**i)** $s*-\text{int}(A)\times s*-\text{int}(B)\subseteq s*-\text{int}(A\times B)$

**ii)** $s*-cl(A)\times s*-cl(B)=s*-cl(A\times B)$.

**Proof:**

**(i)** Let $(x,y)\in s*-\text{int}(A)\times s*-\text{int}(B)$ $\Rightarrow$ $x\in s*-\text{int}(A)$ and $y\in s*-\text{int}(B)$. Hence there are s*-open sets U in X and V in Y such that $x\in U\subseteq A$ and $y\in V\subseteq B$. Therefore $(x,y)\in U\times V\subseteq A\times B$. But by proposition ((1.7),(i)), $U\times V$ is an s*-open set in $X\times Y$. Hence $(x,y)\in s*-\text{int}(A\times B)$, thus $s*-\text{int}(A)\times s*-\text{int}(B)\subseteq s*-\text{int}(A\times B)$.

**(ii)** Since $s*-cl(A)$ and $s*-cl(B)$ are s*-closed sets, then by proposition ((1.7),(ii)), $s*-cl(A)\times s*-cl(B)$ is an s*-closed set in $X\times Y$. Since $A\times B\subseteq$ $s*-cl(A)\times s*-cl(B)$ , then $s*-cl(A\times B)\subseteq s*-cl[s*-cl(A)\times s*-cl(B)]=s*-cl(A)\times s*-cl(B)$. Hence $s*-cl(A\times B)\subseteq s*-cl(A)\times s*-cl(B)$. By the same way we can prove that $s*-cl(A)\times s*-cl(B)\subseteq s*-cl(A\times B)$. Therefore $s*-cl(A)\times s*-cl(B)=s*-cl(A\times B)$.

**Definition (1.9):**

A function $f:X\rightarrow Y$ from a topological space $(X,\tau)$ into a topological space $(Y,\tau')$ is called.

**(i)** s*-continuous [8] if the inverse image of every open set in $Y$ is s*-open in X.

**(ii)** s*-irresolute [6] if the inverse image of every s*-open set in $Y$ is s*-open in X.

**Corollary (1.10):**

Let $(X,\tau)$ and $(Y,\tau')$ be topological spaces. Then the projection functions $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are s*-irresolute functions.

**Proof:**

Let $U$ be an s*-open set in X, then $\pi_X^{-1}(U) = U \times Y$. Since U is s*-open in X and Y is s*-open in Y, then by proposition ((1.7),(i)), $U \times Y$ is an s*-open set in $X \times Y$. Thus $\pi_X : X \times Y \to X$ is s*-irresolute. Similaly we can prove that $\pi_Y : X \times Y \to Y$ is s*-irresolute.

**Definition (1.11)[8]:**

A topological space $(X,\tau)$ is called an s*-$T_2$-space if for any two distinct points x and y of X, there are two s*-open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \phi$.

**Definition (1.12)[8]:**

A topological space $(X,\tau)$ is called an s*-regular space if for any closed subset F of X and any point x of X which is not in F, there are two s*-open sets U and V such that $x \in U$, $F \subseteq V$ and $U \cap V = \phi$.

**Definition (1.13)[10]:**

Let $(x_d)_{d\in D}$ be a net in a topological space $(X,\tau)$. Then $(x_d)_{d\in D}$ s*-converges to $x \in X$ (written $x_d \xrightarrow{s^*} x$) iff for each s*-neighborhood $U$ of x, there is some $d_0 \in D$ such that $d \geq d_0$ implies $x_d \in U$. Thus $(x_d)_{d\in D}$ s*-converges to x iff it is eventually in every s*-neighborhood of x. The point x is called an s*-limit point of $(x_d)_{d\in D}$.

**Definition (1.14)[10]:**

Let $(x_d)_{d\in D}$ be a net in a topological space $(X,\tau)$. Then $(x_d)_{d\in D}$ is said to have $x \in X$ as an s*-cluster point (written $x_d \overset{s^*}{\propto} x$) iff for each s*-neighborhood U of x and for each $d \in D$, there is some $d_0 \geq d$ such that $x_{d_0} \in U$. Thus $(x_d)_{d\in D}$ has x as an s*-cluster point iff $(x_d)_{d\in D}$ is frequently in every s*-neighborhood of x.

**Theorem (1.15)[10]:**

Let $(x_d)_{d\in D}$ be a net in a topological space $(X,\tau)$ and for each $d$ in $D$ let $A_d$ be the set of all points $x_{d_0}$ for $d_0 \geq d$. Then x is an s*-cluster point of $(x_d)_{d\in D}$ if and only if $x$ belongs to the s*-closure of $A_d$ for each $d$ in D.

**Theorem (1.16)[10]:**

Let $(X,\tau)$ be a topological space and $A \subseteq X$. If x is a point of X, then $x \in s^*\text{-}cl(A)$ if and only if there exists a net $(x_d)_{d\in D}$ in $A$ such that $x_d \xrightarrow{s^*} x$.

## 2. s*-compactness

In this section we study a new type of compactness (to the best of our knowledge), namely s*-compactness. We will introduce new definitions, theorems, corollaries, remarks and examples.

**Definition (2.1):**

A collection $\{U_\alpha\}_{\alpha\in\Lambda}$ of s*-open sets in a topological space $(X,\tau)$ is called an s*-open cover of a subset $A$ of $X$ if $A \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha$.

**Definition (2.2):**

A topological space $(X,\tau)$ is said to be s*-compact if every s*-open cover of $X$ has a finite subcover.

**Definition (2.3):**

A subset $A$ of a topological space $(X,\tau)$ is said to be s*-compact if every cover of $A$ by s*-open subsets of $X$ has a finite subcover.

**Theorem (2.4):**

Every s*-compact space is a compact space.

**Proof:**

This is obvious since every open set is s*-open.

The converse of theorem (2.4) is not true in general. As in the following example:

**Example (2.5):**

Let $X$ be any infinite set with indiscrete topology $(X,I)$, then $(X,I)$ is a compact space, but $(X,I)$ is not s*-compact, since $\{\{x\}: x\in X\}$ is an s*-open cover of X which has no finite subcover.

**Examples (2.6):**

**i)** It is clear that any finite topological space is s*-compact.

**ii)** Any infinite set X with the co-finite topology $(X,\tau_{cof.})$ is s*-compact.

**Proof:**

Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be any s*-open cover of $X \Rightarrow X \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha$ and $U_\alpha$ is an s*-open set in $X$ for each $\alpha\in\Lambda$. Choose $U_{\alpha_0}\in\{U_\alpha\}_{\alpha\in\Lambda}$, then $U_{\alpha_0}^c = U'$ is an s*-closed set in X. To prove that $U'$ is closed. Now, let $x\in cl(U')$. Since X is a semi-$T_1$-space (because X is a $T_1$-space), then by theorem (1.5), $\{x\}$ is a semi-closed set in X, if $x\notin U'$, then $\{x\}\subseteq cl(U')\cap(X-U') = cl(U')-U'$. Hence $cl(U')-U'$ contains a non-empty semi-closed set $\{x\}$. This is a contradiction since $U'$ is s*-closed and according to the theorem (1.6) this is not possible. That is $x\in U'$, it follows that $U_{\alpha_0}^c = U'$ is closed, hence $U_{\alpha_0}$ is open .Therefore $\{U_\alpha\}_{\alpha\in\Lambda}$ is an open cover of X. Since X is compact, then $\exists\, \{U_{\alpha_i}\}_{i=0}^{n}$ is a finite subcover of $\{U_\alpha\}_{\alpha\in\Lambda}$. Thus $(X,\tau_{cof.})$ is an s*-compact space.

**iii)** The real line $\Re$ with usual topology $(\Re, \mu)$ is not s*-compact, since it is not compact.

**iv)** Any infinite set X with discrete topology $(X, D)$ is not s*-compact, since $\{\{x\} : x \in X\}$ is an s*-open cover of $X$ which has no finite subcover.

**Theorem (2.7):**

A topological space $(X, \tau)$ is s*-compact if and only if given any family $\{F_\alpha\}_{\alpha\in\Lambda}$ of s*-closed subsets of X such that the intersection of any finite number of the $F_\alpha$ is non-empty, then $\bigcap_{\alpha\in\Lambda} F_\alpha \neq \phi$.

**Proof:**

$\Rightarrow$ Suppose that $(X, \tau)$ is s*-compact and $\{F_\alpha\}_{\alpha\in\Lambda}$ be any family of s*-closed subsets of X such that the intersection of any finite number of the $F_\alpha$ is non-empty. To prove that $\bigcap_{\alpha\in\Lambda} F_\alpha \neq \phi$, if not $\Rightarrow \bigcap_{\alpha\in\Lambda} F_\alpha = \phi$ $\Rightarrow$ $(\bigcap_{\alpha\in\Lambda} F_\alpha)^c = \phi^c$ $\Rightarrow$ $\bigcup_{\alpha\in\Lambda}(X - F_\alpha) = X$. Set $U_\alpha = X - F_\alpha$ for each $\alpha \in \Lambda \Rightarrow$ $\bigcup_{\alpha\in\Lambda} U_\alpha = X$. Each $U_\alpha$ is the complement of an s*-closed set and hence is s*-open, therefore $\{U_\alpha\}_{\alpha\in\Lambda}$ is an s*-open cover of X. Since $(X, \tau)$ is s*-compact $\Rightarrow$ $\exists\{U_{\alpha_i}\}_{i=1}^n$ is a finite subcover of $\{U_\alpha\}_{\alpha\in\Lambda}$ $\Rightarrow$ $X = \bigcup_{i=1}^n U_{\alpha_i}$ $\Rightarrow$ $\bigcap_{i=1}^n F_{\alpha_i} = \phi$, this is a contradiction. Thus $\bigcap_{\alpha\in\Lambda} F_\alpha \neq \phi$.

Conversely, to prove that $(X, \tau)$ is s*-compact. Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be any s*-open cover of X $\Rightarrow$ $X = \bigcup_{\alpha\in\Lambda} U_\alpha$ and $U_\alpha$ is an s*-open set in X for each $\alpha \in \Lambda \Rightarrow$ $X^c = (\bigcup_{\alpha\in\Lambda} U_\alpha)^c$ $\Rightarrow$ $\phi = \bigcap_{\alpha\in\Lambda} U_\alpha^c$, where $U_\alpha^c = F_\alpha$ is an s*-closed set in X for each $\alpha \in \Lambda$ $\Rightarrow$ $\{F_\alpha\}_{\alpha\in\Lambda}$ is a family of s*-closed subsets of X such that $\bigcap_{\alpha\in\Lambda} F_\alpha = \phi$. Hence we can find finitely many of the $F_\alpha$, say $F_{\alpha_1}, F_{\alpha_2}, \ldots\ldots, F_{\alpha_n}$ such that $\bigcap_{i=1}^{n} F_{\alpha_i} = \phi \Rightarrow \bigcup_{i=1}^{n} U_{\alpha_i} = X \Rightarrow \{U_{\alpha_i}\}_{i=1}^{n}$ is a finite sub cover of $\{U_\alpha\}_{\alpha\in\Lambda}$. Thus $(X,\tau)$ is an s*-compact space.

**Theorem (2.8):**

Any s*-closed subset of an s*-compact space $(X,\tau)$ is s*-compact.

**Proof:**

Let $(X,\tau)$ be an s*-compact space and A be any s*-closed subset of X. To prove that A is s*-compact. Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be any cover of A by s*-open subsets of X $\Rightarrow A \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha$. Since A is s*-closed in X, then $A^c$ is s*-open in X. Since $A \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha \Rightarrow A \bigcup A^c \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha \bigcup A^c$ $\Rightarrow X \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha \bigcup A^c \Rightarrow \{\{U_\alpha\}_{\alpha\in\Lambda}, A^c\}$ is an s*-open cover of X. Since X is s*-compact, then $\exists \{\{U_{\alpha_i}\}_{i=1}^{n}, A^c\}$ is a finite sub cover of X. Since $A \bigcup A^c = X$ & $A \bigcap A^c = \phi \Rightarrow A \subseteq \bigcup_{i=1}^{n} U_{\alpha_i} \Rightarrow \{U_{\alpha_i}\}_{i=1}^{n}$ is a finite sub cover of $\{U_\alpha\}_{\alpha\in\Lambda}$. Thus A is s*-compact in X.

**Theorem (2.9):**

Any s*-compact subset of an s*-$T_2$-space $(X,\tau)$ is s*-closed.

**Proof:**

Let A be any s*-compact subset of an s*-$T_2$-space X. To prove that A is s*-closed. Let $x \in A^c$, then $\forall\ y \in A \Rightarrow y \notin A^c \Rightarrow x \neq y$. Since X is an s*-$T_2$-space, then by definition (1.11) there are two s*-open sets $U_x$ and $V_y$ of x and y respectively such that $U_x \bigcap V_y = \phi$. Hence $A \subseteq \bigcup_{y\in A} V_y \Rightarrow \{V_y\}_{y\in A}$ is an s*-open cover of A. But A is s*-compact, and then $\exists \{V_{y_i}\}_{i=1}^{n}$ is a finite subcover of $\{V_y\}_{y\in A}$. Now, let

$V = V_{y_1} \bigcup ...... \bigcup V_{y_n}$ and $U = U_{x_1} \bigcap ...... \bigcap U_{x_n}$, then $U$ and $V$ are s*-open since there are respectively the union and finite intersection of s*-open sets. Furthermore, $A \subseteq V$ and $x \in U$ since x belongs to each $U_{x_i}$. Since $U_{x_i} \bigcap V_{y_i} = \phi$ , $\forall\ i = 1,....,n$ $\Rightarrow$ $U \bigcap V_{y_i} = \phi$, $\forall\ i = 1,.....,n$. Hence $U \bigcap V = \phi$. Since $A \subseteq V$, then $U \bigcap A = \phi$, therefore $x \in U \subseteq A^c$, thus $A^c$ is s*-open. Hence A is s*-closed.

**Remark (2.10):**

If a topological space $(X,\tau)$ is not s*-$T_2$-space, then s*-compact subset in general is not s*-closed. As in the following example:

**Example:**

Let $X = \{a,b,c\}$ & $\tau = \{\phi, X, \{a\}\}$. Since $S*O(X,\tau) = \{\phi, X, \{a\}\}$, then $(X,\tau)$ is not s*-$T_2$-space. Observe that $\{a\}$ is s*-compact, but is not s*-closed.

**Corollary (2.11):**

A subset of an s*-compact s*-$T_2$-space $(X,\tau)$ is s*-compact iff it is s*-closed.

**Proof:**

It is obvious.

**Corollary (2.12):**

Any s*-compact s*-$T_2$-space $(X,\tau)$ is an s*-regular space.

**Proof:**

Let $x \in X$ and A be a closed subset of X such that $x \notin A$. Hence A is an s*-closed subset of X such that $x \notin A$. Since X is s*-compact, then by theorem (2.8), A is s*-compact and $x \notin A$. Hence, $\forall\ y \in A$ $\Rightarrow$ $y \notin A^c$ $\Rightarrow$ $x \neq y$. Since X is an s*-$T_2$-space, then by definition (1.11) there are two s*-open sets $U_x$ and $V_y$ of x and y respectively such that $U_x \bigcap V_y = \phi$. Hence $A \subseteq \bigcup_{y \in A} V_y \Rightarrow \{V_y\}_{y \in A}$ is an s*-open cover of A. But A is s*-compact, then $\exists \{V_{y_i}\}_{i=1}^{n}$ is a finite sub cover of $\{V_y\}_{y \in A}$. Now, let $V = V_{y_1} \bigcup ...... \bigcup V_{y_n}$ and $U = U_{x_1} \bigcap ...... \bigcap U_{x_n}$, then $U$ and $V$ are s*-open since there are respectively the union and finite intersection of s*-open sets. Furthermore, $A \subseteq V$ and $x \in U$ since x belongs to each $U_{x_i}$. Since $U_{x_i} \bigcap V_{y_i} = \phi$, $\forall\ i = 1,....,n \Rightarrow U \bigcap V_{y_i} = \phi$, $\forall\ i = 1,.....,n$. Hence $U \bigcap V = \phi$. Therefore by definition (1.12), $(X, \tau)$ is an s*-regular space.

**Theorem (2.13):**

**(i)** The s*-continuous image of an s*-compact space is compact.

**(ii)** The s*-irresolute image of an s*-compact space is s*-compact.

**Proof:**

**(i)** Let $f : X \to Y$ be an s*-continuous function from an s*-compact space $(X, \tau)$ into a topological space $(Y, \tau')$. To prove that $f(X)$ is compact. Let $\{U_\alpha\}_{\alpha \in \Lambda}$ be any open cover of $f(X) \Rightarrow f(X) \subseteq \bigcup_{\alpha \in \Lambda} U_\alpha$, where $U_\alpha$ is open in Y for each $\alpha \in \Lambda \Rightarrow X \subseteq \bigcup_{\alpha \in \Lambda} f^{-1}(U_\alpha)$. Since $f$ is s*-continuous and $U_\alpha$ is open in Y for each $\alpha \in \Lambda$, then by definition ((1.9),(i)) $f^{-1}(U_\alpha)$ is s*-open in X for each $\alpha \in \Lambda$, hence $\{f^{-1}(U_\alpha)\}_{\alpha \in \Lambda}$ is an s*-open cover of X. Because $(X, \tau)$ is s*-compact, then $\{f^{-1}(U_\alpha)\}_{\alpha \in \Lambda}$ has a finite subcover of X, that is $X \subseteq \bigcup_{i=1}^{n} f^{-1}(U_{\alpha_i})$. Hence $f(X) \subseteq \bigcup_{i=1}^{n} f(f^{-1}(U_{\alpha_i})) \subseteq \bigcup_{i=1}^{n} U_{\alpha_i} \Rightarrow$ $\{U_{\alpha_i}\}_{i=1}^{n}$ is a finite subcover of $\{U_\alpha\}_{\alpha \in \Lambda}$. Thus $f(X)$ is a compact set in Y.

**(ii)** Let $f : X \to Y$ be an s*-irresolute function from an s*-compact space $(X,\tau)$ into a topological space $(Y,\tau')$. To prove that $f(X)$ is s*-compact. Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be any cover of $f(X)$ by s*-open subsets of Y $\Rightarrow$ $f(X) \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha$, where $U_\alpha$ is s*-open in Y for each $\alpha \in \Lambda$. Hence $X \subseteq \bigcup_{\alpha\in\wedge} f^{-1}(U_\alpha)$. Since f is s*-irresolute and $U_\alpha$ is s*-open in Y for each $\alpha \in \Lambda$, then by definition ((1.9),(ii)) $f^{-1}(U_\alpha)$ is s*-open in X for each $\alpha \in \Lambda$, hence $\{f^{-1}(U_\alpha)\}_{\alpha\in\Lambda}$ is an s*-open cover of X. Because X is s*-compact, then $\{f^{-1}(U_\alpha)\}_{\alpha\in\Lambda}$ has a finite subcover of X, that is $X \subseteq \bigcup_{i=1}^{n} f^{-1}(U_{\alpha_i})$. Hence $f(X) \subseteq \bigcup_{i=1}^{n} f(f^{-1}(U_{\alpha_i})) \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$. So, $\{U_{\alpha_i}\}_{i=1}^{n}$ is a finite subcover of $\{U_\alpha\}_{\alpha\in\Lambda}$. Thus $f(X)$ is an s*-compact set in Y.

**Corollary (2.14):**
If $X \times Y$ is s*-compact space, then each of X and Y are s*-compact space.

**Proof:**
By corollary (1.10) the projection functions $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are s*-irresolute functions and by theorem ((2.13),(ii)) X and Y are s*-compact spaces.

**Theorem (2.15):**
The union of two s*-compact subsets of a topological space $(X,\tau)$ is s*-compact.

**Proof:**
Let A and B be s*-compact sets. To prove that $A \bigcup B$ is s*-compact. Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be any cover of $A \bigcup B$ by s*-open subsets of

X $\Rightarrow$ $A\bigcup B\subseteq\bigcup_{\alpha\in\Lambda}U_\alpha$ . Since $A\subseteq A\bigcup B$ and $B\subseteq A\bigcup B$, then $\{U_\alpha\}_{\alpha\in\Lambda}$ is an s*-open cover of $A$ and $B$ respectively. Since $A$ and B are s*-compact sets, then there exist finitely members of $\Lambda$ say $\alpha_1,\alpha_2,......,\alpha_n$ such that $A\subseteq\bigcup_{i=1}^{n}U_{\alpha_i}$ , and finitely members of $\Lambda$ say $\alpha_1,\alpha_2,......,\alpha_m$ such that $B\subseteq\bigcup_{j=1}^{m}U_{\alpha_j}$ . It follows that $A\bigcup B\subseteq\bigcup_{k=1}^{n+m}U_{\alpha_k}$ . Thus $A\bigcup B$ is s*-compact.

**Corollary (2.16):**

The union of any finite collection of s*-compact subsets of a topological space $(X,\tau)$ is s*-compact.

**Proof:**

It is obvious.

Now, we need the following theorem.

**Theorem (2.17):**

A net $(x_d)_{d\in D}$ in a topological space $(X,\tau)$ has x as an s*-cluster point iff it has a subnet which s*-converges to x.

**Proof:**

Let x be an s*-cluster point of $(x_d)$. Define M $=\{$(d,U): $d\in D$, U is an s*-nhood of x such that $x_d\in U\}$. Order M as follows: $(d_1,U_1)\le(d_2,U_2)$ iff $d_1\le d_2$ and $U_2\subseteq U_1$. It is clear that $M$ is a directed set. Define $\varphi:M\to D$ by: $\varphi(d,U)=d$. Then $\varphi$ is increasing and cofinal in D, so $\varphi$ defines a subnet of $(x_d)$. Let $U_0$ be an s*-nhood of x. Since x is an s*-cluster point of $(x_d)$, then $\exists\ d_0\in D$ such that $x_{d_0}\in U_0$. Hence $(d_0,U_0)\in M$ and moreover $(d,U)\ge(d_0,U_0)$ implies $U\subseteq U_0$, so that $(x\circ\varphi)(d,U)=x(d)=x_d\in U\subseteq U_0$. It follows that the subnet defined by $\varphi$ s*-converges to x.

Conversely, suppose $\varphi : M \to D$ defines a subnet of $(x_d)$ which s*-converges to x. To prove that x is an s*-cluster point of $(x_d)$. Let U be any s*-nhood of x and $d_0 \in D$. Since $\varphi$ is cofinal in D, then there is some $m_0 \in M$ such that $\varphi(m_0) \geq d_0$. Since the subnet $(x_{\varphi(m)})$ of $(x_d)$ is s*-converges to x, then there is also some $m_u \in M$ such that $m \geq m_u$ implies $x_{\varphi(m)} \in U$. Since M is a directed set, then there is $m^* \in M$ such that $m^* \geq m_0$ and $m^* \geq m_u$. Since $\varphi$ is increasing and $m^* \geq m_0$, then $\varphi(m^*) \geq \varphi(m_0)$, hence $\varphi(m^*) = d^* \geq d_0$ and $x_{d^*} = x_{\varphi(m^*)} \in U$. Thus for each s*-nhood U of x and each $d_0 \in D$, there is some $d^* \geq d_0$ such that $x_{d^*} \in U$. It follows that x is an s*-cluster point of $(x_d)$.

**Theorem (2.18):**

A topological space $(X, \tau)$ is s*-compact if and only if every net in X has an s*-cluster point.

**Proof:**

Let $(X, \tau)$ be an s*-compact space and let $(x_d)$ be a net in X. For each $d$ in D, let $M_d = \{x_{d_0} : d_0 \geq d\}$. Since D is directed by $\geq$, then the collection $\{M_d : d \in D\}$ has the finite intersection property. Hence $\{s*-cl(M_d) : d \in D\}$ also has the finite intersection property. It follows from theorem (2.7) that $\bigcap\{s*-cl(M_d) : d \in D\} \neq \phi$. Let $x \in \bigcap\{s*-cl(M_d) : d \in D\}$, then $x \in s*-cl(M_d)\ \forall\ d \in D$. Hence by the theorem (1.15) $x$ is an s*-cluster point of $(x_d)$.

Conversely, suppose that every net in X has an s*-cluster point and let $\Omega$ be a collection of s*-closed subsets of X with the finite intersection property. Let $\Omega' = \{$ D: D is the intersection of a finite subcollection of $\Omega\}$. Since the intersection of every two members of $\Omega'$ is a member of $\Omega'$, then $(\Omega', \subseteq)$ is a directed set by inclusion. Since each D is non-empty, then there is a point $x_D$ in D.

Now define the function $x:\Omega' \to X$ by: $x(D) = x_D \ \forall \ D \in \Omega'$. Then $(x_D)_{D\in\Omega'}$ is a net in X. By hypothesis, $(x_D)_{D\in\Omega'}$ must has an s*-cluster point say $x_0$ and by theorem (2.17) there is a subnet $(x_{DU})$ of $(x_D)_{D\in\Omega'}$ which s*-converges to $x_0$. Let E be an arbitrary member of $\Omega'$, then for each $D \geq E$ in $\Omega'$, we have $x(D) = x_D \in D \subseteq E$. Hence $(x_D)_{D\in\Omega'}$ is eventually in the s*-closed set E. Since $(x_{DU})$ is a subnet of $(x_D)_{D\in\Omega'}$, then $(x_{DU})$ is also eventually in E and by theorem (1.16), we get $x_0 \in s^*\!-\!cl(E) = E$. Since E is an arbitrary and $\Omega \subseteq \Omega'$, then we have $x_0 \in \bigcap \Omega' \subseteq \bigcap \Omega$. Hence $\bigcap \Omega \neq \phi$ and by theorem (2.7) $(X,\tau)$ is an s*-compact space.

## 3. Pair-Wise s*-Compactness

In this section we study a new type of compactness (to the best of our knowledge), namely pair-wise s*-compactness. We will introduce new definitions, theorems, corollaries, remarks and examples.

**Definition (3.1)[4]:**

A triple $(X,\tau_1,\tau_2)$ consists of a non-empty set X with two topologies $\tau_1$ and $\tau_2$ on X is said to be bitopological space.

**Definition (3.2)[11]:**

A subset A of a bitopological space $(X,\tau_1,\tau_2)$ is said to be pair-wise clopen set if A is $\tau_1$-open and $\tau_2$-closed or A is $\tau_1$-closed and $\tau_2$-open, that is $(A \in \tau_1 \wedge A^c \in \tau_2)$ or $(A^c \in \tau_1 \wedge A \in \tau_2)$.

**Definition (3.3)[12]:**

Let $(X,\tau_1,\tau_2)$ and $(Y,\tau'_1,\tau'_2)$ be two bitopological spaces, then the topology $W_1$ whose base is $E_1 = \{U \times V : U \in \tau_1 \ and \ V \in \tau'_1\}$ and the topology $W_2$ whose base is $E_2 = \{U' \times V' : U' \in \tau_2 \ and \ V' \in \tau'_2\}$ are called the product topologies for $X \times Y$ and $(X \times Y, W_1, W_2)$ is called the product bitopological space of two bitopological spaces X and Y.

**Definition (3.4)[4]:**

A bitopological space $(X, \tau_1, \tau_2)$ is called a pair-wise $T_2$-space if for any two distinct points x and y of X, there are a $\tau_1$-open set U and a $\tau_2$-open set V such that $x \in U$, $y \in V$ and $U \cap V = \phi$.

**Definition (3.5)[5]:**

Let $(X, \tau_1, \tau_2)$ be a bitopological space and A be a subset of X. By a pair-wise open cover of A, we mean a subcollection of the family $\tau_1 \cup \tau_2$ which contains at least one non-empty element of $\tau_1$ and at least one non-empty element of $\tau_2$ and it covers A.

**Definition (3.6)[5]:**

A bitopological space $(X, \tau_1, \tau_2)$ is said to be pair-wise compact space if every pair-wise open cover of $X$ has a finite subcover.

**Definition (3.7):**

Let $(X, \tau_1, \tau_2)$ be a bitopological space and A be a subset of X. By a pair-wise s*-open cover of A, we mean a subcollection of the family $\tau_1 - S{*}O(X) \cup \tau_2 - S{*}O(X)$ which contains at least one non -empty element of $\tau_1 - S{*}O(X)$ and at least one non-empty element of $\tau_2 - S{*}O(X)$ and it covers A.

**Definition (3.8):**

A bitopological space $(X, \tau_1, \tau_2)$ is said to be pair-wise s*-compact space if every pair-wise s*-open cover of X has a finite subcover.

**Theorem (3.9):**

Every pair-wise s*-compact space is a pair-wise compact space.

The converse of theorem (3.9) is not true in general as shown by the following example:

**Example (3.10):**

The bitopological space $(N, I, D)$ (where N is the set of all natural numbers, and I, D are the indiscrete and discrete topologies on N respectively) is a pair-wise compact space, but is not pair-wise s*-compact, since $\{\{x\}: x \in N\}$ is a pair-wise s*-open cover of N which has no finite subcover.

**Remark (3.11):**

Let $(X, \tau_1, \tau_2)$ be a bitopological space. If $(X, \tau_1)$ and $(X, \tau_2)$ are s*-compact spaces, then $(X, \tau_1, \tau_2)$ need not be a pair-wise s*-compact space. As in the following example:

**Example (3.12):**

Let $X = \Re$ and $\tau_1 = \{U \subseteq \Re : 0 \notin U\} \cup \{\Re\}$ and $\tau_2 = \{U \subseteq \Re : 1 \notin U\} \cup \{\Re\}$. It is clear that $\{\{x\}: x \in \Re\}$ is a pair-wise s*-open cover of $\Re$ (since $\{\{x\}: x \in \Re\}$ is a pair-wise open cover of $\Re$) which has no finite subcover. Therefore $(\Re, \tau_1, \tau_2)$ is not a pair-wise s*-compact space, while $(\Re, \tau_1)$ and $(\Re, \tau_2)$ are s*-compact spaces.

**Proposition (3.13):**

Let $(X, \tau_1, \tau_2)$ be a bitopological space. If $\tau_1 - S^*O(X)$ is a subfamily of $\tau_2 - S^*O(X)$ and $(X, \tau_2)$ is an s*-compact space or $\tau_2 - S^*O(X)$ is a subfamily of $\tau_1 - S^*O(X)$ and $(X, \tau_1)$ is an s*-compact space, then $(X, \tau_1, \tau_2)$ is a pair-wise s*-compact space.

**Proof:**

Suppose that $\tau_1 - S^*O(X)$ is a subfamily of $\tau_2 - S^*O(X)$ and $(X, \tau_2)$ is an s*-compact space. Then $\tau_1 - S^*O(X) \cup \tau_2 - S^*O(X) = \tau_2 - S^*O(X)$, that is every pair-wise s*-open cover of X is an s*-open cover with respect to $\tau_2$. But every s*-open cover of X with respect to $\tau_2$ has a finite subcover, it follows that every pair-wise s*-open cover of X has a finite subcover. Hence $(X,\tau_1,\tau_2)$ is a pair-wise s*-compact space. Similarly we can prove the second case.

**Definition(3.14):**

A subset A of a bitopological space $(X,\tau_1,\tau_2)$ is said to be pair-wise s*-clopen set if $(A\in\tau_1-S^*O(X)\wedge A^c\in\tau_2-S^*O(X))$ or $(A^c\in\tau_1-S^*O(X)\wedge A\in\tau_2-S^*O(X))$.

**Remark(3.15):**

If $(X,\tau_1,\tau_2)$ is a bitopological space. Then every pair-wise clopen subset of X is a pair-wise s*-clopen set. But the converse is not true by the following example:

**Example(3.16):**

Let $X=\{a,b,c\}$, $\tau_1=\{X,\phi,\{a\}\}$ and $\tau_2=I=\{X,\phi\}$, then $\tau_1-S^*O(X)=\{X,\phi,\{a\}\}$ and $\tau_2-S^*O(X)=\{X,\phi,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\}\}$. $\{a\}$ is a pair-wise s*-clopen subset of X, since $\{a\}\in\tau_1-S^*O(X)$ and $\{a\}^c=\{b,c\}\in\tau_2-S^*O(X)$. But $\{a\}$ is not a pair -wise clopen set, since $\{a\}\in\tau_1$, but $\{a\}^c=\{b,c\}\notin\tau_2$.

**Proposition (3.17):**

A pair-wise s*-clopen subset of a pair-wise s*-compact space is a pair-wise s*-compact set.

**Proof:**

Let A be a pair-wise s*-clopen subset of a pair-wise s*-compact space $(X,\tau_1,\tau_2)$ and $\{U_\alpha\}_{\alpha\in\Lambda}$ be any pair-wise s*-open cover of A. It follows by definition (3.14) that $A^c$ is a member of the family $\tau_1-S^*O(X)\cup\tau_2-S^*O(X)$. Then $\{\{U_\alpha\}_{\alpha\in\Lambda},A^c\}$ is a pair-wise s*-open cover of X which is a pair-wise s*-compact space, then there exist finitely many elements $\alpha_1,\alpha_2,.....,\alpha_n$ such that $X = (\bigcup_{i=1}^{n} U_{\alpha_i}) \cup A^c$. Since $A \cup A^c = X$ & $A \cap A^c = \phi$, then $A \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$. Therefore $\{U_{\alpha_i}\}_{i=1}^{n}$ is a finite subcover of $\{U_\alpha\}_{\alpha \in \wedge}$. Thus A is a pair-wise s*-compact subspace of X.

**Corollary (3.18):**
A pair-wise clopen subset of a pair-wise s*-compact space is a pair-wise s*-compact set.

**Corollary (3.19):**
A pair-wise s*-clopen (resp. pair-wise clopen) subset of a pair-wise s*-compact space is a pair-wise compact set.

**Definition (3.20):**
Let $(X,\tau_1,\tau_2)$ and $(Y,\tau_1',\tau_2')$ be two bitopological spaces. Then a function $f : X \to Y$ is said to be:

**(i)** Pair-wise s*-irresolute if $f^{-1}(U) \in \tau_1 - S*O(X)$ for each $U \in \tau_1' - S*O(Y)$ and $f^{-1}(V) \in \tau_2 - S*O(X)$ for each $V \in \tau_2' - S*O(Y)$.

**(ii)** Pair-wise s*-continuous if $f^{-1}(U) \in \tau_1 - S*O(X)$ for each $U \in \tau_1'$ and $f^{-1}(V) \in \tau_2 - S*O(X)$ for each $V \in \tau_2'$.

**(iii)** Pair-wise s*-open if $f(U) \in \tau_1' - S*O(Y)$ for each $U \in \tau_1 - S*O(X)$ and $f(V) \in \tau_2' - S*O(Y)$ for each $V \in \tau_2 - S*O(X)$.

**Proposition (3.21):**
Let $(X \times Y, W_1, W_2)$ be the product bitopological space of bitopological spaces $(X,\tau_1,\tau_2)$ and $(Y,\tau_1',\tau_2')$. Then the biprojection functions $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are pair-wise s*-irresolute.

**Proof:**

Let $U \in \tau_1 - S*O(X)$ and $V \in \tau_2 - S*O(X)$, then $\pi_X^{-1}(U) = U \times Y$ and $\pi_X^{-1}(V) = V \times Y$. Since U is $\tau_1 - S*O(X)$ and Y is $\tau_1' - S*O(Y)$, then by proposition ((1.7),(i)) $U \times Y$ is $W_1 - S*O(X \times Y)$. Again since V is $\tau_2 - S*O(X)$ and Y is $\tau_2' - S*O(Y)$, then by proposition ((1.7),(i)) $V \times Y$ is $W_2 - S*O(X \times Y)$. Thus $\pi_X$ is a pair-wise s*-irresolute function. Similarly $\pi_Y$ is also a pair-wise s*-irresolute function.

**Theorem (3.22):**

**i)** The pair-wise s*-irresolute image of a pair-wise s*-compact space is a pair-wise s*-compact.

**ii)** The pair-wise s*-continuous image of a pair-wise s*-compact space is a pair-wise compact.

**Proof:**

**i)** Let $f:(X,\tau_1,\tau_2) \to (Y,\tau_1',\tau_2')$ be a pair-wise s*-irresolute function and X is a pair-wise s*-compact space. To prove that $f(X)$ is pair-wise s*-compact. Let $\{U_\alpha\}_{\alpha \in \Lambda}$ be any pair-wise s*-open cover of $f(X)$, that is $f(X) \subseteq \bigcup_{\alpha \in \Lambda} U_\alpha$. Since $f$ is pair-wise s*-irresolute, so $\{f^{-1}(U_\alpha)\}_{\alpha \in \Lambda}$ is a pair-wise s*-open cover of X which is a pair-wise s*-compact space, then there exist finitely many elements $\alpha_1, \alpha_2, ....., \alpha_n$ such that $X \subseteq \bigcup_{i=1}^{n} f^{-1}(U_{\alpha_i})$. Hence $f(X) \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$, thus $f(X)$ is a pair-wise s*-compact set.

**ii)** The prove is similar to part (i) hence is omitted.

**Corollary (3.23):**

Let $(X \times Y, W_1, W_2)$ be the product bitopological space of bitopological spaces $(X, \tau_1, \tau_2)$ and $(Y, \tau_1', \tau_2')$. If $X \times Y$ is a pair-wise s*-compact space, then each of the spaces X and Y is a pair-wise s*-compact space.

**Proof:**

By proposition (3.21) the biprojection functions $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are pair-wise s*-irresolute and by theorem ((3.22),(i)), we get X and Y are pair-wise s*-compact spaces.

**Definition (3.24):**

A bitopological space $(X, \tau_1, \tau_2)$ is called a pair-wise s*- $T_2$-space if for any two distinct points x and y of X, there are a $\tau_1$- s*-open set U and a $\tau_2$-s*-open set V such that $x \in U$, $y \in V$ and $U \bigcap V = \phi$.

**Remark (3.25):**

Every pair-wise $T_2$-space is a pair-wise s*-$T_2$- space, but the converse is not true in general. As in the following example.

**Example (3.26):**

Let $X = \{a,b\}$ and $\tau_1 = \tau_2 = I = \{X, \phi\}$. Then $\tau_1 - S*O(X) = \tau_2 - S*O(X) = \{X, \phi, \{a\}, \{b\}\}$.It is clear that $(X, \tau_1, \tau_2)$ is a pair-wise s*-$T_2$-space, but is not a pair-wise $T_2$-space.

**Remark (3.27):**

A pair-wise s*-compact subset of a pair-wise s*-$T_2$-space need not to be pair-wise s*-clopen set. As in the following example.

**Example (3.28):**

Let $X = \{a,b,c\}$ and $\tau_1 = \{X, \phi, \{a,b\}\}$ and let $\tau_2 = \{X, \phi, \{b,c\}\}$. Then $\tau_1 - S*O(X) = \{X, \{a\}, \{b\}, \{a,b\}, \phi\}$ and $\tau_2 - S*O(X) =$

$\{X,\{b\},\{c\},\{b,c\},\phi\}$. It is clear that $(X,\tau_1,\tau_2)$ is a pair-wise s*-$T_2$-space. But $A=\{a,c\}$ is a pair-wise s*-compact subset of X, but it is not a pair-wise s*-clopen set.

To define a pair-wise s*-regular space we introduce the following definition.

**Definition (3.29):**
Let $(X,\tau_1,\tau_2)$ be a bitopological space. Then $\tau_i$ $(i=1,2)$ is called s*-regular with respect to $\tau_j$ $(j=1,2, i\neq j)$ if for each point x in X and each $\tau_i$-closed set $F$ such that $x\notin F$, there exist a $\tau_i$-s*-open set U and a $\tau_j$-s*-open set V such that $x\in U$, $F\subseteq V$ and $U\cap V=\phi$.

**Definition (3.30):**
A bitopological space $(X,\tau_1,\tau_2)$ is called a pair-wise s*-regular space if and only if $\tau_1$ is s*-regular with respect to $\tau_2$ and $\tau_2$ is s*-regular with respect to $\tau_1$.

**Remark (3.31):**
A pair-wise s*-compact pair-wise s*-$T_2$-space need not to be pair-wise s*-regular space. As in the following example.

**Example (3.32):**
Let $X=\{a,b,c\}$ and $\tau_1=\{X,\phi,\{b,c\}\}$ and let $\tau_2=\{X,\phi,\{b,c\},\{a\}\}$. Then $\tau_1-S*O(X)=\{X,\phi,\{b\},\{c\},\{b,c\}\}$ and $\tau_2-S*O(X)=\{X,\phi,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\}\}$. It is clear that $(X,\tau_1,\tau_2)$ is a pair-wise s*-compact pair-wise s*-$T_2$-space, but is not pair-wise s*-regular, since $\{a\}$ is a $\tau_2$-closed set in X and $b\notin\{a\}$, but for each $\tau_2$-s*-open set U with $b\in U$, there is no $\tau_1$-s*-open set containing $\{a\}$ whose intersection with U is empty.

**Theorem (3.33):**

A bitopological space $(X,\tau_1,\tau_2)$ is pair-wise s*-regular iff for each $x \in X$ and each $\tau_1$-open set U of x, there exists a $\tau_1$-s*-open set W of x such that $x \in W \subseteq s^*\!-\!cl_{\tau_2}(W) \subseteq U$ and for each $\tau_2$-open set V of x, there exists a $\tau_2$-s*-open set H of x such that $x \in H \subseteq s^*\!-\!cl_{\tau_1}(H) \subseteq V$.

**Proof:**

Suppose that $(X,\tau_1,\tau_2)$ is pair-wise s*-regular and let $x \in X$ and U be a $\tau_1$-open set such that $x \in U$, it follows that $x \notin U^c$ where $U^c$ is a $\tau_1$-closed set. But $\tau_1$ is s*-regular with respect to $\tau_2$, then there exists a $\tau_1$-s*-open set W and a $\tau_2$-s*-open set V such that $x \in W$ and $U^c \subseteq V$ and $V \cap W = \phi$. Hence, we get $V^c \subseteq U$ and $W \subseteq V^c$ which is a $\tau_2$-s*-closed set. Thus $x \in W \subseteq s^*\!-\!cl_{\tau_2}(W) \subseteq U$. Similarly, we can prove that for each $\tau_2$-open set V of x, there exists a $\tau_2$-s*-open set H of x such that $x \in H \subseteq s^*\!-\!cl_{\tau_1}(H) \subseteq V$.

Conversely, to prove that $(X,\tau_1,\tau_2)$ is pair-wise s*-regular i.e. $\tau_1$ is s*-regular with respect to $\tau_2$ and $\tau_2$ is s*-regular with respect to $\tau_1$.

Let $x \in X$ and F be a $\tau_1$-closed set in X such that $x \notin F$, it follows that $x \in F^c$ which is a $\tau_1$-open set. By hypotheses there exist a $\tau_1$-s*-open set W of x such that $x \in W \subseteq s^*\!-\!cl_{\tau_2}(W) \subseteq F^c$. Hence $x \in W$ and $F \subseteq [s^*\!-\!cl_{\tau_2}(W)]^c$ which is a $\tau_2$-s*-open set and $W \cap [s^*\!-\!cl_{\tau_2}(W)]^c = \phi$. Thus $\tau_1$ is s*-regular with respect to $\tau_2$. Similarly, we can prove that $\tau_2$ is s*-regular with respect to $\tau_1$.

**Definition (3.34):**

Let $(X,\tau_1,\tau_2)$ be a bitopological space. Then $\tau_i$ $(i = 1,2)$ is called s*-normal with respect to $\tau_j$ $(j = 1,2, i \neq j)$ if for each $\tau_i$-closed set $F_1$ and each $\tau_j$-closed set $F_2$ such that $F_1 \cap F_2 = \phi$, there exist a $\tau_i$-

s*-open set U and a $\tau_j$-s*-open set V such that $F_2 \subseteq U$, $F_1 \subseteq V$ and $U \bigcap V = \phi$.

**Definition(3.35):**

A bitopological space $(X, \tau_1, \tau_2)$ is called a pair-wise s*-normal space iff $\tau_1$ is s*-normal with respect to $\tau_2$ and $\tau_2$ is s*-normal with respect to $\tau_1$.

**Theorem (3.36):**

A bitopological space $(X, \tau_1, \tau_2)$ is pair-wise s*-normal iff for each $\tau_1$-open set U and each $\tau_2$-closed set $F_1$ such that $F_1 \subseteq U$, there exists a $\tau_1$-s*-open set W such that $F_1 \subseteq W \subseteq s*{-}cl_{\tau_2}(W) \subseteq U$ and for each $\tau_2$-open set V and each $\tau_1$-closed set $F_2$ such that $F_2 \subseteq V$, there exists a $\tau_2$-s*-open set H such that $F_2 \subseteq H \subseteq s*{-}cl_{\tau_1}(H) \subseteq V$.

**Proof:**

Suppose that $(X, \tau_1, \tau_2)$ is pair-wise s*-normal and let U be a $\tau_1$-open set and $F_1$ be a $\tau_2$-closed set such that $F_1 \subseteq U$, it follows that $U^c \bigcap F_1 = \phi$, where $U^c$ is a $\tau_1$-closed set. But $\tau_1$ is s*-normal with respect to $\tau_2$, then there exists a $\tau_1$-s*-open set W and $\tau_2$-s*-open set V such that $F_1 \subseteq W$ and $U^c \subseteq V$ and $W \bigcap V = \phi$. Hence, we get $V^c \subseteq U$ and $W \subseteq V^c$ which is a $\tau_2$-s*-closed set. Thus $F_1 \subseteq W \subseteq s*{-}cl_{\tau_2}(W) \subseteq U$. Similarly, we can prove that for each $\tau_2$-open set V and each $\tau_1$-closed set $F_2$ such that $F_2 \subseteq V$, there exists a $\tau_2$-s*-open set H such that $F_2 \subseteq H \subseteq s*{-}cl_{\tau_1}(H) \subseteq V$.

Conversely, to prove that $(X, \tau_1, \tau_2)$ is pair-wise s*-normal i.e. $\tau_1$ is s*-normal with respect to $\tau_2$ and $\tau_2$ is s*-normal with respect to $\tau_1$. Let $F_1$ be a $\tau_1$-closed set and $F_2$ be a $\tau_2$-closed set such that $F_1 \bigcap F_2 = \phi$, it follows that $F_2 \subseteq F_1^c$ which is a $\tau_1$-open set. By hypotheses there exist a $\tau_1$-s*-open set W such that $F_2 \subseteq W \subseteq s*-cl_{\tau_2}(W) \subseteq F_1^c$. Hence $F_2 \subseteq W$ and $F_1 \subseteq [s*-cl_{\tau_2}(W)]^c$ which is a $\tau_2$-s*-open set and $W \bigcap [s*-cl_{\tau_2}(W)]^c = \phi$. Thus $\tau_1$ is s*-normal with respect to $\tau_2$.

Similary, we can prove that $\tau_2$ is s*-normal with respect to $\tau_1$.

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# حول ثنائي *s - متراص في الفضاءات التبولوجية الثنائية

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**المستخلص**

الهدف الرئيسي من هذا البحث هو استنباط انواع خاصة من التراص على الفضاء التبولوجي $(X,\tau)$ والفضاء التبولوجي الثنائي $(X,\tau_1,\tau_2)$ حيث درسنا مفهوم التراص من النوع -*s والتراص الثنائي من النوع -*s . كذلك درسنا المكافئات والخواص الأساسية للفضاءات المتراصة - *s و الفضاءات الثنائية المتراصة – *s.

**الكلمات الرئيسية:الفضاء المتراص- *s، الغطاء المفتوح الثنائي – *s، المجموعة المغلقة - المفتوحة الثنائية – *s، الفضاء الثنائي المتراص – *s، الدالة المحيرة الثنائية – *s، الدالة المستمرة الثنائية – *s.**