Exponential Reliability Estimation of (3+1) Cascade Model

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Abstract: In this paper presents the R reliability mathematical formula of the (3 + 1) Exponential cascade model. The reliability of the model is expressed by exponential random variables, which are stress and strength distributions. The reliability model was estimated by three dissimilar methods (ML, Pr and LS) and simulation was performed using MATLAB 2016 program to compare the results of the reliability model estimates using the MSE criterion, the results indicated that the best estimator among the three estimators was ML.

Keywords: Standby redundancy, Parameter, Exponential distribution, Unit, distributed Identically.

1. Introduction

Many researches have been performed on reliability estimation $\mathcal{R} = pr(X > Y)$ in the field of strength and stress models. The cascade is a special kind of stress-strength model. Cascade redundancy is a hierarchical standby redundancy in which a standby unit with different stress substitutes for a system. When a system unit fails, it is replaced by a standby unit and the stress changed $\mathcal{K}$ times the previous stress [1]. In a previous study, Karam and Khaleel (2019) presented a study of (2+1) cascade model, which the model consists of two main components and one redundancy standby .In this paper, we assumed that the (3+1) cascade model consists four components (three basic components and one component redundancy standby). Consider a special model (3+1) of cascade with (U_1, U_2, U_3 *and* U_4) units, in which three units U_1, U_2 *and* U_3 are work and the unit U_4 is a standby unit. Assume that $X_1, X_2, X_3 X_4$ denote the unit strengths(U_1, U_2, U_3 *and* U_4) respectively and Y_1, Y_2, Y_3, Y_4 indicated the enforcement of stress. Here, if the active unit U_1 is a failure then the standby component U_4 is activated, where $X_4 = \mathcal{M} X_1$ and $Y_4 = \mathcal{K} Y_1$, if the active unit U_1 is a failure then the standby component U_4 is activated, where $X_4 = \mathcal{M}X_2$ and $Y_4 = \mathcal{K}Y_2$ and if the active unit U_3 is a failure then the standby component U_4 is activated, where $X_4 = \mathcal{M}X_3$ and $Y_4 = \mathcal{K}Y_3$ where "$\mathcal{K}$" and "$\mathcal{M}$" denote the stress and strength attenuation factors respectively, such that $0 < \mathcal{M} < 1$ and $\mathcal{K} > 1$.

Reddy (2016) [2] presents of $R = pr(X > Y)$ by discussing model stress-strength of a cascade, assuming all the parameters are independent and following exponential stress-strength distributions in one parameter and calculating first four cascade reliability for different stress- strength values. Mutkekar and Munoli (2016) [3], (1 + 1) exponential distribution cascade model is derived with the common effect of the force and stress reduction factors. Kumar and Vaish (2017) [4], discussed that Gompertz distribution is stress and that strength is power distribution parameters. Karam and Khaleel (2018) [1] derived a special (2 + 1) stress- strength reliability Cascade model for the distribution of Weibull. Khaleel and karam (2019) [5] discussed the reliability of the (2 + 1) Cascade inverse distribution Weibull model, reliability can be found when reverse Weibull random variables with unknown parameters scale and known shape parameter are distributed with strength-stress and used six different estimations mothed to estimate reliability. Karam and Khaleel (2019) [6], expression for model confidence is found when strength and stress distribution are generalized in reversed Rayleigh random variable Rayleigh, derived from mathematical formulas for Reliability to Special (2+ 1).

2. The mathematical formula

Suppose, for the four units (three basic and one redundant standby), the random strength-stress variables of the four units j=1,2,3,4 each independently and identically distributed exponential of the parameter scale $a_i; i$ =1,2,3,4 and scale b_j; j =1,2,3,4.

The Cumulative distribution function of Exp(a) is:

$$F(x) = 1 - e^{-ax} \quad x > 0; a > 0 \quad \ldots(1)$$

and

The Cumulative distribution function of Exp(b) is :

$$G(y) = 1 - e^{-by} \quad y > 0; b > 0 \quad \ldots(2)$$

The reliability function for (3+1) cascade model is:

$$\mathcal{R} = pr[x_1 \geq y_1, x_2 \geq y_2, x_3 \geq y_3]$$

$$+pr[x_1 < y_1, x_2 \geq y_2, x_3 \geq y_3, x_4 \geq y_4]$$

$$+pr[x_1 \geq y_1, x_2 < y_2, x_3 \geq y_3, x_4 \geq y_4]$$

$+pr[x_1 \geq y_1, x_2 \geq y_2, x_3 < y_3, x_4 \geq y_4]$

$\mathcal{R} = S_1 + S_2 + S_3 + S_4$ ...(3)

$S_1 = pr\left[x_1 \geq y_1, x_2 \geq y_2, x_3 \geq y_3\right]$

$= pr[x_1 \geq y_1]\, p[x_2 \geq y_2]\, p[x_3 \geq y_3]$

$= \left[\int_0^\infty \left(\overline{F}_{x_1}(y_1)\right) g(y_1) dy_1\right]$

$.\left[\int_0^\infty \left(\overline{F}_{x_2}(y_2)\right) g(y_2) dy_2\right]$

$.\left[\int_0^\infty \left(\overline{F}_{x_3}(y_3)\right) g(y_3) dy_3\right]$

Where $\overline{\mathrm{F}}(\mathrm{x}) = 1 - \mathrm{F}(\mathrm{x})$

$S_1 = [\int_0^\infty (e^{-a_1 y_1}) b_1 e^{-b_1 y_1} dy_1]$

$.[\int_0^\infty (e^{-a_2 y_2}) b_2 e^{-b_2 y_2} dy_2]$

$.[\int_0^\infty (e^{-a_3 y_3}) b_3 e^{-b_3 y_3} dy_3]$

$S_1 = [\int_0^\infty b_1 e^{-(a_1+b_1)y_1} dy_1]$

$.[\int_0^\infty b_2 e^{-(a_2+b_2)y_2} dy_2]$

$.[\int_0^\infty b_3 e^{-(a_3+b_3)y_3} dy_3]$

Now, will get S_1:

$S_1 = \left[\frac{b_1}{a_1+b_1}\right]\left[\frac{b_2}{a_2+b_2}\right]\left[\frac{b_3}{a_3+b_3}\right]$ ...(4)

To derive S_2 will start

$S_2 = pr[x_1 < y_1, x_2 \geq y_2, x_3 \geq y_3, x_4 \geq y_4]$

$= pr\left[x_1 < y_1, x_2 \geq y_2, x_3 \geq y_3, \mathcal{M}x_1 \geq \mathcal{K}y_1\right]$

$S_2 = pr\left[x_1 < y_1, \mathcal{M}x_1 \geq \mathcal{K}y_1\right]$

$p[x_2 \geq y_2]\, p[x_3 \geq y_3]$

$S_2 = \left[\int_0^\infty (F_{x_1}(y_1)) \left(\overline{F}_{x_1}\left(\frac{\mathcal{K}}{\mathcal{M}} y_1\right)\right) g(y_1)\right.$

$\left. dy_1\right] \left[\int_0^\infty \left(\overline{F}_{x_2}(y_2)\right) g(y_2) dy_2\right]$

$.\left[\int_0^\infty \left(\overline{F}_{x_3}(y_3)\right) g(y_3) dy_3\right]$

$S_2 = \left[\int_0^\infty (1 - e^{-a_1 y_1}) \left(e^{-a_1\left(\frac{\mathcal{K}}{\mathcal{M}}\right) y_1}\right)\right.$

$b_1 e^{-b_1 y_1} dy_1][\int_0^\infty (e^{-a_2 y_2})$

$b_2 e^{-b_2 y_2} dy_2][\int_0^\infty (e^{-a_3 y_3})$

$b_3 e^{-b_3 y_3} dy_3]$

$S_2 = \left[\int_0^\infty \left(e^{-a_1\left(\frac{\mathcal{K}}{\mathcal{M}}\right) y_1}\right) b_1 e^{-b_1 y_1} dy_1\right.$

$\left. - \int_0^\infty (e^{-a_1 y_1}) \left(e^{-a_1\left(\frac{\mathcal{K}}{\mathcal{M}}\right) y_1}\right) b_1 e^{-b_1 y_1} dy_1\right]$

$.[\int_0^\infty (e^{-a_2 y_2}) b_2 e^{-b_2 y_2} dy_2][\int_0^\infty (e^{-a_3 y_3})$

$b_3 e^{-b_3 y_3} dy_3]$

$S_2 = \left[\int_0^\infty b_1 e^{-\left(a_1\left(\frac{\mathcal{K}}{\mathcal{M}}\right)+b_1\right) y_1} dy_1\right.$

$\left. - \int_0^\infty b_1 e^{-\left(a_1 + a_1\left(\frac{\mathcal{K}}{\mathcal{M}}\right)+b_1\right) y_1} dy_1\right]$

$.[\int_0^\infty b_2 e^{-(a_2+b_2)y_2} dy_2]$

$.[\int_0^\infty b_3 e^{-(a+b_3)y_3} dy_3]$

Then

$S_2 = \left[\frac{a_1 b_1}{\left(a_1\left(\frac{\mathcal{K}}{\mathcal{M}}\right)+b_1\right)\left(a_1\left(1+\left(\frac{\mathcal{K}}{\mathcal{M}}\right)\right)+b_1\right)}\right]\left[\frac{b_2}{a_2+b_2}\right]$

$.\left[\frac{b_3}{a_3+b_3}\right]$ ... (5)

In a similarly way for S_3 :

$S_3 = pr\left[x_1 \geq y_1, x_2 < y_2, x_3 \geq y_3, x_4 \geq y_4\right]$

$= pr\left[x_1 \geq y_1, x_2 < y_2, x_3 \geq y_3, \mathcal{M}x_2 \geq \mathcal{K}y_2\right]$

$= pr\left[x_1 \geq y_1\right] pr\left[x_2 < y_2, \mathcal{M}x_2 \geq \mathcal{K}y_2\right]\ pr[x_3 \geq y_3]$

$$S_3 = \left[\frac{b_1}{a_1+b_1}\right]\left[\frac{a_2b_2}{\left(a_2\left(\frac{\mathcal{K}}{\mathcal{M}}\right)+b_2\right)\left(a_2\left(1+\left(\frac{\mathcal{K}}{\mathcal{M}}\right)\right)+b_2\right)}\right] \cdot\left[\frac{b_3}{a_3+b_3}\right] \quad \ldots(6)$$

Also for S_4

$$S_4 = pr[x_1 \geq y_1, x_2 \geq y_2, x_3 < y_3, x_4 \geq y_4]$$

$$= pr[x_1 \geq y_1, x_2 \geq y_2, x_3 < y_3, \mathcal{M}x_2 \geq \mathcal{K}y_2]$$

$$= pr[x_1 \geq y_1]pr[x_2 \geq y_2] .pr[x_3 < y_3, \mathcal{M}x_3 \geq \mathcal{K}y_3]$$

$$S_4 = \left[\frac{b_1}{a_1+b_1}\right]\left[\frac{b_2}{a_2+b_2}\right] \cdot\left[\frac{a_3b_3}{\left(a_3\left(\frac{\mathcal{K}}{\mathcal{M}}\right)+b_3\right)\left(a_3\left(1+\left(\frac{\mathcal{K}}{\mathcal{M}}\right)\right)+b_3\right)}\right] \quad \ldots(7)$$

Now, replacement equations (4),(5), (6) and (7) in equation (3) :

$$\mathcal{R} = \left[\frac{b_1}{a_1+b_1}\right]\left[\frac{b_2}{a_2+b_2}\right]\left[\frac{b_3}{a_3+b_3}\right] + \left[\frac{a_1b_1}{\left(a_1\left(\frac{\mathcal{K}}{\mathcal{M}}\right)+b_1\right)\left(a_1\left(1+\left(\frac{\mathcal{K}}{\mathcal{M}}\right)\right)+b_1\right)}\right]\left[\frac{b_2}{a_2+b_2}\right]\left[\frac{b_3}{a_3+b_3}\right]$$
$$+ \left[\frac{b_1}{a_1+b_1}\right]\left[\frac{a_2b_2}{\left(a_2\left(\frac{\mathcal{K}}{\mathcal{M}}\right)+b_2\right)\left(a_2\left(1+\left(\frac{\mathcal{K}}{\mathcal{M}}\right)\right)+b_2\right)}\right]\left[\frac{b_3}{a_3+b_3}\right]$$
$$+ \left[\frac{b_1}{a_1+b_1}\right]\left[\frac{b_2}{a_2+b_2}\right]\left[\frac{a_3b_3}{\left(a_3\left(\frac{\mathcal{K}}{\mathcal{M}}\right)+b_3\right)\left(a_3\left(1+\left(\frac{\mathcal{K}}{\mathcal{M}}\right)\right)+b_3\right)}\right] \quad \ldots(8)$$

3. Parameters Estimation of Exponential distribution

3.1 Maximum likelihood function (ML):

Suppose $x_1, x_2 \ldots, x_n$ random strength sample for an $Exp(a)$ distribution sample size r. The general form of the maximum probability function L will be as :

$$L(x_1, x_2, \ldots x_r, a) = a^r e^{-a\sum_{i=1}^{r} x_i} \quad \ldots(9)$$

Taking the natural logarithm of (9) :

$$LnL = rLna - a\sum_{i=1}^{r} x_i \quad \ldots(10)$$

By deriving equation (10), then will get :

$$\frac{\partial \ln L}{\partial a} = \frac{r}{a} - \sum_{i=1}^{r} x_i \quad \ldots(11)$$

Equating the equation (11) to zero :

$$\hat{a}_{ML} = \frac{r}{\sum_{i=1}^{r} x_i} \quad \ldots(12)$$

and suppose that $X_{1_{i_1}}; i_1 = 1, 2, \ldots, r_1$ $X_{2_{i_2}}; i_2 = 1, 2, \ldots, r_2$, $X_{3_{i_3}}; i_3 = \cdots, r_3$ and $X_{4_{i_4}}; i_4 = 1, 2, \ldots, r_4$ are strength random samples from $\mathrm{Exp}(a_1)$, $\mathrm{Exp}(a_2)$, $\mathrm{Exp}(a_3)$ and $\mathrm{Exp}(a_4)$, with samples size r_1, r_2, r_3 and r_4 respectively :

$$\hat{a}_{\zeta ML} = \frac{r_\zeta}{\sum_{i_\zeta=1}^{r_\zeta} x_{\zeta_{i_\zeta}}}, \zeta = 1,2,3,4 \quad \ldots(13)$$

The maximum likelihood estimator for unknown parameters b_1, b_2, b_3 and b_4 is as follows, for stress random variables:

$$\hat{b}_{\zeta ML} = \frac{v_\zeta}{\sum_{j_\zeta=1}^{v_\zeta} y_{\zeta_{j_\zeta}}}, \zeta = 1,2,3,4 \quad \ldots(14)$$

Now, replacement equations (13) and (14) in equation (8) :

$$\hat{\mathcal{R}}_{ML} = \left[\frac{\hat{b}_{1ML}}{\hat{a}_{1ML}+\hat{b}_{1ML}}\right]\left[\frac{\hat{b}_{2ML}}{\hat{a}_{2ML}+\hat{b}_{2ML}}\right]\left[\frac{\hat{b}_{3ML}}{\hat{a}_{3ML}+\hat{b}_{3ML}}\right] + \left[\frac{\hat{a}_{1ML}\hat{b}_{1ML}}{\left(\hat{a}_{1ML}\left(\frac{\mathcal{K}}{\mathcal{M}}\right)+\hat{b}_{1ML}\right)\left(\hat{a}_{1ML}\left(1+\left(\frac{\mathcal{K}}{\mathcal{M}}\right)\right)+\hat{b}_{1ML}\right)}\right]$$
$$.\left[\frac{\hat{b}_{2ML}}{\hat{a}_{2ML}+\hat{b}_{2ML}}\right]\left[\frac{\hat{b}_{3ML}}{\hat{a}_{3ML}+\hat{b}_{3ML}}\right] + \left[\frac{\hat{b}_{1ML}}{\hat{a}_{1ML}+\hat{b}_{1ML}}\right] .\left[\frac{\hat{a}_{2ML}\hat{b}_{2ML}}{\left(\hat{a}_{2ML}\left(\frac{\mathcal{K}}{\mathcal{M}}\right)+\hat{b}_{2ML}\right)\left(\hat{a}_{2ML}\left(1+\left(\frac{\mathcal{K}}{\mathcal{M}}\right)\right)+\hat{b}_{2ML}\right)}\right]$$

$$\cdot\left[\frac{\hat{b}_{3ML}}{\hat{a}_{3ML}+\hat{b}_{3ML}}\right]+\left[\frac{\hat{b}_{1ML}}{\hat{a}_{1ML}+\hat{b}_{1ML}}\right]\left[\frac{\hat{b}_{2ML}}{\hat{a}_{2ML}+\hat{b}_{2ML}}\right]$$

$$\cdot\left[\frac{\hat{a}_{3ML}\hat{b}_{3ML}}{\left(\hat{a}_{3ML}\left(\frac{\mathcal{K}}{\mathcal{M}}\right)+\hat{b}_{3ML}\right)\left(\hat{a}_{3ML}\left(1+\left(\frac{\mathcal{K}}{\mathcal{M}}\right)\right)+\hat{b}_{3ML}\right)}\right] \quad \ldots(15)$$

3.2 Percentile Estimation Method (Pr):

This method was discovered by Kao in (1958 - 1959). In this method we will start by function CDF [7]:

$$F(x_i) = 1 - e^{-ax_i}$$

$$ln\left(1 - F(x_i)\right) = -ax_i$$

$$x_i = \left(\frac{-\,ln\left(1-F\left(x_{(i)}\right)\right)}{a}\right) \quad \ldots(16)$$

using P_i ; $i = 1,2,\ldots,r$,we get :

$$x_i = \left(\frac{-\,ln(1-P_i)}{a}\right) \quad \ldots(17)$$

The minimizing following equation :

$$\sum_{i=1}^{r}[x_i - F(x_i)]^2 \quad \ldots(18)$$

Substitution (17) in (18), will get as :

$$\sum_{i=1}^{r}\left[x_{(i)} - \left(\frac{-\,ln(1-P_i)}{a}\right)\right]^2 \quad \ldots(19)$$

By derivative partial of equation (19) with respect to a, and equating the value to zero, we get :

$$\sum_{i=1}^{r} 2\left[x_{(i)} - a^{-1}(-\,ln(1-P_i))\right] .\,(a^{-2})(-\,ln(1-P_i)) = 0$$

The Percentile estimator of a; say $\hat{a}_{(\mathrm{Pr})}$ becomes :

$$\hat{a}_{(Pr)} = \left[\frac{\sum_{i=1}^{r}(-ln(1-P_i))^2}{\sum_{i=1}^{r}(x_{(i)})(-\,ln(1-P_i))}\right] \quad \ldots(20)$$

By using the same way of obtaining equations (13) (14) we get :

$$\hat{a}_{\zeta(Pr)} = \left[\frac{\sum_{i_\zeta=1}^{r_\xi}\left(-ln\left(1-P_{i_\zeta}\right)\right)^2}{\sum_{i_\zeta=1}^{r_\xi}\left(x_{\xi_{(i_\zeta)}}\right)\left(-ln\left(1-P_{i_\zeta}\right)\right)}\right]$$

$; \zeta = 1,2,3,4 \quad \ldots(21)$

and

$$\hat{b}_{\zeta(Pr)} = \left[\frac{\sum_{j_\zeta=1}^{v_\zeta}\left(-ln\left(1-P_{j_\zeta}\right)\right)^2}{\sum_{j_\zeta=1}^{v_\zeta}\left(y_{\xi_{(j_\zeta)}}\right)\left(-ln\left(1-P_{j_\zeta}\right)\right)}\right]$$

$; \zeta = 1,2,3,4 \quad \ldots(22)$

Now, replacement equations (21) and (22) in equation (8) :

$$\hat{\mathcal{R}}_{\mathrm{Pr}} = \left[\frac{\hat{b}_{1\mathrm{Pr}}}{\hat{a}_{1\mathrm{Pr}}+\hat{b}_{1\mathrm{Pr}}}\right]\left[\frac{\hat{b}_{2\mathrm{Pr}}}{\hat{a}_{2\mathrm{Pr}}+\hat{b}_{2\mathrm{Pr}}}\right]\left[\frac{\hat{b}_{3\mathrm{Pr}}}{\hat{a}_{3\mathrm{Pr}}+\hat{b}_{3\mathrm{Pr}}}\right]$$

$$+\left[\frac{\hat{a}_{1\mathrm{Pr}}\hat{b}_{1\mathrm{Pr}}}{\left(\hat{a}_{1\mathrm{Pr}}\left(\frac{\mathcal{K}}{\mathcal{M}}\right)+\hat{b}_{1\mathrm{Pr}}\right)\left(\hat{a}_{1\mathrm{Pr}}\left(1+\left(\frac{\mathcal{K}}{\mathcal{M}}\right)\right)+\hat{b}_{1\mathrm{Pr}}\right)}\right]$$

$$\cdot\left[\frac{\hat{b}_{2\mathrm{Pr}}}{\hat{a}_{2\mathrm{Pr}}+\hat{b}_{2\mathrm{Pr}}}\right]\left[\frac{\hat{b}_{3\mathrm{Pr}}}{\hat{a}_{3\mathrm{Pr}}+\hat{b}_{3\mathrm{Pr}}}\right]+\left[\frac{\hat{b}_{1\mathrm{Pr}}}{\hat{a}_{1\mathrm{Pr}}+\hat{b}_{1\mathrm{Pr}}}\right]$$

$$\cdot\left[\frac{\hat{a}_{2\mathrm{Pr}}\hat{b}_{2\mathrm{Pr}}}{\left(\hat{a}_{2\mathrm{Pr}}\left(\frac{\mathcal{K}}{\mathcal{M}}\right)+\hat{b}_{2\mathrm{Pr}}\right)\left(\hat{a}_{2\mathrm{Pr}}\left(1+\left(\frac{\mathcal{K}}{\mathcal{M}}\right)\right)+\hat{b}_{2\mathrm{Pr}}\right)}\right]$$

$$\cdot\left[\frac{\hat{b}_{3\mathrm{Pr}}}{\hat{a}_{3\mathrm{Pr}}+\hat{b}_{3\mathrm{Pr}}}\right]+\left[\frac{\hat{b}_{1\mathrm{Pr}}}{\hat{a}_{1\mathrm{Pr}}+\hat{b}_{1\mathrm{Pr}}}\right]\left[\frac{\hat{b}_{2\mathrm{Pr}}}{\hat{a}_{2\mathrm{Pr}}+\hat{b}_{2\mathrm{Pr}}}\right]$$

$$\cdot\left[\frac{\hat{a}_{3\mathrm{Pr}}\hat{b}_{3\mathrm{Pr}}}{\left(\hat{a}_{3\mathrm{Pr}}\left(\frac{\mathcal{K}}{\mathcal{M}}\right)+\hat{b}_{3\mathrm{Pr}}\right)\left(\hat{a}_{3\mathrm{Pr}}\left(1+\left(\frac{\mathcal{K}}{\mathcal{M}}\right)\right)+\hat{b}_{3\mathrm{Pr}}\right)}\right] \quad \ldots(23)$$

3.3 Least Square Method (LS):

In this method, used the minimize equation to reduce the non-parametric $(\hat{\mathrm{F}})$ and parametric (F) distribution functions :

$$S(a) = \sum_{i=1}^{r}\left(\hat{F}(x_i) - F(x_i)\right)^2$$

$$= \sum_{i=1}^{r}\left(\hat{F}(x_i) - (1 - e^{-ax_i})\right)^2 \quad ...(24)$$

Now, get a linear form to CDF of Exponential distribution by using the following :

$$F(x_i) = 1 - e^{-ax_i}$$

$$-ln\big(1 - F(x_i)\big) = ax_i \quad ...(25)$$

Since $\hat{F}(x_i)$ is unknown, we will use :

$$\hat{F}\big(x_{(i)}\big) = \frac{i}{r+1}\ ; i = 1,2,\ldots,r$$

The equation S(a) becomes :

$$S(a) = \sum_{i=1}^{r}\big(z_{(i)} - ax_{(i)}\big)^2 \quad ...(26)$$

Where $z_{(i)} = -\ln\left(1 - \hat{F}\big(x_{(i)}\big)\right)$

$$= -\ln(1 - P_i)$$

and P_i is the plotting position.

By partially deriving equation (26) for the parameter a, we get :

$$\frac{\partial S(a)}{\partial a} = \sum_{i=1}^{r} 2\big(z_{(i)} - ax_{(i)}\big)\, x_{(i)} = 0$$

$$\sum_{i=1}^{r} z_{(i)}x_{(i)} - a\sum_{i=1}^{r} x_{(i)}^2 = 0$$

Then we get $\hat{a}_{LS}$:

$$\hat{a}_{LS} = \frac{\sum_{i=1}^{r} z_{(i)}x_{(i)}}{\sum_{i=1}^{r} x_{(i)}^2} \quad ...(27)$$

Will get as

$$\hat{a}_{\zeta LS} = \frac{\sum_{i_\zeta=1}^{r_\zeta} z_{\zeta_{(i_\zeta)}} x_{\zeta_{(i_\zeta)}}}{\sum_{i_\zeta=1}^{r_\zeta} x^2_{\zeta_{(i_\zeta)}}}\ , \zeta = 1,2,3,4 \quad ...(28)$$

and

$$\hat{b}_{\zeta LS} = \frac{\sum_{j_\zeta=1}^{v_\zeta} z_{\zeta_{(j_\zeta)}} y_{\zeta_{(j_\zeta)}}}{\sum_{j_\zeta=1}^{v_\zeta} y^2_{\zeta_{(j_\zeta)}}}\ , \zeta = 1,2,3,4 \quad ...(29)$$

Where $\hat{G}\big(y_{(j)}\big) = \frac{j}{v+1}\ ; j = 1,2,\ldots,v$

and $z_{(j)} = -\ln\left(1 - \hat{G}\big(y_{(j)}\big)\right)$

$$= -\ln\big(1 - P_j\big)$$

Now, replacement equations (28) and (29) in equation (8) :

$$\hat{\mathcal{R}}_{LS} = \left[\frac{\hat{b}_{1LS}}{\hat{a}_{1LS}+\hat{b}_{1LS}}\right]\left[\frac{\hat{b}_{2LS}}{\hat{a}_{2LS}+\hat{b}_{2LS}}\right]\left[\frac{\hat{b}_{3LS}}{\hat{a}_{3LS}+\hat{b}_{3LS}}\right] + \left[\frac{\hat{a}_{1LS}\hat{b}_{1LS}}{\left(\hat{a}_{1LS}\left(\frac{\mathcal{K}}{\mathcal{M}}\right)+\hat{b}_{1LS}\right)\left(\hat{a}_{1LS}\left(1+\left(\frac{\mathcal{K}}{\mathcal{M}}\right)\right)+\hat{b}_{1LS}\right)}\right]$$
$$\cdot\left[\frac{\hat{b}_{2LS}}{\hat{a}_{2LS}+\hat{b}_{2LS}}\right]\left[\frac{\hat{b}_{3LS}}{\hat{a}_{3LS}+\hat{b}_{3LS}}\right] + \left[\frac{\hat{b}_{1LS}}{\hat{a}_{1ML}+\hat{b}_{1ML}}\right] \cdot\left[\frac{\hat{a}_{2LS}\hat{b}_{2LS}}{\left(\hat{a}_{2LS}\left(\frac{\mathcal{K}}{\mathcal{M}}\right)+\hat{b}_{2LS}\right)\left(\hat{a}_{2LS}\left(1+\left(\frac{\mathcal{K}}{\mathcal{M}}\right)\right)+\hat{b}_{2LS}\right)}\right]$$
$$\cdot\left[\frac{\hat{b}_{3LS}}{\hat{a}_{3LS}+\hat{b}_{3LS}}\right] + \left[\frac{\hat{b}_{1LS}}{\hat{a}_{1LS}+\hat{b}_{1LS}}\right]\left[\frac{\hat{b}_{2LS}}{\hat{a}_{2LS}+\hat{b}_{2LS}}\right] \cdot\left[\frac{\hat{a}_{3LS}\hat{b}_{3LS}}{\left(\hat{a}_{3LS}\left(\frac{\mathcal{K}}{\mathcal{M}}\right)+\hat{b}_{3LS}\right)\left(\hat{a}_{3LS}\left(1+\left(\frac{\mathcal{K}}{\mathcal{M}}\right)\right)+\hat{b}_{3LS}\right)}\right] \quad ...(30)$$

4. The experimental study

We simulate the outputs of all three estimating methods by using MSE. Study of simulation is replicated several times (10000) so that the samples of three sizes (small, moderate and large) are independently collected.

4.1 Algorithm of Simulation:

The simulation algorithms are written for estimating $\mathcal{R}$ using MATLAB program, according to the following steps:

1-The random samples $(x_{11}, x_{12}, \ldots, x_{1r_1}), (x_{21}, x_{22}, \ldots, x_{2r_2}), (x_{31}, x_{32}, \ldots, x_{3r_3})$, and $(y_{11}, y_{12}, \ldots, y_{1v_1}), (y_{21}, y_{22}, \ldots, y_{2v_2}), (y_{31}, y_{32}, \ldots, y_{3v_3})$

of sizes $(r_1, r_2, r_3, v_1, v_2, v_3) = (15,15,15$ $15,15), (40,40,40,40,40)$ *and* $(90,90,90,$ $90,90)$ are generated from exponential distribution.

2- Selected the values of parameters for 6 experiments $(a_1, a_2, a_3, b_2, b_2, b_3)$ in the table (1):

Table(1): Values of parameters and Reliability

Experiment	$\mathcal{K}$	$\mathcal{M}$	a_1	a_2	a_3	b_1	b_2	b_3	$\mathcal{R}$
1	1.5	0.5	2	2	2	2	2	2	0.1625
2	1.5	0.5	2	2	2	1.8	1.8	1.8	0.1380
3	1.5	0.5	1	1	1	4	4	4	0.6491
4	1.8	0.4	2	2	2	2	2	2	0.1460
5	1.1	0.95	2	2	2	2	2	2	0.2351
6	1.1	0.99	1	1	1	4	4	4	0.7579

3- Parameters $a_1, a_2, a_3, b_2, b_2, b_3$ were estimated (ML, Pr, and LS) in equations: (13),(14), (21),(22),(28) and (29) respectively.

4- $\mathcal{R}$ was estimated in equations: (15),(23) and (30).

5- Calculate the mean by Mean $= \frac{\sum_{i=1}^{L} \hat{\mathcal{R}}_i}{L}$

6- The last stage is to use the "Mean square Error" to assess the results of the three estimation methods.: $\text{MSE}(\hat{\mathcal{R}}) = \frac{1}{\text{L}} \sum_{i=1}^{\text{L}} (\hat{\mathcal{R}}_i - \mathcal{R})^2$

4.3 Simulation Results

After applying the previous steps of $\mathcal{R}$ for sample size $(r_1, r_2, r_3, v_2, v_2, v_3)$: (15,15, 15,15,15,15) , (40,40,40,40,40,40) and (90,90,90,90,90,90) :

Table(2): Values MSE for 6 experiments

Exp	Simple size	ML	Pr	LS	Best
1	(15,15, 15,15,15,15)	0.0027	0.0029	0.0031	ML
	(40,40,40,40,40,40)	0.0010	0.0011	0.0012	ML
	(90,90,90,90,90,90)	0.0002	0.007	0.009	ML
2	(15,15, 15,15,15,15)	0.0017	0.0019	0.0020	ML
	(40,40,40,40,40,40)	0.0008	0.0009	0.0010	ML
	(90,90,90,90,90,90)	0.0002	0.003	0.005	ML
3	(15,15, 15,15,15,15)	0.0061	0.0068	0.0074	ML
	(40,40,40,40,40,40)	0.0022	0.0025	0.0027	ML
	(90,90,90,90,90,90)	0.0009	0.0011	0.0012	ML
4	(15,15, 15,15,15,15)	0.0022	0.0024	0.0026	ML
	(40,40,40,40,40,40)	0.0008	0.0009	0.0010	ML
	(90,90,90,90,90,90)	0.0001	0.003	0.006	ML
5	(15,15, 15,15,15,15)	0.0044	0.0049	0.0052	ML
	(40,40,40,40,40,40)	0.0017	0.0020	0.0021	ML
	(90,90,90,90,90,90)	0.0004	0.009	0.0012	ML
6	(15,15, 15,15,15,15)	0.0049	0.0055	0.0060	ML
	(40,40,40,40,40,40)	0.0016	0.0019	0.0021	ML

	(90,90,90,90,90,90)	0.0006	0.008	0.0010	ML

5. Conclusions

This conclusion according to the simulation study results:

1. We concluded from the table (1)

I-With increasing value of parameter a, reliability is decreasing.

II-With the increasing value of parameter b, reliability is increased.

III- With the increasing value of $\frac{\mathcal{K}}{\mathcal{M}}$, reliability is decreasing.

2. We concluded from the table (2) the best estimator for $\mathcal{R}$ is ML for 6 experiments and different sample sizes

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