# On Harmonic Univalent Functions Involving q-Poisson Distribution Series

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**Abstract:** Lately, the q- derivative operator has been used to investigate several subclasses of harmonic functions in different ways with different perspectives by many researchers and many interesting results were obtained. The q-derivative operator are also used to construct some subclasses of harmonic functions. In this paper, we define involving of q-Poisson distribution three harmonic functions and we aim to find the conditions for these functions to belong to the subclasses of q-starlike and q-convex harmonic univalent functions.



***Keywords*:** Complex harmonic functions, univalent functions, q-calculus, q-starlike functions, q-convex functions.

## 1. Introduction

Let $H$ denote the class of continuous complex-valued harmonic functions which are harmonic in the open unit disk $U = \{z \epsilon \mathbb{C} : |z| < 1\}$ and let $A$ be the subclass of $H$ consisting of functions which are analytic in $U$. A function harmonic in $U$ may be written as $f = h + \overline{g}$, where $h$ and $g$ are analytic in $U$. We call $h$ the analytic part and $g$ co-analytic part of $f$. A necessary and sufficient condition for $f$ to be locally univalent and sense-preserving in $U$ is that $|h'(z)| > |g'(z)|$ (see [4]). To this end, without loss of generality, we may write

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k \text{ and } g(z) = \sum_{k=1}^{\infty} b_k z^k . (1)$$

Let $SH$ denote the class of functions $f = h + \overline{g}$ which are harmonic, univalent and sense-preserving in $U$ for which $h(0) = h'(0) - 1 = 0 = g(0)$. One shows easily that the sense-preserving property implies that $|b_1| < 1$. The subclass $SH^0$ of $SH$ consist of all functions in $SH$ which have the additional property $b_1 = 0$. Clunie and Sheil-Small [4] investigated the class $SH$ as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on $SH$ and its subclasses.

The theory of quantum calculus known as q-calculus is equivalent to traditional infinitesimal calculus without the notion of limits. Throughout this article, we will use basic notations and definitions of the q-calculus as follows: Let $0 < q < 1$. For any non-negative integer $k$, the q-integer number $k$, denoted by $[k]_q$ is

$$[k]_q = \frac{1 - q^k}{1 - q}. \qquad (k = 1,2,3, \dots), [0]_q = 0.$$

Similarly, the q-differential operator of a function $f$ analytic in $U$ is defined by

$$D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z}, \qquad z \in U.$$

One can easily show that $D_q f(z) \to f(z)$ as $q \to 1^-$. Also, clearly $\lim_{q\to 1^-}[k]_q = k$. For details on q-calculus, one can refer to [7].

In 1990, Ismail et. al. [6] used q-calculus, in the theory of analytic univalent functions by defining a class of complex valued functions that are analytic on the open unit disk $U$ with the normalizations $f(0) = 0, f'(0) = 1$, and $|f(qz)| \leq |f(z)|$ on $U$ for every $q$, $q \in (0,1)$. Motivated by these authors, several researches used the theory of analytic and harmonic univalent functions and q-calculus; for example see [1] and [10]. The q-difference operator of analytic functions $h$ and $g$ given by (1) are by definiton, given as follows [7]

$$D_q h(z) = \begin{cases} \frac{h(z)-h(qz)}{(1-q)z} & ; z \neq 0 \\ h'(0) & ; z = 0 \end{cases} \quad \text{and}$$

$$D_q g(z) = \begin{cases} \frac{g(z)-g(qz)}{(1-q)z} & ; z \neq 0 \\ g'(0) & ; z = 0 \end{cases}. \qquad (2)$$

Thus, for the function $h$ and $g$ of the form (1), we have

$$D_q h(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1}$$

and

$$D_q g(z) = \sum_{k=1}^{\infty} [k]_q b_k z^{k-1}.$$

A harmonic function $f = h + \overline{g}$ defined by (1) is said to be q-harmonic, locally univalent and sense-preserving in $U$ denoted by $SH_q$, if and only if the second dilatation $w_q$ satisfies the condition

$$|w_q(z)| = \left|\frac{D_q g(z)}{D_q h(z)}\right| < 1 \qquad (3)$$

where $0 < q < 1$ and $z \in U$. Note that as $q \to 1^-$, $SH_q$ reduces to the family $SH$ (see [1]).

Denote by $SH_q^*(\alpha)$ the subclass of $SH_q$ consisting of functions $f$ of the form (1) that satisfy the condition

$$\text{Re}\left[\frac{zD_q h(z) - \overline{zD_q g(z)}}{h(z) + \overline{g(z)}}\right] > \alpha \qquad (4)$$

and denote by $KH_q(\alpha)$ the subclass of $SH_q$ consisting of functions $f = h + \overline{g}$ of the form (1) that satisfy the condition

$$\text{Re}\left[\frac{zD_q(zD_q h(z)) - \overline{zD_q(zD_q g(z))}}{zD_q h(z) - \overline{zD_q g(z)}}\right] > \alpha \quad (5)$$

where is $0 < q < 1$, $z \in U$, $0 \leq \alpha < 1$, $D_q h(z)$ and $D_q g(z)$ are defined by (2) (see for details [10]). We will call respectively, q-starlike and q-convex harmonic functions of order $\alpha$.

Define $TSH_q^*(\alpha) = SH_q^*(\alpha) \cap T^1$ and $TKH_q(\alpha) = KH_q(\alpha) \cap T^2$ where $T^m, (m = 1,2)$ consisting of the functions $f = h + \overline{g}$ in $H$ so that $h(z)$ and $g(z)$ are of the form

$$h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k, \quad g(z) = (-1)^{m-1} \sum_{k=1}^{\infty} |b_k| z^k, \quad (m = 1,2). \qquad (6)$$

By suitably specializing the parameters, the classes $SH_q^*(\alpha)$ and $KH_q(\alpha)$ reduce to the various subclasses of harmonic univalent functions. Such as,

*(i)* $SH_q^*(\alpha) = SH^*(\alpha)$ for $q \to 1^-$ ([8]),

*(ii)* $KH_q(\alpha) = KH(\alpha)$ for $q \to 1^-$ ([9]),

*(iii)* $SH_q^*(0) = SH^*$ for $q \to 1^-$ ([16]),
*(iv)* $KH_q(0) = KH$ for $q \to 1^-$ ([17]).

The elementary distributions such as the Poisson, the Logarithmic, the Binomial have been partially studied in the Geometric Function Theory from a theoretical point of view see ([2], [3], [5], [11], [12], [13], [14], [15]).

A variable $x$ is said to q-Poisson Distribution if it takes the values $0,1,2,3, \dots$ with probabilities $e_q^{-r}$, $\frac{r}{[1]_q!}e_q^{-r}$, $\frac{r^2}{[2]_q!}e_q^{-r}$, $\dots$, respectively, where $r$ a parameter and

$$e_q^x = 1 + \frac{x}{[1]_q!} + \frac{x^2}{[2]_q!} + \cdots + \frac{x^k}{[k]_q!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!} \qquad (7)$$

is q-analogue of the exponential function $e^x$ and

$$[k]_q! = [1]_q[2]_q \dots [k]_q$$

is q-analogue of the factorial function $k! = 1.2.3 \dots k$. Thus, for q-Poisson Distribution, we have

$$P_q^r(x = k) = \frac{r^k}{[k]_q!} e_q^{-r}, \ \ k = 1,2,3, \cdots.$$

Now, we introduce a power series whose coefficients are probabilities of the q-Poisson Distribution, that is

$$P_q^r(z) = z + \sum_{k=2}^{\infty} \frac{r^{k-1} e_q^{-r}}{[k-1]_q!} z^k \qquad (z \in U).$$

Note that, by using ratio test we conclude that the radius of convergence of the above power series is infinity. Now, for $r, s > 0$ and $0 < q < 1$, we introduce the analytic functions

$$P_q^r(z) = z + \sum_{k=2}^{\infty} \frac{r^{k-1} e_q^{-r}}{[k-1]_q!} z^k \quad \text{and} \quad P_q^s(z) = z + \sum_{k=2}^{\infty} \frac{s^{k-1} e_q^{-s}}{[k-1]_q!} z^k. \qquad (8)$$

Let us define harmonic functions $P_q^{r,s}$ and $T^m P_q^{r,s}$ as

$$P_q^{r,s}(z) = P_q^r(z) + \overline{P_q^s(z) - z} \ \ and \ \ T^m P_q^{r,s} = 2z - P_q^r(z) + (-1)^{m-1}\overline{P_q^s(z) - z}. \quad (9)$$

It is clear that $P_q^{r,s} \in H^0$ and $T^m P_q^{r,s} \in T^{m,0}$.

In this study, we define two functions $P_q^{r,s}$ and $T^m P_q^{r,s}$ by q-Poission Distribution and we aim to find the conditions for these functions to belong to the classes of q-starlike and q-convex harmonic functions.

## 2. Preliminary Lemmas

To prove our theorems we will use the following lemmas.

**Lemma 1.** [10] Let $f = h + \overline{g}$ be given by (1). If for some $\alpha$ $(0 \le \alpha < 1)$ and the inequality

$$\sum_{k=2}^{\infty} \left([k]_q - \alpha\right)|a_k| + \sum_{k=1}^{\infty} \left([k]_q + \alpha\right)|b_k| \le 1 - \alpha \qquad (10)$$

is hold, then $f$ is harmonic, sense-preserving, univalent in $U$ and $f \in SH_q^*(\alpha)$.

**Remark 2.** [10] Let $f = h + \overline{g}$ be given by (6). Then $f \in TSH_q^*(\alpha)$ if and only if the coefficient condition (10) is satisfied.

**Lemma 3.** [10] Let $f = h + \overline{g}$ be given by (1). If for some $\alpha$ $(0 \le \alpha < 1)$ and the inequality

$$\sum_{k=2}^{\infty}[k]_q([k]_q-\alpha)|a_k| + \sum_{k=1}^{\infty}[k]_q([k]_q+\alpha)|b_k| \leq 1-\alpha \qquad (11)$$

is hold, then $f$ is harmonic, sense-preserving, univalent in $U$ and $f \in KH_q(\alpha)$.

**Remark 4.** [10] Let $f = h + \overline{g}$ be given by (6). Then $f \in TKH_q(\alpha)$ if and only if the coefficient condition (11) is holds.

## 3. Main Results

**Theorem 5.** If $r, s > 0, 0 < q < 1, 0 \leq \alpha < 1$ and the inequality

$$r + s + e_q^{-r(1-q)} + e_q^{-s(1-q)} \leq (1-\alpha)\left(1+e_q^{-r}\right) + (1+\alpha)e_q^{-s} \qquad (12)$$

is satisfied then $P_q^{r,s} \in SH_q^{*,0}(\alpha)$.

**Proof.** Let $r, s > 0, 0 < q < 1, 0 \leq \alpha < 1$. Referring Lemma 1, it is sufficient to show that the inequality

$$\sum_{k=2}^{\infty}\left\{([k]_q-\alpha)\frac{r^{k-1}e_q^{-r}}{[k-1]_q!}\right\} + \sum_{k=2}^{\infty}\left\{([k]_q+\alpha)\frac{s^{k-1}e_q^{-s}}{[k-1]_q!}\right\} \leq 1-\alpha \qquad (13)$$

is satisfied to show that the function $P_{p,q}^{r,s}(z) = P_q^r(z) + \overline{P_q^s(z) - z}$ belongs to the class $SH_q^{*,0}(\alpha)$ where $P_q^r$ and $P_q^s$ are given by (8). Then, using the inequality (7), we obtain

$$\sum_{k=2}^{\infty}\left\{([k]_q-\alpha)\frac{r^{k-1}e_q^{-r}}{[k-1]_q!} + ([k]_q+\alpha)\frac{s^{k-1}e_q^{-s}}{[k-1]_q!}\right\}$$

$$= \sum_{k=2}^{\infty}\left\{([k-1]_q + q^{k-1} - \alpha)\frac{r^{k-1}e_q^{-r}}{[k-1]_q!} + ([k-1]_q + q^{k-1} + \alpha)\frac{s^{k-1}e_q^{-s}}{[k-1]_q!}\right\}$$

$$= \sum_{k=2}^{\infty}\frac{r^{k-1}e_q^{-r}}{[k-2]_q!} + \sum_{k=2}^{\infty}(q^{k-1}-\alpha)\frac{r^{k-1}e_q^{-r}}{[k-1]_q!} + \sum_{k=2}^{\infty}\frac{s^{k-1}e_q^{-s}}{[k-2]_q!} + \sum_{k=2}^{\infty}(q^{k-1}+\alpha)\frac{s^{k-1}e_q^{-s}}{[k-1]_q!}$$

$$= re_q^{-r}\sum_{k=0}^{\infty}\frac{r^k}{[k]_q!} + e_q^{-r}\sum_{k=1}^{\infty}\frac{(qr)^k}{[k]_q!} - \alpha e_q^{-r}\sum_{k=1}^{\infty}\frac{r^k}{[k]_q!} + se_q^{-s}\sum_{k=0}^{\infty}\frac{s^k}{[k]_q!} + e_q^{-s}\sum_{k=1}^{\infty}\frac{(qs)^k}{[k]_q!} + \alpha e_q^{-s}\sum_{k=1}^{\infty}\frac{s^k}{[k]_q!}$$

$$= r + e_q^{-r}\left(e_q^{rq}-1\right) - \alpha e_q^{-r}\left(e_q^{r}-1\right) + s + e_q^{-s}\left(e_q^{sq}-1\right) + \alpha e_q^{-s}\left(e_q^{s}-1\right)$$

Therefore, inequality (13) holds true if

$$r + e_q^{-r}\left(e_q^{rq}-1\right) - \alpha e_q^{-r}\left(e_q^{r}-1\right) + s + e_q^{-s}\left(e_q^{sq}-1\right) + \alpha e_q^{-s}\left(e_q^{s}-1\right) \leq 1-\alpha$$

which is equivalent to (12). Thus, the proof of Theorem 5.

**Corollary 6.** If $r, s > 0, 0 < q < 1, 0 \leq \alpha < 1$, then the function $T^1P_q^{r,s}$ defined by (9) belongs to the class $TSH_q^{*,0}(\alpha)$ if and only if satisfied inequality (12).

**Theorem 7.** If $r,s>0, 0<q<1, 0\le\alpha<1$ and the inequality

$$r^2+s^2+\alpha(s-r)+e_q^{-r(1-q^2)}+e_q^{-s(1-q^2)} + [r(1+2q)-\alpha]e_q^{-r(1-q)} + [s(1+2q)+\alpha]e_q^{-s(1-q)} \le (1-\alpha)\left(1+e_q^{-r}\right) + (1+\alpha)e_q^{-s} \qquad (14)$$

is satisfied then $P_q^{r,s}\in KH_q^0(\alpha)$.

**Proof.** Let $r,s>0, 0<q<1, 0\le\alpha<1$. Referring Lemma 2, it is sufficient to show that the inequality

$$\sum_{k=2}^{\infty}[k]_q\left\{([k]_q-\alpha)\frac{r^{k-1}e_q^{-r}}{[k-1]_q!} + ([k]_q+\alpha)\frac{s^{k-1}e_q^{-s}}{[k-1]_q!}\right\} \le 1-\alpha. \qquad (15)$$

is satisfied to show that the function $P_{p,q}^{r,s}(z)=P_q^r(z)+\overline{P_q^s(z)-z}$ belongs to the class $KH_q^0(\alpha)$ where $P_q^r$ and $P_q^s$ are given by (8). Then, using the inequality (7), we obtain

$$\sum_{k=2}^{\infty}[k]_q\left\{([k]_q-\alpha)\frac{r^{k-1}e_q^{-r}}{[k-1]_q!} + ([k]_q+\alpha)\frac{s^{k-1}e_q^{-s}}{[k-1]_q!}\right\}$$

$$= \sum_{k=2}^{\infty}\{([k-1]_q[k-2]_q + (2q^{k-1}+q^{k-2} - \alpha)[k-1]_q)\}\frac{r^{k-1}e_q^{-r}}{[k-1]_q!} + \sum_{k=2}^{\infty}q^{k-1}(q^{k-1}-\alpha)\frac{r^{k-1}e_q^{-r}}{[k-1]_q!}$$

$$+ \sum_{k=2}^{\infty}\{([k-1]_q[k-2]_q + (2q^{k-1}+q^{k-2} + \alpha)[k-1]_q)\}\frac{s^{k-1}e_q^{-s}}{[k-1]_q!} + \sum_{k=2}^{\infty}q^{k-1}(q^{k-1}-\alpha)\frac{s^{k-1}e_q^{-s}}{[k-1]_q!}$$

$$= r^2e_q^{-r}\sum_{k=0}^{\infty}\frac{r^k}{[k]_q!} + r(1+2q)e_q^{-r}\sum_{k=0}^{\infty}\frac{(qr)^k}{[k]_q!}$$

$$-\alpha re_q^{-r}\sum_{k=0}^{\infty}\frac{r^k}{[k]_q!} + e_q^{-r}\sum_{k=1}^{\infty}\frac{(q^2r)^k}{[k]_q!}$$

$$-\alpha e_q^{-r}\sum_{k=1}^{\infty}\frac{(qr)^k}{[k]_q!} + s^2e_q^{-s}\sum_{k=0}^{\infty}\frac{s^k}{[k]_q!}$$

$$+s(1+2q)e_q^{-s}\sum_{k=0}^{\infty}\frac{(qs)^k}{[k]_q!} + \alpha se_q^{-s}\sum_{k=0}^{\infty}\frac{s^k}{[k]_q!}$$

$$+e_q^{-s}\sum_{k=1}^{\infty}\frac{(q^2s)^k}{[k]_q!} + \alpha e_q^{-s}\sum_{k=1}^{\infty}\frac{(qs)^k}{[k]_q!}$$

$$= r^2+r(1+2q)e_q^{-r(1-q)}-\alpha r+e_q^{-r(1-q^2)} - e_q^{-r}-\alpha e_q^{-r(1-q)}+\alpha e_q^{-r}+s^2 + s(1+2q)e_q^{-s(1-q)}+\alpha s + e_q^{-s(1-q^2)}-e_q^{-s}+\alpha e_q^{-s(1-q)} - \alpha e_q^{-s}.$$

Therefore, inequality (15) holds true if

$$r^2+r(1+2q)e_q^{-r(1-q)}-\alpha r+e_q^{-r(1-q^2)} - e_q^{-r}-\alpha e_q^{-r(1-q)}+\alpha e_q^{-r}+s^2 + s(1+2q)e_q^{-s(1-q)}+\alpha s + e_q^{-s(1-q^2)}-e_q^{-s}+\alpha e_q^{-s(1-q)} - \alpha e_q^{-s} \le 1-\alpha$$

which is equivalent to (14). Thus, the proof of Theorem 7.

**Corollary 8.** If $r, s > 0, 0 < q < 1, 0 \leq \alpha < 1$, then the function $T^2 P_q^{r,s}$ defined by (9) belongs to the class $KH_q^0(\alpha)$ if and only if satisfied inequality (14).

## 4. Conclusion

The novelty of the above results consists in the fact that using some recent results we found sufficient conditions such that the function $P_q^{r,s}$ defined by (9) belongs to the classes $SH_q^{*,0}(\alpha)$ and $KH_q^0(\alpha)$ respectively.

Moreover, for appropriate choices of the parameters we found a few interesting special cases of the above main results.

Finally, new subclass analysis can be done using this method in the future.

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