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Point and Interval Estimation of Stress-Strength Model for Exponentiated Inverse Rayleigh distribution

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Abstract

This paper deals with finding a formula for the stress-strength reliability function $P(T < X < Z)$ for complete data when the strength (X) falls between the stress (T) and the stress (Z) ; where X,T,Z are independent random variables and follow the Exponentiated Inverse Rayleigh Distribution with unknown shape parameters and common known scale parameter , and estimate this formula with the Maximum Likelihood Estimate method (MLE) and the Bayesian method using Non-informative priors and informative priors under Weighted Square Error Loss Function (WSELF) ,Also the interval estimation had been done for the reliability function that based on the Maximum Likelihood Estimator .

Simulation study is used to determine the best estimator; the results showed that Bayesian estimation using informative priors based on Weighted Square Error Loss Function is the best estimator For the equal sizes , and Bayesian estimation using Non-informative priors based on Weighted Square Error Loss Function is the best estimator when the size of the stress sample (Z) larger than the size of (X,T) , and Maximum Likelihood Estimator is the best estimator For the rest sizes.

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Introduction

Exponentiated Inverse Rayleigh distribution (EIR) is a life time distribution used in reliability estimation and statistical quality control techniques. it's a generalization of inverse Rayleigh distribution that developed by Nadarajah and Kotz (1). they suggested a method of generating new exponential type distribution by using reliability function:

$$F(x) = 1 - \{R(x)\}^{\alpha}$$

Where R(x) is the reliability function of Inverse Rayleigh distribution.

The C.D.F of Exponentiated Inverse Rayleigh distribution is : $F(x) = 1 - \left(1 - e^{-\left(\frac{\lambda}{x}\right)^2}\right)^{\alpha}, \ \alpha > 0$

And the P.D.F of EIR distribution is: $f(x) = \frac{2\alpha\lambda^2}{x^3} e^{-\left(\frac{\lambda}{x}\right)^2} \left(1 - e^{-\left(\frac{\lambda}{x}\right)^2}\right)^{\alpha-1}, \quad x > 0, \alpha > 0, \lambda > 0$

where λ indicates the scale parameter and α indicates the shape parameter.

Note that When $\alpha = 1$ the Exponentiated Inverse Rayleigh distribution (EIR) distribution turns into Inverse Rayleigh (IR) distribution.

As for the reliability of the stress-strength (S-S.R.), it has two types (classical and modern) stress-strength, the classical stress-strength explained the life of the component and describe the ability (strength (x)) of the component to still functional

when it subject to random stress (T). and interest to estimate the probability of the component's strength (X) exceed the stress (T); $P(T < X)$.
And the component either fail or the system containing the component might malfunction when $(T \geq X)$.
The second type is P(T<X<Z); which the current study concerned with evaluating and estimating, P(T<X<Z) represent that the strength of the component (X) should not be only greater than the component's stress(T) but also should be smaller than the other component's stress (Z).
For example, blood pressure which has two limits (systolic and diastolic) and the person's blood pressure should be between these limits (2).
In the past 45 years; a case of stress-strength reliability P(T<X<Z) considered when the cumulative functions of T and Z are known and pdf of X is unknown but its observation is available (3). The reliability estimated where X, T and Z are independent and follow a Weibull distribution with different unknown scale parameters and commonly known shape parameter, in presence of k outliers in the strength X, the moment estimator and maximum likelihood (MLE) estimators and mixture estimators of the reliability are derived (4). Then the reliability R = P(X<T<Z) was estimated using Monte-Carlo simulation (MCS) for n-standby system when both of stress and strength follows a particular continuous distribution (5). And the stress-strength reliability estimated using Maximum Likelihood, Method of Moment, Least Square Method, and Weighted Least Square Method when X, T, Z are followed New Weibull-Pareto Distribution with unknown shape parameter (6).

Reliability formula

Deriving The formula of the reliability of stress-strength function P(T<X<Z) under complete data for a component's strength (X) that falls in between the stresses T and Z respectively , will be as follows (7) :

$$R = P(T < X < Z) = \int_0^\infty P(T < X, X < Z) f(x) dx$$

where X, T, Z are all independent

$$= \int_0^\infty H_t(x)\, \overline{G_Z}(x)\, f(x)\, dx \qquad \text{,where } \overline{G_Z}(x) = \left(1 - G_z(x)\right)$$

$$R = \int_0^\infty H_t(x) \left(1 - G_z(x)\right) f(x) dx$$, where $H_t(x)$ and $G_z(x)$ are cumulative distribution functions .

$$R = \int_0^\infty H_t(x) f(x)\, dx - \int_0^\infty H_t(x)\, G_z(x)\, f(x) dx$$

Suppose that : $\int_0^\infty H_t(x) f(x)\, dx = A_1$, $\int_0^\infty H_t(x)\, G_z(x)\, f(x) dx = A_2$

And suppose that T and Z are independently random stresses following $EIR(\alpha_1, \lambda)$, $EIR(\alpha_2, \lambda)$ Respectively,where λ is the scale parameter and α_1, α_2 are the shape parameters ,and (X) is random strength and independent from T and Z and follow EIR (α, λ) then :

$$H_t(x) = 1 - \left(1 - e^{-\left(\frac{\lambda}{x}\right)^2}\right)^{\alpha_1}, \quad x > 0, \lambda;\ \alpha_1 > 0 \quad \text{Is a cum}$$

$$G_z(x) = 1 - \left(1 - e^{-\left(\frac{\lambda}{x}\right)^2}\right)^{\alpha_2}, \quad x > 0, \lambda;\ \alpha_1 > 0.$$

$$A_1 = \int_0^\infty f(x) dx - \int_0^\infty \left(1 - e^{-\left(\frac{\lambda}{x}\right)^2}\right)^{\alpha_1} f(x)\, dx$$

$$= 1 - \int_0^\infty \frac{2\,\alpha\,\lambda^2}{x^3}\, e^{-\left(\frac{\lambda}{x}\right)^2} \left(1 - e^{-\left(\frac{\lambda}{x}\right)^2}\right)^{\alpha - 1} \left(1 - e^{-\left(\frac{\lambda}{x}\right)^2}\right)^{\alpha_1} dx$$

$$A_1 = 1 - \frac{\alpha}{\alpha + \alpha_1}(1) = \frac{\alpha_1}{\alpha_1 + \alpha}$$

$$A_2 = \int_0^\infty H_t(x)\, G_z(x) f(x) dx,$$

$$A_2 = \int_0^\infty \left[1 - \left(1 - e^{-\left(\frac{\lambda}{x}\right)^2}\right)^{\alpha_1}\right] * \left[1 - \left(1 - e^{-\left(\frac{\lambda}{x}\right)^2}\right)^{\alpha_2}\right] \frac{2\alpha\lambda^2}{x^3} e^{-\left(\frac{\lambda}{x}\right)^2} \left(1 - e^{-\left(\frac{\lambda}{x}\right)^2}\right)^{\alpha - 1} dx$$

$$A_2 = 1 - \frac{\alpha}{\alpha + \alpha_1} - \frac{\alpha}{\alpha + \alpha_2} + \frac{\alpha}{\alpha + \alpha_1 + \alpha_2}$$

Substituting the result of A_1 *and* A_2 in R to get the formula of R

$$R = \frac{\alpha\alpha_1}{(\alpha + \alpha_2)(\alpha + \alpha_1 + \alpha_2)} \quad , \quad 0 < R < 1$$

Point Estimation

Point estimation is a process of finding an approximate value of unknown parameters from statistics taken from one or several samples of the population . this section shall discuss two types of point estimation (maximum likelihood estimation , Bayesian estimation).

Maximum likelihood estimation

Let $\{x_i \ , i = 1,2,\dots,n\}$ be a sample of random observations of strength taken from EIR (α,λ) With known scale parameter $\lambda > 0$ and unknown shape parameter $\alpha > 0$, then the likelihood function of the sample $\underline{x}$ is given by:

$$L\left(\underline{x}|\alpha\right) = B\,\alpha^n \prod_{i=1}^{n}\left(1 - e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{\alpha}$$

Where: $B = 2^n\lambda^{2n}\left(\prod_{i=1}^{n} x_i^{-3}\right) e^{-\sum_{i=1}^{n}\left(\frac{\lambda}{x_i}\right)^2} \prod_{i=1}^{n}\left(1 - e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-1}$

And let $\{t_j \ , j = 1,2,\dots,m\}$ and $\{z_k \ , k = 1,2,\dots,w\}$ be a samples of random stresses observation taken from $EIR(\alpha_1,\lambda)$, $EIR(\alpha_2,\lambda)$ Respectively, that their scale parameter $\lambda > 0$ is known and equal ;and shape parameters $\alpha_1, \alpha_2 > 0$ are unknown, t_j and z_k are independent from each other and from x_i, then the likelihood functions of the samples $\underline{t}$, $\underline{z}$ are given by :

$$L\left(\underline{t}|\alpha_1\right) = B_1\,\alpha_1^m \prod_{j=1}^{m}\left(1 - e^{-\left(\frac{\lambda}{t_j}\right)^2}\right)^{\alpha_1}$$

Where: $B_1 = 2^m\lambda^{2m}\left(\prod_{j=1}^{m} t_j^{-3}\right) e^{-\sum_{j=1}^{m}\left(\frac{\lambda}{t_j}\right)^2} \prod_{j=1}^{m}\left(1 - e^{-\left(\frac{\lambda}{t_j}\right)^2}\right)^{-1}$

$$L\left(\underline{z}|\alpha_2\right) = B_2\,\alpha_2^w \prod_{k=1}^{w}\left(1 - e^{-\left(\frac{\lambda}{z_k}\right)^2}\right)^{\alpha_2}$$

Where: $B_2 = 2^w\lambda^{2w}\left(\prod_{k=1}^{w} z_k^{-3}\right) e^{-\sum_{k=1}^{w}\left(\frac{\lambda}{z_k}\right)^2} \prod_{k=1}^{w}\left(1 - e^{-\left(\frac{\lambda}{z_k}\right)^2}\right)^{-1}$

The maximum likelihood estimators of the parameters $(\alpha, \alpha_1, \alpha_2)$:

$$\hat{\alpha}_{mle} = \frac{-n}{\sum_{i=1}^{n}\ln\left(1 - e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)} \ , \ \hat{\alpha}_{1mle} = \frac{-m}{\sum_{j=1}^{m}\ln\left(1 - e^{-\left(\frac{\lambda}{t_j}\right)^2}\right)} \ , \ \hat{\alpha}_{2mle} = \frac{-w}{\sum_{k=1}^{w}\ln\left(1 - e^{-\left(\frac{\lambda}{z_k}\right)^2}\right)} \quad , \ \hat{\alpha}_{mle} \ ; \ \hat{\alpha}_{1mle} \ ; \ \hat{\alpha}_{2mle} > 0$$

The MLE for the (S-S.R.) can be found by applying the invariance property on R for the MLE of $\hat{\alpha}, \hat{\alpha}_1, \hat{\alpha}_2$

$$\hat{R}_{mle} = \frac{\hat{\alpha}_{mle}\,\hat{\alpha}_{1mle}}{(\hat{\alpha}_{mle}+\hat{\alpha}_{2mle})(\hat{\alpha}_{mle}+\hat{\alpha}_{1mle}+\hat{\alpha}_{2mle})} \quad , \quad 0 < \hat{R}_{mle} < 1$$

Bayesian estimation

This part estimates the stress- strength reliability using Bayesian estimation method and under consideration that it performed for complete data by using informative and non-informative priors based on Weighted Squared Error loss function (W.S.E.L.F)

***A. Bayesian estimation using Non-informative Jeffrey's prior based on* Weighted Squared Error loss function**

The non-informative Jeffrey's prior for the shape parameter α is (8):

$P(\alpha) \propto \sqrt{I_X(\alpha)}$, where $I_X(\alpha)$ is the Fisher information for the parameter α

The non-informative prior for $(\alpha, \alpha_1, \alpha_2)$: $P(\alpha) \propto \frac{1}{\alpha}$, $P(\alpha_1) \propto \frac{1}{\alpha_1}$, $P(\alpha_2) \propto \frac{1}{\alpha_2}$

The posterior distribution for α is

$$P_{BWN}(\alpha|\underline{x}) \propto \alpha^{n-1} e^{\alpha \sum_{i=1}^{n} \ln\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)}$$

Which is the kernel of gamma distribution G (n , D) Where D = $-\sum_{i=1}^{n} \ln\left(1- e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)$

Then the complete posterior distribution Of $P_{BWN}(\alpha|\underline{x})$ is:

$$P_{BWN}(\alpha|\underline{x}) = \frac{D^n}{\Gamma n} \alpha^{n-1} e^{-\alpha D}$$

Similarly, the posterior distribution for $\alpha_1 and\ \alpha_2$ are

$$P_{BWN}(\alpha_1|\underline{t}) = \frac{D_1^m}{\Gamma m} {\alpha_1}^{m-1} e^{-\alpha_1 D_1} \quad \text{Where } D_1 = -\sum_{j=1}^{m} \ln\left(1- e^{-\left(\frac{\lambda}{t_j}\right)^2}\right)$$

$$P_{BWN}(\alpha_2|\underline{z}) = \frac{D_2^w}{\Gamma w} \alpha_2^{w-1} e^{-\alpha_2 D_2} \quad \text{Where } D_2 = -\sum_{k=1}^{w} \ln\left(1- e^{-\left(\frac{\lambda}{z_k}\right)^2}\right)$$

Since $\underline{x}, \underline{t}\ and\ \underline{z}\ are\ independent$ The joint posterior can be found as follows:

$$P_{BWN}(\alpha, \alpha_1, \alpha_2|\underline{x}, \underline{t}, \underline{z}) = \frac{D^n}{\Gamma n} \frac{D_1^m}{\Gamma m} \frac{D_2^w}{\Gamma w} \alpha^{n-1} {\alpha_1}^{m-1} \alpha_2^{w-1} e^{-\alpha D} e^{-\alpha_1 D_1} e^{-\alpha_2 D_2}$$

The Weighted Squared Error loss function (9)takes the following form:

$$L(R, \hat{R}) = \frac{(\hat{R}-R)^2}{R}$$

To find the Bayesian estimation $(\hat{R})$ for (S-S.R.) based on Weighted Squared Error loss function we solved the following equation:

$$\frac{\partial E[L(R,\hat{R})]}{\partial \hat{R}} = 0 \quad , \quad \hat{R}_w = \frac{1}{E(R^{-1}|\underline{x}, \underline{t}, \underline{z})}$$

The expectation in the denominator using Non-informative Jeffrey's prior based on Weighted Squared Error loss function is:

$$E_{BNW}(R^{-1}|\underline{x}, \underline{t}, \underline{z}) = \int_0^\infty \int_0^\infty \int_0^\infty R^{-1} P_{BWN}(\alpha, \alpha_1, \alpha_2|\underline{x}, \underline{t}, \underline{z})\, d\alpha\, d\alpha_1 d\alpha_2$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty (1 + 2\alpha_1^{-1}\alpha_2 + \alpha^{-1}\alpha_2 + \alpha^{-1}\alpha_1^{-1}\alpha_2^2 + \alpha\alpha_1^{-1}) \frac{D^n}{\Gamma n} \frac{D_1^m}{\Gamma m} \frac{D_2^w}{\Gamma w} \alpha^{n-1} {\alpha_1}^{m-1} \alpha_2^{w-1} e^{-\alpha D} e^{-\alpha_1 D_1} e^{-\alpha_2 D_2}\, d\alpha\, d\alpha_1 d\alpha_2$$

$$E_{BNW}(R^{-1}|\underline{x}, \underline{t}, \underline{z}) =$$

$$\frac{D^n}{\Gamma n} \frac{D_1^m}{\Gamma m} \frac{D_2^w}{\Gamma w} \Big[\int_0^\infty \int_0^\infty \int_0^\infty \alpha^{n-1}\, {\alpha_1}^{m-1} {\alpha_2}^{w-1}\, e^{-D\alpha}\ e^{-D_1\alpha_1}\, e^{-D\alpha_2}\ d\alpha\, d\alpha_1 d\alpha_2 +$$
$$2\int_0^\infty \int_0^\infty \int_0^\infty \alpha^{n-1}\, {\alpha_1}^{(m-1)-1} {\alpha_2}^{(w+1)-1}\, e^{-D\alpha}\ e^{-D_1\alpha_1}\, e^{-D_2\alpha_2}\, d\alpha\, d\alpha_1 d\alpha_2 +$$
$$\int_0^\infty \int_0^\infty \int_0^\infty \alpha^{(n-1)-1}\, {\alpha_1}^{m-1} {\alpha_2}^{(w+1)-1}\, e^{-D\alpha}\ e^{-D_1\alpha_1}\, e^{-D_2\alpha_2}\, d\alpha\, d\alpha_1 d\alpha_2 +$$
$$\int_0^\infty \int_0^\infty \int_0^\infty \alpha^{(n-1)-1}\, {\alpha_1}^{(m-1)-1} {\alpha_2}^{(w+2)-1}\ e^{-D\alpha}\ e^{-D_1\alpha_1}\, e^{-D_2\alpha_2}\, d\alpha\, d\alpha_1 d\alpha_2 +$$
$$\int_0^\infty \int_0^\infty \int_0^\infty \alpha^{(n+1)-1}\, {\alpha_1}^{(m-1)-1} {\alpha_2}^{w-1}\, e^{-D\alpha}\ e^{-D_1\alpha_1}\, e^{-D_2\alpha_2}\ d\alpha\, d\alpha_1 d\alpha_2 \Big],$$

By solving the integrations which is kernels of gamma distribution

$$E_{BNW}(R^{-1}|\underline{x}, \underline{t}, \underline{z}) = 1 + 2\frac{(\mathrm{w})D_1}{(\mathrm{m}-1)D_2} + \frac{(\mathrm{w})D}{(n-1)D_2} + \frac{(\mathrm{w}+1)(\mathrm{w})DD_1}{(n-1)(\mathrm{m}-1)D_2^2} + \frac{(\mathrm{n})D_1}{(\mathrm{m}-1)D}$$

Substituting equation above in $\hat{R}_w$ to get the Bayesian estimation using non- informative prior based on Weighted Square Error Loss Function:

$$\hat{R}_{BNW} = \left[1 + 2\frac{w\, D_1}{(m-1)D_2} + \frac{w\, D}{(n-1)D_2} + \frac{(w+1)(w)DD_1}{(n-1)(m-1)D_2^2} + \frac{n\, D_1}{(m-1)D}\right]^{-1}$$

B. Bayesian estimation using informative priors based on **Weighted Squared Error loss function**

The prior distribution of the parameters $(\alpha, \alpha_1, \alpha_2)$ is gamma distribution with hyper – parameters $(a, a_1, a_2, b, b_1, b_2)$ with pdf's as follows (10) :

$\Pi(\alpha) = \frac{b^a}{\Gamma a}\alpha^{a-1}e^{-b\alpha}$, $\Pi(\alpha_1) = \frac{b_1^{a_1}}{\Gamma a_1}\alpha_1^{a_1-1}e^{-b_1\alpha_1}$, $\Pi(\alpha_2) = \frac{b_2^{a_2}}{\Gamma a_2}\alpha_2^{a_2-1}e^{-b_2\alpha_2}$

Then the posterior for $(\alpha, \alpha_1, \alpha_2)$ will be as follows:

Since $\underline{x}, \underline{t}$ *and* $\underline{z}$ *are independent* random variables The joint posterior distribution For $(\alpha, \alpha_1, \alpha_2)$ can be found as:

$$P(\alpha, \alpha_1, \alpha_2 | \underline{x}, \underline{t}, \underline{z}) = \frac{Q^{n+a}}{\Gamma(n+a)}\frac{Q_1^{m+a_1}}{\Gamma(m+a_1)}\frac{Q_2^{w+a_2}}{\Gamma(w+a_2)}\alpha^{n+a-1}\,\alpha_1^{m+a_1-1}\alpha_2^{w+a_2-1}\,e^{-Q\alpha}\,e^{-Q_1\alpha_1}\,e^{-Q_2\alpha_2}.$$

the posterior distribution for each parameter is :

$P(\alpha|\underline{x}) = \frac{Q^{n+a}}{\Gamma(n+a)}\,\alpha^{n+a-1}e^{-Q\alpha}$, where $Q = -\sum_{i=1}^{n}\ln\left(1 - e^{-\left(\frac{\lambda}{x_i}\right)^2}\right) + b$

$P(\alpha_1|\underline{t}) = \frac{Q_1^{m+a_1}}{\Gamma(m+a_1)}\,\alpha_1^{m+a_1-1}e^{-Q_1\alpha_1}$, Where $Q_1 = -\sum_{j=1}^{m}\ln\left(1 - e^{-\left(\frac{\lambda}{t_j}\right)^2}\right) + b_1$

$P(\alpha_2|\underline{z}) = \frac{Q_2^{w+a_2}}{\Gamma(w+a_2)}\,\alpha_2^{w+a_2-1}e^{-Q_2\alpha_2}$, where $Q_2 = -\sum_{k=1}^{w}\ln\left(1 - e^{-\left(\frac{\lambda}{z_k}\right)^2}\right) + b_2$

the estimated reliability function based on Weighted Square Error Loss Function when the priors are informative is defined as:

$$\hat{R}_{BN} = \frac{1}{E(R^{-1}|\underline{x}, \underline{t}, \underline{z})}$$

Where

$$E(R^{-1}|\underline{x}, \underline{t}, \underline{z}) = \int_0^\infty\int_0^\infty\int_0^\infty (1 + 2\alpha_1^{-1}\alpha_2 + \alpha^{-1}\alpha_2 + \alpha^{-1}\alpha_1^{-1}\alpha_2^2 + \alpha\alpha_1^{-1})\frac{Q^{n+a}}{\Gamma(n+a)}\frac{Q_1^{m+a_1}}{\Gamma(m+a_1)}\frac{Q_2^{w+a_2}}{\Gamma(w+a_2)}\alpha^{n+a-1}\,\alpha_1^{m+a_1-1}\alpha_2^{w+a_2-1}e^{-Q\alpha}\,e^{-Q_1\alpha_1}\,e^{-Q_2\alpha_2}d\alpha\,d\alpha_1 d\alpha_2$$

$$\begin{aligned} E(R^{-1}|\underline{x}, \underline{t}, \underline{z}) = {} & \frac{Q^{n+a}}{\Gamma(n+a)}\frac{Q_1^{m+a_1}}{\Gamma(m+a_1)}\frac{Q_2^{w+a_2}}{\Gamma(w+a_2)}\Big[\int_0^\infty\int_0^\infty\int_0^\infty \alpha^{n+a-1}\,\alpha_1^{m+a_1-1}\alpha_2^{w+a_2-1}e^{-Q\alpha}\,e^{-Q_1\alpha_1}\,e^{-Q_2\alpha_2}\,d\alpha\,d\alpha_1 d\alpha_2 \\ & + 2\int_0^\infty\int_0^\infty\int_0^\infty \alpha^{n+a-1}\,\alpha_1^{(m+a_1-1)-1}\alpha_2^{(w+a_2+1)-1}\,e^{-Q\alpha}\,e^{-Q_1\alpha_1}\,e^{-Q_2\alpha_2}\,d\alpha\,d\alpha_1 d\alpha_2 \\ & + \int_0^\infty\int_0^\infty\int_0^\infty \alpha^{(n+a-1)-1}\,\alpha_1^{m+a_1-1}\alpha_2^{(w+a_2+1)-1}e^{-Q\alpha}\,e^{-Q_1\alpha_1}\,e^{-Q_2\alpha_2}\,d\alpha\,d\alpha_1 d\alpha_2 \\ & + \int_0^\infty\int_0^\infty\int_0^\infty \alpha^{(n+a-1)-1}\,\alpha_1^{(m+a_1-1)-1}\alpha_2^{(w+a_2+2)-1}\,e^{-Q\alpha}\,e^{-Q_1\alpha_1}\,e^{-Q_2\alpha_2}\,d\alpha\,d\alpha_1 d\alpha_2 \\ & + \int_0^\infty\int_0^\infty\int_0^\infty \alpha^{(n+a+1)-1}\,\alpha_1^{(m+a_1-1)-1}\alpha_2^{w+a_2-1}e^{-Q\alpha}\,e^{-Q_1\alpha_1}\,e^{-Q_2\alpha_2}\,d\alpha\,d\alpha_1 d\alpha_2\Big] \end{aligned}$$

By solving the integrations which is kernels of gamma distribution

$$E(R^{-1}|\underline{x}, \underline{t}, \underline{z}) = 1 + 2\frac{(w+a_2)Q_1}{(m+a_1-1)Q_2} + \frac{(w+a_2)Q}{(n+a-1)Q_2} + \frac{(w+a_2+1)(w+a_2)QQ_1}{(n+a-1)(m+a_1-1)Q_2^2} + \frac{(n+a)Q_1}{(m+a_1-1)Q}.$$

Substituting equation above in $\hat{R}_w$ to get the Bayesian estimation using informative prior based on Weighted Square Error Loss Function:

$$\hat{R}_{BW} = \left[1 + 2\frac{(w+a_2)Q_1}{(m+a_1-1)Q_2} + \frac{(w+a_2)Q}{(n+a-1)Q_2} + \frac{(w+a_2+1)(w+a_2)QQ_1}{(n+a-1)(m+a_1-1)Q_2^2} + \frac{(n+a)Q_1}{(m+a_1-1)Q}\right]^{-1}$$

Interval Estimation

The confidence interval can be defined as a numerical range that is expected to contain the true value of an unknown parameter, As for interval estimation; it is the estimate of the unknown parameter within a certain range (period) of values with a certain probability. This probability is called the confidence level and is symbolized by the symbol (1- estimation error) .

To find the estimated confidence interval (interval estimation) of the stress-strength reliability function (R), the asymptotic variances of the estimated parameters $(\hat{\alpha}, \hat{\alpha}_1, \hat{\alpha}_2)$ must be found first; Then the interval estimation of the reliability is generated based on these variances, in this section interval estimation of the stress-strength reliability function of the model P(T<X<Z) will be found based on estimated reliability by the Maximum Likelihood method and Assuming to be for large samples.

And the formula for the asymptotic variances of reliability function will be found according to the following theorem :

Theorem : Let $T_{1n}, \dots, T_{kn}$ be statistics for the parameters $\theta_1, \dots, \theta_k$ such that as $(n \to \infty)$ then the probability distribution of the difference between the statistics and the parameters is in the following form:

$$\sqrt{n}[T_{1n} - \theta_1, \dots, T_{kn} - \theta_k] \xrightarrow{D} N(\mathbf{0}, \mathbf{\Sigma})$$

Where $\xrightarrow{D}$ means "converges in distribution to" , and $\mathbf{\Sigma} = (\sigma_{ij})_{k*k}$ is a matrix with k*k dimension, which represent variance - covariance matrix for the estimated parameters and that **0** represents a zero vector with dimension k*1.

If $g(T_{1n}, \dots, T_{kn})$ is a function in terms of statistics such that all its first derivatives with respect to parameters $\theta_1, \dots, \theta_k$ exist ; and $g(\theta_1, \dots, \theta_k)$ is a function in terms of the parameters when $(n \to \infty)$ then the Asymptotic distribution of $g(T_{1n}, \dots, T_{kn})$ is :

$$\sqrt{n}[\, g(T_{1n}, \dots, T_{kn}) - g(\theta_1, \dots, \theta_k)] \xrightarrow{D} N\left(0, AsyVar\big(g(T_{1n}, \dots, T_{kn})\big)\right)$$

Where $AsyVar\big(g(T_{1n}, \dots, T_{kn})\big)$ represents the value of the asymptotic variance of function $g(T_{1n}, \dots, T_{kn})$ which can be found by the formula:

$$AsyVar\big(g(T_{1n}, \dots, T_{kn})\big) = \big(d^T(\theta_{ij})\, Asycov(T_{in}, T_{jn})\, d(\theta_{ij})\big) \qquad i, j = 1,2, \dots, k$$

Where $Asycov(T_{in}, T_{jn})$ represent variance - covariance matrix for the statistics T_{ij} , And $d^T(\theta_{ij})$ is a row vector with a dimension of 1*k and it represents the derivative of the function in terms of parameters with respect to its parameters:

$$d^T(\theta_{ij}) = \left[\frac{\partial g(\theta_1, \dots, \theta_k)}{\partial \theta_i}\right] \qquad i = 1,2, \dots k \ .$$

By Applying this theorem to the stress-strength reliability function of the model P(T<X<Z), the asymptotic variance of the stress-strength reliability function $(\hat{R})$ will be:

$$AsyVar(\hat{R}) = \big(d^T(\alpha, \alpha_1, \alpha_2)\, Asycov(\hat{\alpha}, \hat{\alpha}_1, \hat{\alpha}_2)\, d(\alpha, \alpha_1, \alpha_2)\big)$$

Where $Asycov(\hat{\alpha}, \hat{\alpha}_1, \hat{\alpha}_2)$ represent variance-covariance matrix for $(\hat{\alpha}, \hat{\alpha}_1, \hat{\alpha}_2)$, And $d^T(\alpha, \alpha_1, \alpha_2)$ is a row vector with a dimension of 1*k and it represents the derivative of the stress-strength reliability function with respect to $(\alpha, \alpha_1, \alpha_2)$.

To find the interval estimation of the stress-strength reliability function $(\hat{R})$ for the model P (T<X<Z) based on estimated reliability by the Maximum Likelihood method for large samples and it is necessary to find the variance-covariance matrix for the estimated parameters $(\hat{\alpha}, \hat{\alpha}_1, \hat{\alpha}_2)$ that have been estimated by Maximum Likelihood method and it can be found using C. R. lower bound note that Maximum likelihood estimators are unbiased $(E\,\hat{\alpha} = \alpha, E\,\hat{\alpha}_1 = \alpha_1, E\,\hat{\alpha}_2 = \hat{\alpha}_2)$ for large samples (n→∞,m→∞,W→∞) ; As a result $Asycov(\hat{\alpha}, \hat{\alpha}_1, \hat{\alpha}_2)$ will be (11) :

$$Asycov(\hat{\alpha}, \hat{\alpha}_1, \hat{\alpha}_2) = I^{-1}(\alpha, \alpha_1, \alpha_2)$$

$I^{-1}(\alpha, \alpha_1, \alpha_2)$ is the inverse of $(\alpha, \alpha_1, \alpha_2)$ and can be found as follows:

$$I^{-1}(\hat{\alpha}, \hat{\alpha}_1, \hat{\alpha}_2) = \begin{bmatrix} -E\left[\frac{\partial^2 \ln L\,(\underline{x}|\alpha)}{\partial \alpha^2}\right] & -E\left[\frac{\partial^2 \ln L\,(\underline{x}|\alpha)}{\partial \alpha \partial \alpha_1}\right] & -E\left[\frac{\partial^2 \ln L\,(\underline{x}|\alpha)}{\partial \alpha \partial \alpha_2}\right] \\ -E\left[\frac{\partial^2 \ln L\,(\underline{t}|\alpha_1)}{\partial \alpha_1 \partial \alpha}\right] & -E\left[\frac{\partial^2 \ln L\,(\underline{t}|\alpha_1)}{\partial \alpha_1^2}\right] & -E\left[\frac{\partial^2 \ln L\,(\underline{t}|\alpha_1)}{\partial \alpha_1 \partial \alpha_2}\right] \\ -E\left[\frac{\partial^2 \ln L\,(\underline{z}|\alpha_2)}{\partial \alpha_2 \partial \alpha}\right] & -E\left[\frac{\partial^2 \ln L\,(\underline{z}|\alpha_2)}{\partial \alpha_2 \partial \alpha_1}\right] & -E\left[\frac{\partial^2 \ln L\,(\underline{z}|\alpha_2)}{\partial \alpha_2^2}\right] \end{bmatrix}^{-1}$$

$$I^{-1}(\alpha, \alpha_1, \alpha_2) = \begin{bmatrix} \frac{\alpha^2}{n} & 0 & 0 \\ 0 & \frac{\alpha_1^2}{m} & 0 \\ 0 & 0 & \frac{\alpha_2^2}{w} \end{bmatrix}$$

The asymptotic variance ($AsyVar$) of the stress-strength reliability function $(\hat{R})$ of the model P(T<X<Z) based on estimated reliability by the Maximum Likelihood method will be :

$$AsyVar(\hat{R}) = \big(d^T(\alpha, \alpha_1, \alpha_2)\, I^{-1}(\alpha, \alpha_1, \alpha_2)\, d(\alpha, \alpha_1, \alpha_2)\big)$$

$$= \frac{\alpha^2}{n} * \left(\frac{\partial R}{\partial \alpha}\right)^2 + \frac{\alpha_1^2}{m} * \left(\frac{\partial R}{\partial \alpha_1}\right)^2 + \frac{\alpha_2^2}{w} * \left(\frac{\partial R}{\partial \alpha_2}\right)^2$$

Where :

$$\left[\frac{\partial R}{\partial \alpha}\right]^2 = \frac{(-\alpha^2\alpha_1 + \alpha_1^2\alpha_2 + \alpha_1\alpha_2^2)^2}{(\alpha^2 + \alpha\alpha_1 + 2\alpha\alpha_2 + \alpha_1\alpha_2 + \alpha_2^2)^4}$$

$$\left[\frac{\partial R}{\partial \alpha_1}\right]^2 = \frac{(\alpha^3 + 2\alpha^2\alpha_2 + \alpha\alpha_2^2)^2}{(\alpha^2 + \alpha\alpha_1 + 2\alpha\alpha_2 + \alpha_1\alpha_2 + \alpha_2^2)^4}$$

$$\left[\frac{\partial R}{\partial \alpha_2}\right]^2 = \frac{(-2\alpha^2\alpha_1 - \alpha\alpha_1^2 - 2\alpha\alpha_1\alpha_2)^2}{(\alpha^2 + \alpha\alpha_1 + 2\alpha\alpha_2 + \alpha_1\alpha_2 + \alpha_2^2)^4}$$

The interval of the stress-strength reliability function for large samples using the reliability function estimated by the maximum likelihood method take the following form:

$$P[\hat{R}(mle) - Z_{\frac{\alpha}{2}}\sqrt{AsyVar(\hat{R})} < R < \hat{R}(mle) + Z_{\frac{\alpha}{2}}\sqrt{AsyVar(\hat{R})}\,] = 1 - estimation\ error$$

Then The interval estimation of the stress-strength reliability function will be :

$$\hat{R}(mle) \mp Z_{\frac{\alpha}{2}}\sqrt{AsyVar(\hat{R})}$$

By applying $AsyVar(\hat{R})$ in the equation above ; the lower limit and the upper limit will be respectively as follows:

$$\hat{R}_L = (\hat{R}(mle)) - Z_{\frac{\alpha}{2}}\sqrt{\frac{\alpha^2}{n} * \left(\frac{\partial R}{\partial \alpha}\right)^2 + \frac{\alpha_1^2}{m} * \left(\frac{\partial R}{\partial \alpha_1}\right)^2 + \frac{\alpha_2^2}{w} * \left(\frac{\partial R}{\partial \alpha_2}\right)^2}$$

$$\hat{R}_U = (\hat{R}(mle)) + Z_{\frac{\alpha}{2}}\sqrt{\frac{\alpha^2}{n} * \left(\frac{\partial R}{\partial \alpha}\right)^2 + \frac{\alpha_1^2}{m} * \left(\frac{\partial R}{\partial \alpha_1}\right)^2 + \frac{\alpha_2^2}{w} * \left(\frac{\partial R}{\partial \alpha_2}\right)^2}$$

Simulation study

In this section , the simulation study was used to determine best estimator for the stress- strength reliability (S-S.R.) of Exponentiated Inverse Rayleigh distribution from three estimators which are (Maximum likelihood estimator $\hat{R}(mle)$, Bayesian estimator using Non-informative Jeffrey's prior based on Weighted Square Error Loss Function ($\hat{R}_{BNW}$) , Bayesian estimation using informative gamma prior based on Weighted Square Error Loss Function ($\hat{R}_{BW}$), and the mean square error for the estimators had been evaluated with different sample sizes (25,50,100) when ($\lambda = 40$, $\alpha = 20$, $\alpha_1 = 18$, $\alpha_2 = 16$, $R = 0.1851852$)and for gamma priors ($a = 1.7$, $a_1 = 1.5$, $a_2 = 1.2$, $b = 0.99$, $b_1 = 0.81$, $b_2 = 0.7$) for 1000 replicates and the simulation study calculated by (R Studio). And to compute the execution of the (S-S.R.) estimator as in steps:

A. Generate random values for $\underline{x}, \underline{t}$ *and* $\underline{z}$ by the inverse function according to:

$x = \lambda/[-\ln(1-(1-u)^{\frac{1}{\alpha}})]^{1/2}$ where u is generated from the uniform distribution.

B. Calculate the mean of the estimators by $\frac{\sum_{i=1}^{n}\hat{R}_i}{length(\hat{R}_i)}$

C. Evaluate the mean square error (MSE) for the estimators $MSE = \frac{\sum_{i=1}^{n}(\hat{R}_i - R)^2}{length(\hat{R}_i)}$,

And the estimator with smallest Mean square error (MSE) considered the best estimator under that size.

Table 1: Simulation results when $\lambda = 40$, $\alpha = 20$, $\alpha_1 = 18$, $\alpha_2 = 16$, $R = 0.1851852$

(n,m,w)		$\hat{R}(mle)$	$\hat{R}_{BW}$	$\hat{R}_{BNW}$	Interval	
					Lower	APar
(25,25,25)	Mean	0.184584317	0.164417829	0.176681509	0.12042947	0.24873915
	MSE	0.001527285	0.001048327	0.001528945		
50,50,50)	Mean	0.186202242	0.174131039	0.182202162	0.14083792	0.23156656
	MSE	0.000718744	0.000557947	0.000710385		

(100,100,100)	Mean	0.184534952	0.178257293	0.18254518	0.15245753	0.21661237
	MSE	0.000388508	0.000348124	0.000390614		
(25,25,50)	Mean	0.185781972	0.145876463	0.178296983	0.12884985	0.24271409
	MSE	0.001060386	0.001939215	0.001026011		
(25,25,100)	Mean	0.186850981	0.1359154	0.179556714	0.13389838	0.23980357
	MSE	0.000767187	0.002699673	0.000757229		
(25,50,50)	Mean	0.184952622	0.166039524	0.179933962	0.13945289	0.23045235
	MSE	0.00068232	0.00079885	0.00068633		
(25,100,50)	Mean	0.183886009	0.178066435	0.180117901	0.14535414	0.22241787
	MSE	0.000552438	0.000587057	0.000578431		
(25,100,100)	Mean	0.183999333	0.16549432	0.180461586	0.15163552	0.21636314
	MSE	0.000365037	0.000689656	0.000399329		
(50,25,25)	Mean	0.185441981	0.169539427	0.178535151	0.12138310	0.24950086
	MSE	0.001311157	0.00145187	0.001322262		
(50,25,50)	Mean	0.186606875	0.152011108	0.180136913	0.12978291	0.24343083
	MSE	0.001025344	0.001494132	0.001049792		
(50,25,100)	Mean	0.186869292	0.1420354	0.180626489	0.13403299	0.23970558
	MSE	0.000860419	0.002159555	0.000833483		
(50,100,50)	Mean	0.185695982	0.187800412	0.182942888	0.14732410	0.22406786
	MSE	0.000486642	0.000552427	0.000489908		
(50,100,100)	Mean	0.184454042	0.174602358	0.181948287	0.15228087	0.21662720
	MSE	0.00034235	0.000386029	0.00034768		
(100,25,25)	Mean	0.184187804	0.169683182	0.177817778	0.12017696	0.24819864
	MSE	0.001375285	0.001444861	0.00137899		
(100,25,50)	Mean	0.186530954	0.153787513	0.180581522	0.12976114	0.24330076
	MSE	0.001015693	0.001376026	0.001037458		
(100,25,100)	Mean	0.18677032	0.144365563	0.181045696	0.13399227	0.23954836
	MSE	0.000828736	0.001958911	0.000835621		
(100,50,50)	Mean	0.185000338	0.175929686	0.181539631	0.13970387	0.23029681
	MSE	0.000548374	0.000579833	0.000549518		
(100,50,100)	Mean	0.186065711	0.165842967	0.182830893	0.14588510	0.22624632
	MSE	0.000481623	0.000653763	0.000488444		

Discussion

The results of simulation showed that the reliability value is $\boldsymbol{R = 0.1851852}$, The Bayesian estimation using informative priors based on Weighted Square Error Loss Function is the best estimator For the equal sizes , and Bayesian estimation using nun-informative priors based on Weighted Square Error Loss Function is the best estimator when the size (w) of the stress sample (Z) larger than the sizes of (X,T) , and Maximum Likelihood Estimator is the best estimator For the rest cases. The experiment was also applied on anther values and it showed the same results.

Conclusion

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The authors are sincerely grateful to the University of Mosul and In this paper, point and interval estimation was presented to estimate the reliability function P (T< X< Z) when each of X, Z and T follows Exponentiated Inverse Rayleigh Distribution with different shape parameters for complete data, the point estimation included maximum likelihood method and Bayesian estimation using informative Gamma priors and nun-informative priors based on Weighted Square Error Loss Function (WSELF) for interval estimation confidence interval were estimated for the reliability function P (T< X< Z) based on maximum likelihood estimator of the reliability , Simulation results that appeared confirm that the value of the reliability and confidence intervals is between (0,1) which match the statistical theory and the Bayesian estimation using informative priors based on Weighted Square Error Loss Function is the best estimator for the equal sizes , and Bayesian estimation using

nun-informative priors based on Weighted Square Error Loss Function is the best estimator when the size (w) of the stress sample (Z) larger than the sizes of (X,T) , and Maximum Likelihood Estimator is the best estimator For the rest cases.
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Conflict of interest

The authors have no conflict of interest.

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التقدير النقطي والفتروي لنموذج القوة-الإجهاد لتوزيع معكوس رايلي الاسي

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الخلاصة: في هذه الدراسة تم ايجاد صيغة رياضية لدالة موثوقية القوة–الإجهاد للنموذج $P(T < X < Z)$ للبيانات الكاملة في حال كون متغير القوة **(X)** واقع بين اجهادين **(T)** و **(Z)** ، حيث ان **X,T,Z** متغيرات عشوائية مستقلة تتبع توزيع معكوس رايلي الاسي بمعلمات شكل مجهولة ومعلمة قياس معلومة ومشتركة **;** ومن ثم تم تقدير هذه الصيغة باستخدام طرائق التقدير النقطي والفتروي ، في التقدير النقطي تم التقدير باستخدام طريقة الامكان الاعظم واسلوب بيز في حال كون التوزيعات الاولية غنية بالمعلومات وقليلة المعلومات وتحت دالة خسارة الخطأ التربيعية الموزونة اما في التقدير النقطي فقد تم تقدير فترات الثقة لدالة موثوقية القوة–الإجهاد بالاعتماد على مقدر دالة الموثوقية بطريقة الامكان الاعظم .

تم اجراء دراسة محاكاة بطريقة المونتي كارلو لإيجاد قيم المقدرات وتحديد افضلية مقدرات التقدير النقطي بالاعتماد على مجموع متوسط مربعات الخطأ ، وقد اظهرت نتائج المحاكاة افضلية التقدير بإسلوب بيز في حال كون التوزيعات الاولية غنية بالمعلومات و تحت دالة خسارة الخطأ التربيعية الموزونة في حال تساوي احجام العينات في حين تكون الأفضلية للتقدير بإسلوب بيز في حال كون التوزيعات الاولية قليلة المعلومات وتحت دالة خسارة الخطأ التربيعية الموزونة في حال كون حجم المتغير (Z) اكبر من احجام بقية المتغيرات ، اما مقدرات الامكان الاعظم فتكون هي الأفضل في بقية الحالات.

الكلمات المفتاحية: تقدير النقطة والفترة الزمنية الزمنية. نموذج قوة الإجهاد(P(T<X<Z ، دالة خسارة الخطأ المربع المرجح. توزيع رايلي العكسي الأسي.