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Some Results in Modified Probabilistic Inner Product Space

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Abstract:

The purpose of this paper is to give some results such as Schwarz-inequality**,** Parallelogram law, Parallelogram identity and some other properties by using the modified definition of probabilistic inner product space.

Key words: Probabilistic Hilbert space**,** Schwarz-inequality, Parallelogram law, Parallelogram identity.

Introduction:

Menger [1] has introduced in 1942 the concept of probabilistic metric space (PMS) and since then many researchers have studied this concept and give modified definition given in [2]. The basic idea of Menger was to replace the distance between two points by using distribution functions instead of nonnegative real numbers values as the value of metric between two points. Such a probabilistic generalization of fundamental importance in probabilistic functional analysis, nonlinear analysis and applications [4,5,6]. In [5,7] the authors use the same approach to define probabilistic normed spaces and probabilistic inner product spaces. In 1994 S.S.Chang [4] introduced a definition of the probabilistic inner product space. In 2001 Yongfu Su [7] gave modification to the Chang's definition. Finally, in 2007 Yongfu Su [8] presented an explanation the concept of Probabilistic inner product space with mathematical expectation.

This paper contains two sections in section one we give basic definitions we need. While in section two we give the main results.

Preliminaries

This section presents some basic and fundamental concepts related to this paper.

Definition (2-1), [1]:

Let f be real function and let c is an accumulation point of Dom f, then f is *left continuous* at c if and only if $f(c-)=f(c)$, and f is *right continuous* at c if and only if $f(c+)=f(c)$, and f is continuous at c if $f(c+)=f(c-)=f(c)$.

Definition (2-2), [1]:

A distribution function $F:\bar{R}\longrightarrow[0,1]$, that is, non-decreasing and left continuous on R; moreover, $F(-\infty)=0$ and $F(+\infty)=1$, where $\bar{R}=R\cup\{-\infty,+\infty\}$ represent the set of extended real number.

Definition (2-3), [2]:

Let $\mathbb{R}$ be the set of real numbers, define the set D to be the set of all left continuous distributions, such that:

$$D=\{F: F\ is\ left\ continous\ distibution\ function\}$$

and let $H(t)=\begin{cases}1 \ , \ t>0\\ 0 \ , \ t\le 0\end{cases}$
be distribution function which belongs to D.

Definition (2-4), [7]:

Let E be a real linear space and let $F: E\times E\to D$ be a function, then the Probabilistic inner product space is the triple $(E,F,*)$ where F is assumed to satisfy the following conditions:
($F_{x,y}(t)$ will represent the value of $F_{x,y}\ at\ t\in R$)

(MPIP-1)
$$F_{x,x}(0)=0,$$
(MPIP-2)
$$F_{x,y}=F_{y,x},$$
(MPIP-3)
$$F_{x,x}(t)=H(t)\leftrightarrow x=0,$$
(MPIP-4)
$$F_{\lambda x,y}(t)=\begin{cases}F_{x,y}\left(\frac{t}{\lambda}\right) & , \ \lambda>0\\ H(t) & , \ \lambda=0\\ 1-F_{x,y}\left({}^{t}/_{\lambda}+\right) & , \ \lambda<0\end{cases}$$
where λ is real number, $F_{x,y}\left({}^{t}/_{\lambda}+\right)$is the right hand limit of $F_{x,y}$ at $\frac{t}{\lambda}$.

(MPIP-5)
if x,y are linearly independent then
$$F_{x+y,z}(t)=\left(F_{x,z}*F_{y,z}\right)(t)$$
Where
$$\left(F_{x,z}*F_{y,z}\right)(t)=\int_{-\infty}^{\infty}F_{x,z}(t-u)dF_{z,y}(u)$$
Note:
If x,y *are linearly dependent then let* $y=\alpha x\,,(\alpha\in R),then$:
$x+y=x+\alpha x=(1+\alpha)x$, now let $\lambda=1+\alpha$ then
$x+y=\lambda x$
$$F_{x+y,z}(t)=F_{\lambda x,z}(t)$$
$$=\begin{cases}F_{x,z}\left(\frac{t}{\lambda}\right) & , \ \lambda>0\\ H(t) & , \ \lambda=0\\ 1-F_{x,z}\left({}^{t}/_{\lambda}+\right) & , \ \lambda<0\end{cases}$$
which is $(\text{MPIP}-4)$.
Then $(E,F,*)$ is called the modified probabilistic inner product Space.

Definition:(2-5), [8]:

A PIP – space $(E,F,*)$ *is called with mathematical expectation if*:
$$\int_{-\infty}^{\infty}tdF_{x,y}(t)<\infty \qquad ,\forall x,y\in E$$

Theorem(2-6), [8]:

Let $(E,F,*)$ be a modified probabilistic inner product space with mathematical expectation then
$$<x,y>=\int_{-\infty}^{\infty}tdF_{x,y}(t) \quad ,\forall x,y\in E$$
then $(E,<.,.>$) is called inner product space, $(E,\parallel\ \parallel)$, is called the normed space where
$$\parallel x\parallel=\sqrt{<x,x>}$$
Note: $<x,x>\ \ge 0 \quad \forall x\in E.$

Main Results

In this section some of the main results will be presented.

Proposition (3-1):

Let $(E,<.,.>)$ be an ordinary inner product space, let $F: E\times E\to D$ be a function defined by
$$F_{x,y}(t)=H(t-<x,y>)$$
then $(E,F,*)$ is a modified probabilistic inner product space.

Proof:
$(PIP-1)$
$$F_{x,x}(0)=H(0-<x,x>)=0$$
$(PIP-2)$
$$F_{x,y}(t)=H(t-<x,y>)$$
$$=H(t-<x,y>)$$
$$=F_{y,x}(t)$$
$(PIP-3)$
$$F_{x,x}(t)=H(t-<x,x>)$$
$$=H(t)$$
$$\rightarrow<x,x>=0\rightarrow x=0$$
$if\ x=0\ then$
$$F_{0,0}(t)=H(t-<0,0>)=H(t)$$
$(PIP-4) \qquad >0, then$
$$F_{\lambda x,y}(t)=H(t-<\lambda x,y>)$$
$$=H(t-\lambda<x,y>$$
$$=H(\lambda(\frac{t}{\lambda}-<x,y>)$$

$= H\left(\frac{t}{\lambda} - < x, y >\right) = F_{x,y}(\frac{t}{\lambda})$

1. $if\ \lambda = 0, then$

$F_{\lambda x,y}(t) = H(t - < \lambda x, y >)$

$= H(t - < 0, y >) = H(t)$

2. $if\ \lambda < 0, then$

$F_{\lambda x,y}(t) = H(t - < \lambda x, y >)$

$= H(t - (-\lambda) < x, y >)$

$= (-\lambda(-\frac{t}{\lambda} + < x, y >)$

$= \left(-\frac{t}{\lambda} + < x, y >\right)$

$= 1 - H\left(\frac{t}{\lambda} - < x, y >\right)$

$=1 - F_{x,y}(\frac{t}{\lambda} +)$

$(PIP - 5)$

1. If

$x, y\ are\ linearly\ dependent\ then\ let\ y = \alpha x\ , \alpha\ is\ scalar\ (\in R)\ then$

$x + y = x + \alpha x$

$= (1 + \alpha)x\ \ , now\ let$

$\lambda = \alpha x\ then\ x + y = hx$

$F_{x+y,z}(t) = F_{\lambda x,z}(t)$

$$= \begin{cases} F_{x,z}\left(\frac{t}{\lambda}\right) & , \quad \lambda > 0 \\ H(t) & , \quad \lambda = 0 \\ 1 - F_{x,z}\left(t/_{\lambda} +\right) & , \quad \lambda < 0 \end{cases}$$

which has been shown in $(PI - 4)$

2. If

$x, y\ are\ linearly\ independent\ then$

$F_{x+y,z}(t) = H(t - < x + y, z >)$

$$\left(F_{x,z} * F_{y,z}\right)(t) = \int_{-\infty}^{\infty} F_{x,z}(t - u) dF_{z,y}(u)$$

$$= \int_{-\infty}^{\infty} H(t - u) - < x, z >) dH(u - < z, y >)$$

$$= \int_{-\infty}^{\infty} H(t - u) - < x + y, z >) dH(u)$$

$$= F_{x+y,z}(t)$$

■

Theorem (3-2):

Let

$(E, F, *)\ be\ PIP - space\ with\ Mathematical\ expectation\ then$:

1. $< x, y + z > = \int_{-\infty}^{\infty} t dF_{x,y+z}(t)$
2. $< x, \lambda z > = \lambda < x, z >$
3. $< 0, y > = < x, 0 > = 0$
4. $< x - y, z > = < x, z > - < y, z >$

Proof:

1.

$by (MPIP - 2)$

$$= \int_{-\infty}^{\infty} t dF_{y+z,x}(t)$$

$$= \int_{-\infty}^{\infty} t dF_{x,y}(t) + \int_{-\infty}^{\infty} t dF_{x,z}(t)$$

$$= < x, y > + < x, z >$$

2.

$$< x, \lambda z > = \int_{-\infty}^{\infty} t dF_{x,\lambda z}(t)$$

$$= \int_{-\infty}^{\infty} t dF_{\lambda z,x}(t) = \lambda < z, x > \quad by\ (PIP - 4)$$

3.

$$< 0, y > = \int_{-\infty}^{\infty} t dF_{0,y}(t)$$

$$= \int_{-\infty}^{\infty} t dH(t) = 0$$

Similarly for $< x, 0 >$

4.

$$< x - y, z > = \int_{-\infty}^{\infty} t dF_{x-y,z}(t)$$

$$= \int_{-\infty}^{\infty} t dF_{x,z}(t) - \int_{-\infty}^{\infty} t dF_{y,}(t)$$

$$= < x, z > - < y, z >$$

5.

$< x, z > = < y, z >$

$$= \int_{-\infty}^{\infty} t dF_{x,z}(t)$$

$$= \int_{-\infty}^{\infty} t dF_{y,z}(t)\ \ \forall z \in E$$

$$0 = \int_{-\infty}^{\infty} tdF_{x,z}(t) - \int_{-\infty}^{\infty} tdF_{y,z}(t) = \int_{-\infty}^{\infty} tdF_{x-y,z}(t) = 0$$

$F_{x-y,z}(t) = 0 \; iff \; x - y = 0 \; \to x = y \; \forall z \in E \;\; by \; (PI - 3)$ ■

Theorem (3-3): (**Schwarz- Inequality**)

For all $x, y \in E$,where $(E, F, *)$ is PIP-space with Mathematical Expectation then:

$(< x, y >)^2 \leq \| x \|^2 \| y \|^2 \;\; \forall x, y \; \in E$

Proof:

$Let \; x \neq 0 \,, y \neq 0$ [$if \; x = 0$ or $y = 0$ then $< 0, y > = < x, 0 > = 0$]

For any λ , λ is real number $(\lambda \in R)$, $\lambda > 0$

1. If x, y are linearly dependent then

$$< x - \lambda y, x - \lambda y > = \int_{-\infty}^{\infty} tdF_{x-\lambda y, x-\lambda y}(t) = \int_{-\infty}^{\infty} tdH(t) \; = 0$$

2. If x, y are linearly independent then $x - \lambda y \neq \hat{0}$ then

$< x - \lambda y, x - \lambda y > \geq 0$

$$< x - \lambda y, x - \lambda y >= \int_{-\infty}^{\infty} tdF_{x,x}(t) - \int_{-\infty}^{\infty} tdF_{\lambda x,y}(t) - \int_{-\infty}^{\infty} tdF_{x,\lambda y}(t) + \int_{-\infty}^{\infty} tdF_{\lambda y,\lambda y}(t) \geq 0$$

$$\to \int_{-\infty}^{\infty} tdF_{x,x}(t) - \lambda \int_{-\infty}^{\infty} tdF_{x,y}(t) - \lambda \int_{-\infty}^{\infty} tdF_{x,y}(t) + \lambda^2 \int_{-\infty}^{\infty} tdF_{y,y}(t) \geq 0$$

$$\to \int_{-\infty}^{\infty} tdF_{x,x}(t) - 2\lambda \int_{-\infty}^{\infty} tdF_{x,y}(t) - \lambda^2 \int_{-\infty}^{\infty} tdF_{y,y}(t) \geq 0$$

Let $\lambda = \frac{<x,y>}{<y,y>} = \frac{\int_{-\infty}^{\infty} tdF_{x,y}(t)}{\int_{-\infty}^{\infty} tdF_{y,y}(t)}$

$$\int_{-\infty}^{\infty} tdF_{x,x}(t) - 2\frac{\int_{-\infty}^{\infty} tdF_{x,y}(t)}{\int_{-\infty}^{\infty} tdF_{y,y}(t)} \int_{-\infty}^{\infty} tdF_{x,y}(t) - \left(\frac{\int_{-\infty}^{\infty} tdF_{x,y}(t)}{\int_{-\infty}^{\infty} tdF_{y,y}(t)}\right)^2 \int_{-\infty}^{\infty} tdF_{y,y}(t) \geq 0$$

$$\int_{-\infty}^{\infty} tdF_{x,x}(t) - \frac{(\int_{-\infty}^{\infty} tdF_{x,y}(t))^2}{\int_{-\infty}^{\infty} tdF_{y,y}(t)} \geq 0$$

$$\int_{-\infty}^{\infty} tdF_{x,x}(t) \int_{-\infty}^{\infty} tdF_{y,y}(t) \geq \left(\int_{-\infty}^{\infty} tdF_{x,y}(t)\right)^2$$

$< x, x >< y, y > \geq (< x, y >)^2$

$(< x, y >)^2 \leq \| x \|^2 \| y \|^2 \;\; where \; \| x \| = \sqrt{< x, x >}$

■.

Theorem (3-4): (**Parallelogram Law**)

For all $x, y \in E$,where $(E, F, *)$ is PIP-space with Mathematical Expectation then:

$$\| x+y \|^2 + \| x-y \|^2 = 2\| x \|^2 + 2\| y \|^2$$

Proof:

$$\| x+y \|^2 + \| x-y \|^2 = \;< x+y, x+y > + < x-y, x-y >$$

$$\int_{-\infty}^{\infty} t dF_{x+y,x+y}(t) + \int_{-\infty}^{\infty} t dF_{x-y,x-y}(t) = \int_{-\infty}^{\infty} t dF_{x,x}(t) + \int_{-\infty}^{\infty} t dF_{x,y}(t) + \int_{-\infty}^{\infty} t dF_{y,y}(t)$$

$$+ \int_{-\infty}^{\infty} t dF_{y,x}(t) + \int_{-\infty}^{\infty} t dF_{y,y}(t) + \int_{-\infty}^{\infty} t dF_{x,x}(t) - \int_{-\infty}^{\infty} t dF_{x,y}(t) - \int_{-\infty}^{\infty} t dF_{y,x}(t) + \int_{-\infty}^{\infty} t dF_{y,y}(t)$$

$$= 2\int_{-\infty}^{\infty} t dF_{x,x}(t) + 2\int_{-\infty}^{\infty} t dF_{y,y}(t) = 2 < x,x > + 2 < y,y > = 2\| x \|^2 + 2\| y \|^2$$

Where $\| x \| = \sqrt{< x,x >}$ ■.

Note:

The purpose of this theorem is to give a necessarily condition that a probabilistic normed space (PN-space) is (PIP-space) if it is satisfied the parallelogram Law.

Theorem (3-5): (**Parallelogram Identity**)

For all $x, y \in E$,where $(E, F, *)$ is PIP-space with Mathematical Expectation then:

$$< x, y > = \frac{1}{4}\{\| x+y \|^2 - \| x-y \|^2\}$$

Proof:

$$\| x+y \|^2 = < x+y, x+y > = \int_{-\infty}^{\infty} t dF_{x+y,x+y}(t) = \int_{-\infty}^{\infty} t dF_{x,x}(t) + \int_{-\infty}^{\infty} t dF_{x,y}(t) + \int_{-\infty}^{\infty} t dF_{y,x}(t) + \int_{-\infty}^{\infty} t dF_{y,y}(t)$$

Similar

$$\| x-y \|^2 = < x-y, x-y > = \int_{-\infty}^{\infty} t dF_{x-y,x-y}(t) = \int_{-\infty}^{\infty} t dF_{x,x}(t) - \int_{-\infty}^{\infty} t dF_{x,y}(t) - \int_{-\infty}^{\infty} t dF_{y,x}(t) + \int_{-\infty}^{\infty} t dF_{y,y}(t)$$

$$\| x+y \|^2 - \| x-y \|^2 = 2\int_{-\infty}^{\infty} t dF_{x,y}(t) + 2\int_{-\infty}^{\infty} t dF_{y,x}(t)$$

By (PIP-2) we get

$$< x, y > = \frac{1}{4}\left\{4\int_{-\infty}^{\infty} t dF_{x,y}(t)\right\} = \int_{-\infty}^{\infty} t dF_{x,y}(t) = < x, y >$$

■.

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بعض النتائج في فضاء الضرب الداخلي الاحتمالي المحسن

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الخلاصة:
ان الهدف من هذا البحث هو الحصول على بعض النتائج مثل: متباينة شوارتز وقانون التوازي ومتطابقة التوازي ونتائج اخرى من خلال استخدام تعريف فضاء الضرب الداخلي الاحتمالي المحسن.

الكلمات المفتاحية: فضاء هلبرت الاحتمالي، متراجحة اشوارتز، قانون متوازي الاضلاع ومتطابقة الاضلاع.