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Solving Two-Points Singular Boundary Value Problem Using Hermite Interpolation

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Abstract:

In this paper, we have been used the Hermite interpolation method to solve second order regular boundary value problems for singular ordinary differential equations. The suggest method applied after divided the domain into many subdomains then used Hermite interpolation on each subdomain, the solution of the equation is equal to summation of the solution in each subdomain. Finally, we gave many examples to illustrate the suggested method and its efficiency.

Keywords: Singular ordinary differential equations, Boundary value problems, Hermite interpolation.

Introduction:

Singular boundary value problems (SBVP's) for ordinary differential equations (ODE) arise very frequently in several areas of science and engineering. For example, in analysis of heat conduction through a solid with heat generation, Thomas–Fermi model in atomic physics, electro hydrodynamics and the theory of thermal explosions. These arise in Physiology as well in the study of various tumor growth problems, in the study of the distribution of heat sources in the human head[1-3]. Singular boundary value problems are always very important, there exists many method for solving. For example, modified Homotopy perturbation method [4], differential transform method [5], cubic trigonometric B-spline method[6], Adomian decomposition method [7], shooting method [8], variation method[9].

Hermite interpolation method which was mooted by charts Hermite is often used in interpolation of the data points when the derivative of the function f(x) in the given points are available this technique has superiority on the other types of interpolation polynomial [10].

In this paper we will use Hermite interpolation method for solving singular boundary value problems of ODE after divided the interval [0,1] into many subdomains equal distance. Numerical examples show that present method is efficiency.

1. Hermite Interpolation [11]

Weierstrass approximation theorem guarantees that one can always find a polynomial that is arbitrarily close to a given function on some finite interval. This means that the approximation error is bounded and can be reduced by the choice of the adequate polynomial. Unfortunately this theorem is not a constructive one, i.e., it does not present a way how to obtain such a polynomial, i.e., the interpolation problem can also be formulated in another way, viz. as the answer to the following question : How to find a good representative of a function that is not knew explicitly, but only at some points of the domain of interest. In this paper we will consider Hermite interpolation where the interpolation polynomial also matches the first derivatives $f^{(1)}(x)\ at\ x = x_k$. This interpolation technique is important since it has the property that gives high order of accuracy.

Theorem 1:[11] Suppose that $f(x) \in C^1[a,b]$, and that $x_{o,} x_1, \dots, x_n \in [a,b]$ are distinct, then the unique polynomial of degree (at most) 2n + 1 denoted by H_{2n+1} , and such that : $H_{2n+1}(x_j) = f(x_j), H'_{2n+1}(x_j) = f'(x_j) \quad j \in Z_{n+1}$ is given by :

$$H_{2n+1}(x) = \sum_{k=0}^{n}\left[1 - 2L'_k(x_k)(x - x_k)\right][L_k(x)]^2 f(x_k) + \sum_{k=0}^{n}(x - x_k)\,[L_k(x)]^2 f'(x_k) \quad \dots(1)$$

$$L_k(x) = \prod_{\substack{i=0\\ i\neq j}}^{n} \frac{x - x_i}{x_k - x_i}$$

The error bound for Hermite interpolation is provided by the expression:

E=$(x - x_o)^2 (x - x_1)^2 \dots (x - x_n)^2 f^{(2n+1)}(x)/(2n+1)!$,where $f(x) \in C^{2n+2}[a,b]$.

2. Singular Boundary Value Problem

The general form of the 2^{nd}order two point boundary value problems (TPBVP) is:

$$y'' + p(x)y' + Q(x)y = 0\ , a \le x \le b \ \dots (2)$$

With the boundary conditions (BC):y(a) = A and y(b) = B, where A, B ϵ R

There are two types of a point $x_o \in [0,1]$.Ordinary point and Singular point.A function y(x) is analytic at $x_o$if it has a power series expansion at $x_o$that converges to y(x) on an open interval containing x_o. A point $x_o$is an ordinary point of the ODE (2), if the functions P(x)and Q(x) are analytic at x_o. Otherwise $x_o$is a singular point of the ODE. On the other hand if P(x) or Q(x) are not analytic at $x_o$then $x_o$is said to be a singular point [12-13].There is at present, numerical method for solving problems with regular singular points using Hermite interpolation method with interval [0 1].

3. Description of the Method

In this section ,we apply the Hermite interpolation method H_{2n+1} and Taylor series to solve regular differential equations. A general form of the 2^{nd} order SBVP's is:

$$x^m y'' = f(x, y, y'), \qquad 0 \le x \le 1 \ \dots(3)$$

Subject to the boundary condition (BC):

In the case Dirichlet BC: y(0)= A, y(1) = B, where A, B ϵ R

In the case Neumann BC: y′(0)= A, y′(1) = B, where A, B ϵ R

In the case Cauchy or mixed BC: y(0)= A, y′(1) = B, where A, B ϵ R

Or y′(0)= A, y(1) = B, where A, B ϵ R

where f is a general nonlinear function . Now, to solve the problem by the suggested method we will doing the following steps:

Step one: Evaluate Taylor series of y(x) about x = 0:

$$y(x) = \sum_{i=0}^{\infty} a_i x^i = a_o + a_1 x + \sum_{i=2}^{\infty} a_i x^i \ \dots (4)$$

Where y(0)= a_o,y'(0)= a_1 ,$\frac{y''(0)}{2!} = a_2, \dots, \frac{y^{(i)}(0)}{i!} = a_i$,i=3,4,…

Evaluate Taylor series of y(x) about x = 1/3:

$$y(x) = \sum_{i=0}^{\infty} b_i (x - 1/3)^i = b_o + b_1(x - 1/3) + \sum_{i=2}^{\infty} b_i (x - 1/3)^i \dots (5)$$

Where y(1/3)= b_o,y'(1/3)= b_1 ,$\frac{y''(1/3)}{2!} = b_2, \dots, \frac{y^{(i)}(1/3)}{i!} = b_i$,i=3,4,…

Evaluate Taylor series of y(x) about x = 2/3:

$y(x) = \sum_{i=0}^{\infty} c_i(x-2/3)^i = c_o + c_1(x-2/3) + \sum_{i=2}^{\infty} c_i(x-2/3)^i \ldots (6)$

Where y(2/3)= c_o,y'(2/3)= c_1 ,$\frac{y''(2/3)}{2!} = c_2$,…,$\frac{y^{(i)}(2/3)}{i!} = c_i$,i=3,4,…

And evaluate Taylor series of y(x) about x = 1:

$y(x) = \sum_{i=0}^{\infty} d_i(x-1)^i = d_o + d_1(x-1) + \sum_{i=2}^{\infty} d_i(x-1)^i \ldots (7)$

Where y(1)= d_o,y'(1)= d_1 ,$\frac{y''(1)}{2!} = d_2$,…,$\frac{y^{(i)}(1)}{i!} = d_i$,i=3,4,…

Step two: Insert the series form (4) with derivatives into equation (3) and put x = 0, then equate the coefficients of powers of x to obtain a_2 .Then derive equation(3) with respect to x,to get new form of equation say(8) as following:

$x^m y'''(x) + m y''(x) x^{m-1} = \frac{df(x,y,y')}{dx} \ldots (8)$

Then insert the series form (4) with derivatives into equation (8) and put x=0 equate the coefficients of power of x to obtain a_3 .Iterate this process many times to obtain a_4 then a_5 and so on.

Step three: Make up x=1/3 into equation (4) to obtain y(1/3)= b_o ,to find b_1 derive the equation(4)and requite x=1/3,and insert the series (5) into equation (3) and put x=1/3,then equate the coefficients of power of (x-1/3) to obtain b_2 .to find b_3 insert the series (5) into equation (8) and put x=1/3 and equate the coefficient of power of (x-1/3). Iterate this process many times to obtain b_4 then b_5 and so on.

Step four: Make up x=2/3 into equation (5) to obtain y(2/3)= c_o ,to find c_1 derive the equation(5)and requite x=2/3,and insert the series (6) into equation (3) and put x=2/3,then equate the coefficients of power of (x-2/3) to obtain c_2 .to find c_3 insert the series (6) into equation (8) and put x=2/3 and equate the coefficient of power of (x-2/3). Iterate this process many times to obtain c_4 then c_5 and so on.

Step five: Insert the series form (7) with derivatives into equation (3) and put x = 1, then equate the coefficients of powers of(x-1) to obtain d_2 ,to find d_3 insert the series(7) with derivatives into equation (8) and put x=1 and equate the coefficient of power of (x-1) . Iterate this process many times to obtain d_4 then d_5 and so on.

Step six: The notation implies that the coefficients depend only on the indicated unknowns $a_o, a_1, d_o and\ d_1$ where $c_i, b_i\ \forall i \geq 1$ dependes on the indicated unknowns a_o, a_1

When the substitute (BC) we get two unknown coefficients and then substitute for coefficients (a_i, b_i, c_i, d_i) that we have obtained the previous steps in Hermite interpolation polynomial H_{2n+1} equation(1).

Step seven: To find the unknown coefficients by reduction order equation and use H_{2n+1} as a replacement of y(x) and substitute the boundary conditions , we have only two unknown coefficients from $a_o, a_1, d_o and\ d_1$ and two equation, we can find this for any n by solving this system of algebraic equations. So insert the value of the unknown coefficients into equation (1),Thus equation (1) represent the solution of the problem.

4. Error Estimation for SBVP's

Every known of BVPs software package reports an estimate of either the relative error or the maximum relative defect. The weights used to scale either the error or the maximum defect differ among BVPs software. Therefore, the BVPs allows users to select the weights they wish to use. The default weights depend on whether an estimate of the error or maximum defect is being used. If the error is being uses estimated, in this paper we modify this package to consist SBVPs, defined as :

E=|| y(x)- H_{2n+1} (x)$||_\infty$ / (1 + || H_{2n+1} (x) $||_\infty$) ; $0 \le x \le 1$

where y(x) is exact solution and H_{2n+1} (x) is suggested solution of SBVPs .

If the exact solution does not find then the component error of SBVPs is

E=|| H_{2n+1}''(x) − f(x, H_{2n+1} (x), H_{2n+1}'(x)) $||_\infty$ / (1 + ||f(x, H_{2n+1} (x), H_{2n+1}'(x)) $||_\infty$)

The relative estimate of both the error and the maximum defect are slightly modified from the one used in SBVP SOLVER[14] .

5. Numerical Examples:

In this section ,we used Hermite interpolation and Taylar series to solve singular boundary value problems (SPVBs).After the domain [0 ,1] is divided into two points with points 0,1 where you found polynomial solution $H_7(x)$ and presented the results the Variational iteration method and the exact solution, and the domain [0 ,1] is divided into four points with points 0,1 where you found polynomial solution $H_{11}(x)$,also the domain[0 ,1] is divided into night points with points 0,1 where you found polynomial solution $H_{21}(x)$.Then examples calculated maximum error in each case n=4,6,11 with figure polynomials to find a good solution.

Example1. Consider the following SBVP :

$$y'' + \frac{2}{x}y' + (x^2-1)y = -x^4 + 2x^2 - 7 \ , \quad 0 \le x \le 1$$

with mixed BC:y'(0) = 0, y(1) = 1. The exact solution is : y(x) =1-x^2 [15]

Using Taylor polynomials ,we have

y(0)=1 ,y(1/3)= 0.8888888888888889, y(2/3)= , 0.5555555555555555

y'(1/3)= -0.6666666666666667, y'(2/3)= -1.333333333333333, y'(1)= -2

Now, we solve this equation using these data

$H_7(x)$= 9.148237723 $10^{-14}x^7$- 3.268496584 $10^{-13}x^6$ + 4.227729278 $10^{-13}x^5$- 2.664535259 10^{-13} x^4+ 7.860379014 $10^{-14}x^3$- 1.0x^2+ 1.0

Geng [15] solved this example by (VIM) and give the following series solution:

p (x) = 0.999997 - x^2+ 1.2347 $10^{-7}x^4$+ 1.46988 $10^{-8}x^6$+ 0.0000165329x^8- 0.0000201421x^{10}+7.95888 $10^{-6}x^{12}$- 1.03072 $10^{-6}x^{14}$

The numerical results are given in following Table 1 gives $H_7(x)$ and result VIM also exact solution. Table 2 gives the maximum error for number point n = 4 , 6, 11. Figure 1 gave the accuracy of the suggested method.

Table 1: Numerical results for n=4 of example 1

x_i	Exact Solution y(x)	Hermite interpolation H_7(x)	VIM p(x)
0	1	1	0.999997000000000
0.1	0.990000000000000	0.990000000000000	0.989997000012525
0.2	0.960000000000000	0.960000000000000	0.959997000238787
0.3	0.910000000000000	0.910000000000001	0.909997001980789
0.4	0.840000000000000	0.840000000000001	0.839997012074748
0.5	0.750000000000000	0.750000000000002	0.749997054738341
0.6	0.640000000000000	0.640000000000003	0.639997189102191
0.7	0.510000000000000	0.510000000000004	0.509997518669322
0.8	0.360000000000000	0.360000000000005	0.359998167043863
0.9	0.190000000000000	0.190000000000007	0.189999194602654
1	0	8.43769498715119e-15	4.57128800035456e-07

Table 2: The result maximum error when n=4,6,11of example 1

Error	Hermite interpolation H_7(x)	Hermite interpolation H_{11}(x)	Hermite interpolation H_{21}(x)
M.E	4.21884749357560e-15	2.31020758079126e-12	3.03624319630824e-07

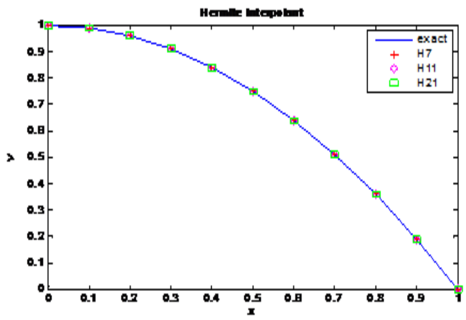

Fig.1: Comparison between the exact and suggested solution when n= 4, 6,11

Example2. Consider the following SBVP : $y'' + \frac{2}{x}y' - 4y = -2$, $0 \le x \le 1$
with mixed BC:y'(0) = 0, y(1) = 5.5.
The exact solution is : y(x) =0.5 + $\frac{5\sinh(2x)}{x\sinh(2)}$ [15]
Using Taylor polynomials, we have
y(0)= 3.257205700320381 ,y(1/3)= 3.466030095098947, y(2/3)= 4.1499242047031
y'(1/3)= 1.280759726616932, y'(2/3)= 2.915149400050196, y'(1)= 5.373147016441
Now, we solve this equation using these data
H_7(x)= $0.009130738291121613x^7$ + $0.01920841575611729x^6$+ $0.01361484275073988x^5$+ $0.3613904331003707x^4$+ $0.001445668149072448x^3$+ $1.838004201632197x^2$+ 3.257205700320381
Geng [15] solved this example by (VIM) and give the following series solution:
p (x) = 3.25721 + $1.83814x^2$+ $0.367628x^4$+ $0.0350121x^6$+$0.00194512x^8$+ $0.0000707316x^{10}$
The numerical results are given in and Table 3 gives H_7(x), results VIM exact solution. Table 4 gives the maximum error for number point n = 4, 6, 11. Figure 2 gives the accuracy of the suggested method.

Table 3: Numerical results for n=4 of example 2

x_i	Exact Solution y(x)	Hermite interpolation H_7(x)	VIM p(x)
0.1	3.27562381647618	3.27562348331808	3.27562819783156
0.2	3.33132158129189	3.33132136138556	3.33132605056115
0.3	3.42564142056487	3.42564145791085	3.42564603865789
0.4	3.56086353732463	3.56086354389276	3.56086836853219
0.5	3.74027136831943	3.74027129054089	3.74027648126133
0.6	3.96824614512855	3.96824615761122	3.96825161157128
0.7	4.25039346768551	4.25039351936955	4.25039935164274
0.8	4.59370586068823	4.59370563007463	4.59371217247398
0.9	5.00676642428200	5.00676603087337	5.00677296919957
1	5.50000000000000	5.50000000000000	5.50000595160000

Table 4: The result maximum error when n=4,6,11of example 2

Error	Hermite interpolation H_7(x)	Hermite interpolation H_{11}(x)	Hermite interpolation H_{21}(x)
M.E	6.05244039501958e-08	3.26871913686616e-06	1.95978813952553e-05

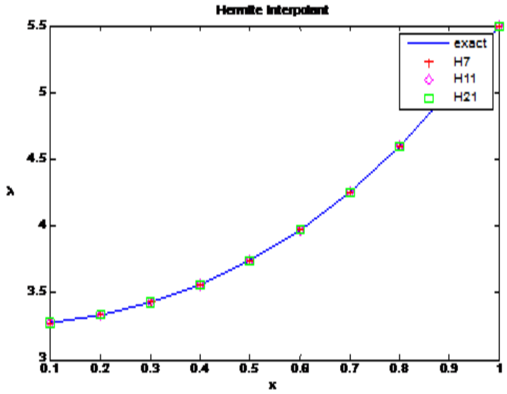

Fig.2: Comparison between the exact and suggested solution when n=4,6,11

Example 3. consider the following SBVP: (1- x^2) y''+ x y'+ y = 0 , $0 \le x \le 1$
With Neumann BC : y'(0) = 0, y'(1) = - y(1) [11]
Using Taylor polynomials, we have
y(0)= 1.85779174711865 ,y(1/3)= 2.076407841859629, y(2/3)= 2.02488427571492,y(1)= 1.65096790760441614054343517637,
y'(1/3)= 0.2793960039864789, y'(2/3)= -0.6144298559221, y'(1)= -1.650967907604
Now, we solve this equation using these data
H_7(x) =-0.0014996028446319087379379197955132x^7+ 0.008758728078627768x^6 -

0.0274267339641482976730912923811287x^5+ 0.07514319897959248x^4- 0.332863322735647670924663543701 17x^3 - 0.928936107025947421789169311523 44x^2 + x + 1.857791747118655

The numerical results are given in and Table 5 gives $H_7(x)$, results VIM exact solution. Table 6gives the maximum error for number point n = 4, 6, 11. Figure 3 gives the accuracy of the suggested method.

Table 5: Numerical results for n=4 of example 3

x_i	P_7	Hermite interpolation $H_7(x)$
0	1.857784296228232	1.85779174711866
0.1	1.948169418187120	1.94817677138695
0.2	2.018075735440336	2.01808339018278
0.3	2.065740050052486	2.06574825787000
0.4	2.089526555233489	2.08953495227793
0.5	2.087906467474147	2.08791430899834
0.6	2.059442143114885	2.05944879456718
0.7	2.002774819034724	2.00278016273118
0.8	1.916615117147540	1.91661963799949
0.9	1.799735452392664	1.79973987068057
1	1.650963483906875	1.65096790760442

Table 6:the results maximum error when n=4 of example 3

Xi	H7"	f(x, H7(x), H7'(x))	Error |H7" − f(x, H7(x), H7'(x))|
0	-1.85787221405916	-1.85779174711866	8.04669405023439e-05
0.1	-2.04909591215124	-2.04911939063003	2.34784787940256e-05
0.2	-2.22522747898047	-2.22522921308460	1.73410413539798e-06
0.3	-2.38870665496927	-2.38869807607405	8.57889522176691e-06
0.4	-2.54149371208515	-2.54149298894841	7.23136737157404e-07
0.5	-2.68514503382407	-2.68514933766494	4.30384087213298e-06
0.6	-2.82088869519379	-2.82088767638881	1.01880497460627e-06
0.7	-2.94970004269726	-2.94969437953626	5.66316099970265e-06
0.8	-3.07237727431599	-3.07238020853248	2.93421649200099e-06
0.9	-3.18961701949341	-3.18962492996904	7.91047562120895e-06
	Max. Error	1.85779174711866	8.04669405023439e-05

Where n=4,6,10, then max. error:

$$\|H_7''(x) - f(x, H_7(x), H_7'(x))\|_\infty / (1 + \|f(x, H_7(x), H_7'(x))\|_\infty = 2.815703e-05$$

$$\|H_{11}''(x) - f(x, H_{11}(x), H_{11}'(x))\|_\infty / (1 + \|f(x, H_{11}(x), H_{11}'(x))\|_\infty = 0.000767$$

$$\|H_{21}''(x) - f(x, H_{21}(x), H_{21}'(x))\|_\infty / (1 + \|f(x, H_{21}(x), H_{21}'(x))\|_\infty = 0.003949$$

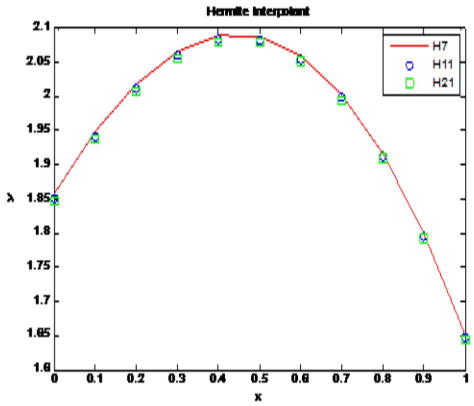

Fig. 3: Comparison the suggested solution when n=4,6,11

Conclusion:

In this paper Hermite interpolation method was used to solving singular boundary value problem. The result shown that the divided the domain into a number point follow the same steps as the previous is a very powerful and efficient in finding accurate solution for a large class of regular singular point.

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حل مسألة القيم الحدودية الشاذة ذات النقطتين باستخدام الاندراج هيرمت

هبة عواد عبد الرزاق

قسم الرياضيات، كلية العلوم للبنات، جامعة بغداد

الخلاصة:

في هذا البحث،استخدامنا طريقة اندراج هيرمت لحل مسائل القيم الحدودية النظامية من الرتبة الثانية للمعادلات التفاضلية الاعتيادية الشاذة . اقترح طريقة تطبيقها بعد تقسيم المجال الى العديد من المجالات الفرعية ومن ثم استخدام الاندراج هيرمت على كل مجال فرعي . الحل للمعادلة يساوي جمع الحل في كل مجال فرعي. اخيرا،قدمنا العديد من الامثلة لتوضح الاسلوب المقترح وكفاءته.

الكلمات المفتاحية: المعادلات التفاضلية الاعتيادية الشاذة،مسائل القيم الحدودية، اندراج هيرمت.