Behavior of Complementary Component Results for a Simple Harmonic Eddy-Current Problem

Mr. Essam M. Abdul –Baki*

Received on: 8/10/1998
Accepted on: 14/4/1999

Abstract

The behavior of complementary solutions is investigated through a comprehensive set of computational results for simple harmonic eddy-current problem.

It is found that despite the absence of bounds the two solutions complement each other in a tangible and useful sense. A simple one-dimensional problem was chosen for application on a thin conducting sheet.

The over-all accuracy due to the effect of various factors was examined and was much better than the available solution.

الخلاصة

تم استقصاء سلوك النتائج المتكاملة من خلال نتائج حسابية معمقة لمشكلة تيار دوام بسيطة.

لقد وجد أنه، وعلى الرغم من غياب القيود، فان كلا الحلين يكمل أحدهما الآخر بطريقة ملموسة وعملية. وقد أختيرت مشكلة بسيطة وببعد واحد لتطبيقها على صفيحة موصلة خفيفة.

أختبرت الدقة العمومية بالاستناد إلى تأثير عوامل مختلفة، وكانت أفضل بكثير من الحل المتوفر.

Notation

U^c, U^* = complex phasor and its conjugate;

U^r, U^i = real and imaginary components of U^c;

$\overline{U}$ = Pre-specified quantity;

$\underline{n}$ =outward unit normal to a surface;

$< \underline{U}, \underline{V} >$ = Volume integral of $\underline{U}.\underline{V}$ over the region of the problem;

$[\underline{U}, \underline{V}]$ = Surface integral of $\underline{U}.\underline{V}$ over the boundary.

*Laser and Optoelectronics Engineering Department, University of Technology, Baghdad, Iraq.

1. Introduction

A recent theoretical investigation [1,2] has shown that complementary formulations for the harmonic eddy current problem, proposed by a number of workers, are essentially equivalent despite apparent differences. The investigation also showed that the complementary formulations are non-bounding. As a natural sequel, the present paper examines the actual behavior of complementary computational results. A simple one-dimensional problem, whose exact solution is available, was chosen for the purpose. The paper examines the effects of various factors on the relative behavior of complementary results, and on the over-all solution accuracy as measured by the Ligurian error. The paper also highlights the advantages of averaging complementary estimates in preference to settling for either estimate alone.

The error-based approach adopted here in the derivation produces, in addition to independently solvable complementary variation principles, a single variational principle that involves both complementary systems simultaneously. Computational results obtained from this single solution are also presented and examined; the over-all accuracy is found to compare favorably with that of independent complementary solutions.

2. The Problem

The problem to be solved is a conducting sheet oriented as shown in figure1. It has a thickness of 2b in the y-directions, and extends indefinitely along the x- and z- axes. It carries a specified current $\overline{I^c}$ per unit width, following on the z-direction. The conductivity of the sheet material is σ and the permeability is μ; the angular frequency is ω.

At low frequencies, the governing equations are:

$$\nabla \times \underline{H}^C = \underline{J}^C \qquad (1)$$

and

$$\nabla \times \underline{E}^C = j\omega B^C \qquad (2)$$

All fields vary with y only, so that the problem is one-dimensional. Moreover, each field has only one component: $\underline{H}^C = H^C(y)\underline{u}_x$, $\underline{J}^C = J^C(y)\underline{u}_z$, $\underline{E}^C = E^C(y)\underline{u}_z$, and $\underline{B}^C = B^C(y)\underline{u}_x$, equations (1) and (2) thus are simplified to :

$$-\frac{dH^C}{dy} = J^C \qquad (3)$$

and

$$\frac{dE^C}{dy} = -j\omega B^C \qquad (4)$$

Boundary conditions may be defined in relation to a brick of unit width and unit depth. At the front and rear surfaces, where $\underline{n} \times \underline{E^C} = 0$, and at the sides $\underline{n} \times \underline{H^C} = 0$. Non-homogeneous conditions arise at the top and bottom surfaces, where $\underline{n} \times \underline{H^C}$ is constant on each; symmetry and Ampere's circuiral law yield:

$$H^C(-b) = -H^C(b) = \frac{1}{2}\overline{I^C} \qquad (5)$$

It is convenient to divide variables and governing equations into two complementary systems: $\underline{H}^C, \underline{J}^C$, and equations (1) and (3) belong to the H-system, while $\underline{E}^C, \underline{B}^C$ and equations (2) and (4) belong to the E-system. The non-homogenous boundary conditions of equation (5), which force the solution, belong to the H-system in this problem. The two systems are related through the two constitutive relationships.

$$\underline{B}^C = \mu \underline{H}^C \text{ or } \underline{H}^C = \nu \underline{B}^C$$

and

$$\underline{E}^C = \rho \underline{J}^C \quad \text{or} \quad \underline{J}^C = \sigma \underline{E}^C$$

3. Trail Variables

In the simple numerical solutions to be performed, $H^C(y)$ and $E^C(y)$ will be represented as polynomials in y. Symmetry about the midplane makes H^C and odd function of y and E^C an even function. Solution will be performed at two levels of refinement: third and fifth order polynomials for H^C, and second and fourth order polynomials for E^C; the corresponding results will be labeled H_3, H_5, E_2, and E_4 respectively. Each of the trial functions for H_5 and E4 involves six real parameters:

$$H^C(y) = (p_1 y + P_3 y^3 + P_5 y^5) + j(q_1 y + q_3 y^2 + q_5 y^5) \quad (8)$$

and

$$E^C(y) = (r_0 + r_2 y^3 + r_4 y^4) + j(S_0 + S_2 y^2 + S_4 y^4) \quad (9)$$

The trial functions for H3 are obtained by setting P_5 and q_5 to zero, and those for E2 by setting r_4 and s_4 to zero.

The solution of the problem is basically a process of enforcing the governing equations on the trial variables. The H-system equation (3) is enforced by substituting for H^C from equation (8) to express j^C in terms of the H-system parameters.

$$J^C(y) = -(p_1 + 3P_3 y^3 + 5P_5 y^4) - j(q_1 + 3q_3 y^2 + 5q_5 y^4) \quad (10)$$

Similarly, the E-system equation (4) is enforced by substituting for E^C from equation (9) to express B^C in terms of the E-system parameters:

$$B^C(y) = -\frac{1}{\omega}(2r_2 y + 4r_4 y^3) + j\frac{1}{\omega}(2s_2 y + 4s_4 y^3) \quad (11)$$

The boundary conditions (5) eliminate two of the H-system parameters since application to equation (8) yields.

$$-(p_1 b + P_3 b^3 + P_5 b^5) - j(q_1 b + q_3 b^3 + q_5 b^5) = \frac{1}{2}\overline{I}^r + j\frac{1}{2}\overline{I}^i \quad (12)$$

In actual formulation, it is convenient to set $\overline{I}^i = 0$ with no loss of generality.

Equations (10)-(12) account for the independent H- and E-system governing equations (3)-(5). This leaves the constitutive relationships (6) and (7) link the two systems. These will be considered in the following section.

4. Solution Formulation

Constitutive relationships may be enforced by minimizing constitutive errors; the process yields variational solution formulations [1,2] which will be used to determine the remaining unknown parameters in the trail functions.

The instantaneous magnetic and conduction constitutive errors, or Ligurians, are:

$$\Lambda_M = \frac{1}{2} < \mu \underline{H}, \underline{H} > + \frac{1}{2} < \nu \underline{B}, \underline{B} > - < \underline{H}, \underline{B} > \quad (13)$$

and

$$\Lambda C = \frac{1}{2} < \rho \underline{J}, \underline{J} > + \frac{1}{2} < \sigma \underline{E}, \underline{E} > - < \underline{J}, \underline{E} > \quad (14)$$

Respectively, Λ_M and Λ_C are strictly nonnegative, and zero only if their respective variables satisfy the corresponding constitutive relationships. A total Ligurian is defined by:

$$\Lambda(t) = \Lambda_M(t) + \alpha \int_{t-\tau}^{t} \Lambda_C(t)dt \quad (15)$$

where α and τ may be assigned arbitrary positive values. Again, Λ is strictly nonnegative, and zero only if the trial variables satisfy both constitutive relationships simultaneously. With the trial variables constrained to satisfy all the other governing equations, enforcement of the constitutive relationships by error minimization through the variational principle

$$\delta\Lambda = 0 \quad (16)$$

yields the requisite solution formulation.

Because the Ligurain involve products of fields and under steady harmonic conditions, they are each composed of a time-invariant component together with a sinusoidal component at double frequency; for example:

$$\Lambda(t) = \Lambda^O + \Lambda^r Cos2\omega t - \Lambda^i Sin2\omega t$$
$$= \Lambda^O + Re\{\Lambda^C e^{2j\omega t}\} \quad (17)$$

Representing the magnetic and conducting Ligurian, Λ_M and Λ_C of equations (13) and (14), in the same way, when substituting into equation (15), and performing the algebra, the following general expressions for Λ^o and Λ^c are obtained:

$$\Lambda^O(H,E) = +\Lambda^O_M(H,E) + \alpha\tau\Lambda^O_C(H,E) \quad (18)$$

with

$$\Lambda^O_M(H,E) = \frac{1}{2} < \mu\underline{H}^C, \underline{H}^* > + \frac{1}{2} < \nu\underline{B}^C, \underline{B}^* > - Re\{< \underline{H}^C, \underline{B}^* >\}$$

$$\Lambda^O_C(H,E) = \frac{1}{2} < \rho\underline{J}^C, \underline{J}^* > + \frac{1}{2} < \sigma\underline{E}^C, \underline{E}^* > - Re\{< \underline{J}^C, \underline{E}^* >\}$$

and

$$\Lambda^C(H,E) = \Lambda^C_M(H,E) + \frac{\alpha}{2j\omega}(1 - e^{-2j\omega\tau})\Lambda^c_C \quad (19)$$

with

$$\Lambda^C_M(H,E) = \frac{1}{2} < \mu\underline{H}^C, \underline{H}^C > + \frac{1}{2} < \nu\underline{B}^C, \underline{B}^C > - < \underline{H}^C, \underline{B}^C >$$

$$\Lambda^c_C(H,E) = \frac{1}{2} < \rho\underline{J}^C, \underline{J}^C > + \frac{1}{2} < \sigma\underline{E}^C, \underline{E}^C > - < \underline{J}^C, \underline{B}^C >$$

As the instantaneous Ligurian $\Lambda(t)$ is strictly nonnegative, equation (17) implies that:

$$\Lambda^o \geq |\Lambda^c| \geq 0 \quad (20)$$

Substituting for $\Lambda(t)$ from equation (17) into the general variational principles of equation (16), we obtain the variational principles.

$$\delta\Lambda^o = 0 \quad (21)$$

and

$$\delta\Lambda^r = 0, \ \delta\Lambda^i = 0 \quad (22a)$$

The variational principles of equation (22a) are equivalent and may be expressed concisely as:

$$\delta\Lambda^c = 0 \quad (22b)$$

Equations (21) and (22) describe two distinct variational principles. The first corresponds to the time invariant component Λ^o and is minimal, while the second corresponds to the complex phasor Λ^c associated with the sinusoidal component, and is merely stationary. The two principles yield identical results in an exact solution. In numerical formulation, however, they yield somewhat different values for the unknown parameters, and hence different approximations to the exact solution.

4.1 Complementary Formulations

The variational principle of equation (22) is of particular interest as it can be split into complementary variational principles. To initiate the split the multiplier α and the integration interval τ must be assigned the following values.

$$\alpha = 1 \ , \ \tau = \frac{\pi}{2\omega} \quad (23)$$

Substituting into equation (19) yield, after some rearrangement we get:

$$\Lambda^C(H,E) = \{\frac{1}{2} < \mu\underline{H}^C, \underline{H}^C > + \frac{1}{2j\omega} < \rho\underline{J}^C, \underline{J}^C >\}$$
$$+ \{\frac{1}{2} < \nu\underline{B}^C, \underline{B}^C > + \frac{1}{2j\omega} < \sigma\underline{E}^C, \underline{E}^C >\}$$
$$\{< \underline{H}^C, \underline{B}^C > + \frac{1}{j\omega} < \underline{J}^C, \underline{E}^C >\} \quad (24)$$

The integrals in the first and second pairs of brackets belong to the H- and E-systems respectively. The integrals in the third pair of brackets have factors from both systems, and can be manipulated as follows [1,2]: $\nabla \times \underline{H}^C$ is substituted for $\underline{J}^C$, a vector identify is applied followed by the divergence theorem, $-j\omega\underline{B}^C$ is substituted for $\nabla \times \underline{E}^C$, and appropriate terms are eliminated. That is:

$$<\underline{H}^C,\underline{B}^C>+\frac{1}{j\omega}<\underline{J}^C,\underline{E}^C>=<\underline{H}^C,\underline{B}^C>+\frac{1}{j\omega}<\nabla\times\underline{H}^C,\underline{E}^C>$$

$$<\underline{H}^C,\underline{B}^C>+\frac{1}{j\omega}<\underline{H}^C,\nabla\times\underline{E}^C>+\frac{1}{j\omega}[n\times\underline{H}^C,\underline{E}^C]$$

$$=<\underline{H}^C,\underline{B}^C>-<\underline{H}^C,\underline{B}^C>+\frac{1}{j\omega}[n\times\underline{H}^C,\underline{E}^C]$$

$$=\frac{1}{j\omega}[n\times\underline{H}^C,\underline{E}^C]$$

The volume integrals in the third pair of brackets of equation (24) thus reduce to a surface integral on the boundary. Recalling the boundary conditions as described in conjunction with equation (5), it will be seen that the only nonzero contributions to the surface integral come from the top and bottom boundary sections where tangential $\underline{H}^C$ is known. Substituting back equation (24), we can write:

$$\Lambda^C(H,E)=\theta^C(H)+\Xi^C(E) \quad (25)$$

where

$$\theta^C(H)=\frac{1}{2}<\mu\underline{H}^C,\underline{H}^C>+\frac{1}{2j\omega}<\rho\underline{J}^C,\underline{J}^C> \quad (26)$$

and

$$\Xi^C(E)=\frac{1}{2}<\nu\underline{B}^C,\underline{B}^C>+\frac{1}{2j\omega}<\sigma\underline{E}^C,\underline{E}^C>-\frac{1}{j\omega}[n\times\underline{H}^C,\underline{E}^C] \quad (27)$$

These are the complementary functionals. The surface terms in equation (27) introduces the H-system boundary conditions naturally into the E-systems functional $\Xi^C(E)$. Substituting from equation (25) into equation (22c), we get the complementary variational principles.

$$\delta\theta^C(H)=0 \quad (28)$$

$$\delta\Xi^C(E)=0 \quad (29)$$

These can be solved independently of each other because the H- and E-system trial variables are independent of each other.

5. Circuit Parameters

The behavior of complementary solutions will be described mainly in terms of the estimated values of resistance and inductance. For the specified-current conditions assumed, these are given by:

$$R(H)=<\rho\underline{J}^C,\underline{J}^*>/\left|\overline{I^C}\right|^2 \quad (30)$$

$$L(H)=<\mu\underline{H}^C,\underline{H}^*>/\left|\overline{I^C}\right|^2 \quad (31)$$

$$R(E)=<\sigma\underline{E}^C,\underline{E}^*>/\left|I^C\right|^2 \quad (32)$$

$$L(E)=<\nu\underline{B}^C,\underline{B}^*>/\left|\overline{I^C}\right|^2 \quad (33)$$

R(H) and L(H) are H-system estimates, and R(E) and L(E) are E-system estimates. At the exact solution, corresponding complementary estimates are equal to each other and to the exact values given by[3].

$$R=\frac{1}{2\sigma\Delta}\frac{Sinh\gamma+Sin\gamma}{Cosh\gamma-Cos\gamma} \quad (34)$$

and

$$L=\frac{1}{2\omega\sigma\Delta}\frac{Sinh\gamma-Sin\gamma}{Cosh\gamma-Cos\gamma} \quad (35)$$

where Δ is the depth of penetration

$$\Delta=\sqrt{2/\omega\sigma\mu} \quad (36)$$

and

$$\gamma=2b/\Delta$$

In the present numerical formulation, the integrals in equations (30)-(33) can be evaluated once the parameter p, q, r, and s are obtained from the solution. The estimates will be normalized with respect to the exact dc values of resistance and inductance, which are:

$$R_{dc}=\frac{1}{2b\sigma} \quad (38)$$

and

$$L_{dc}=\frac{\mu b}{6} \quad (39)$$

It is of interest to observe that the integrals in the expressions for the circuit parameters, equations (30)-(33), appear explicitly in the expression for the time-invariant Ligurian Λ^o, equation

(18), but not in the complex Ligurian Λ^C, equation (19), nor in the complementary functional θ^C and Ξ^C, equations (26) and(27).

6. Complementary Solutions

The sheet problem was solved using the complementary formulations of section 4.1, with the trial functions of section 3. Figures 2 and 3 show the resulting resistance and inductance estimates at different values of sheet thickness (normalized with respect to depth of penetration). The curves are labeled according to system and order of polynomial; thus H5 denotes and H-system estimates are found by using fifth order polynomials, and so forth.

For the infinitesimally thin sheet, numerical estimates coincide with exact values. As sheet thickness increases (or, alternatively, as frequency increases for a fixed thickness), the field distribution develops into patterns of increasing complexity. Each of the numerical estimates follows the exact curve quite closely up to a point beyond which the simple trail functions are no longer able to approximate the exact distribution. The different points of departure, most clearly seen in Fig.2, are predictably higher for higher-order polynomials. H-system estimates are more accurate than E-system estimates because the boundary conditions of equation (5), which force the solution, belong to the H-system and are enforced explicitly on H-system variables, equation (12); they enter the E-solution only naturally through the last term in equation (27).

The boundness of complementary eddy-current solutions has been shown in figures (2) and (3) which provide graphical confirmation in that the solutions are non-bounding. The complementary functions of equations (26) and (27), derived here by the error-based approach, are essentially the same as those given by other workers [4,8]. It follows that the absence of bounds is not a peculiarity of the error-based derivation, but is fundamental to the harmonic eddy-current problem.

Bounded behavior is characterized by two main features. Firstly, the upper and lower bounds always bracket the exact solution irrespective of how crude the approximation may be. Secondly, numerical estimates converge to exact values as the trial functions are refined. While it is evident that the curves of figures 2 and 3 do not exhibit such behavior, it will be observed that the resistance curves of fig.2 appear to exhibit bounded behavior up to b/Δ just under 5. It is probably this kind of apparent boundedness that encouraged workers to expect true boundedness. Such behavior may be a fortuitous consequence of a combination of factors relating to the particular problem considered, the type of trail functions used, and the quantity being computed. The inductance curves, for example, do not even appear bounding, as the various curves cross each other at a number of points (for example, H5 and E4 at $b/\Delta \cong 6.6$).

Despite the non-boundedness of the solutions, careful examination of the curves suggests that the average of complementary estimates lies more consistently in the vicinity of the exact results rather than of the estimates. While rather weak, this observation does allow one to rely on the average with more confidence than either of the complementary estimates taken alone, especially in the range where the approximations are known to be relatively good. Of course the availability of the exact solution for the present simple problem makes it easy to identify the range of good approximation; however; this range can also be identified by comparing the

Ligurian error with the estimated values of the complementary functional, as in fig.4. The Ligurian is a comprehensive measure of the numerical error in properly formulated solutions where the only approximations are in the restrictions on the trail functions (as distinct from approximations in the modeling of the original problem).

7. Conclusions

The paper has shown that complementary formulations of the harmonic eddy-current problem are non-bounding. However, if the trends observed for the particular problem considered here prove to be general, then the absence of bounds is not a serious drawback: the two sets of estimates do complement each other about the exact solution in a tangible and useful sense. The averages of complementary estimates lie more consistently close to the exact values than either of the complements taken alone. This is particularly true where the approximation is relatively good, a condition that can be monitored through the Ligurian error.

The paper has also show that complementary solutions do not yield the best values for the unknown parameters. Better values, leading to a smaller over-all Ligurian error, can be obtained from a single solution, based on the real time-invariant Ligurtian that involves both complementary systems simultaneously.

The paper indicates that the analyst has, in principle, two alternatives. The first is to perform wither of the complementary solutions alone; this is by far the most common policy. The second is to perform both complementary solutions, the economical disadvantage of performing two solutions, as opposed to only one in the first alternative, is offset by the additional information provided, particularly with regard to accuracy. In general, for a given level of confidence, complementary solutions taken together can do with less refined approximation than either solution taken alone.

References

1. J. RIKABI: "An investigation of proposed complementary formulations and bounds for the harmonic eddy current problem". Submitted to Proc IEE, Pt A, August 1989.

2. J. RIKBI, C.F. BRYANT, and E.M. FREEMAN: "Error-based derivation of complementary formulations for the eddy-current problem" Proc IEE, Pt A, 1988, 135, (4), PP 208-216.

3. P. HAMMOND: "Applied electromagnetism". Pergamon Press, Oxford, 1971.

4. J.R. FRASER: "Complementary and Dual finite element principles". Ph.D. thesis, University of Aberdeen, UK, 1982.

5. J. PENMAN: "Dual and complementary variational techniques for the calculation of electromagnetic fields". Advances in Electronic and Electron Physics, 1980, 70, pp 315-364.

6. R.L. Ferrari: "Complementary variational formulation for eddy-current problems using the field variables E and H directly". Proc IEE, Pt A, 1985, 132 (4), PP 157-164.

7. P. Hammond: "Energy methods in electromagnetism". Clarendon Press, Oxford, 1981.

8. P. Hammond: "Upper and lower bounds in eddy-current calculations". Proc IEE, Pt A, 1989, 136, (4), pp 207-216.

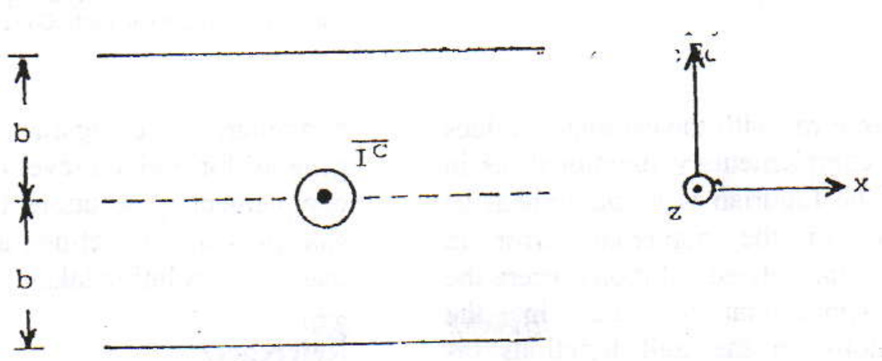

Figure 1

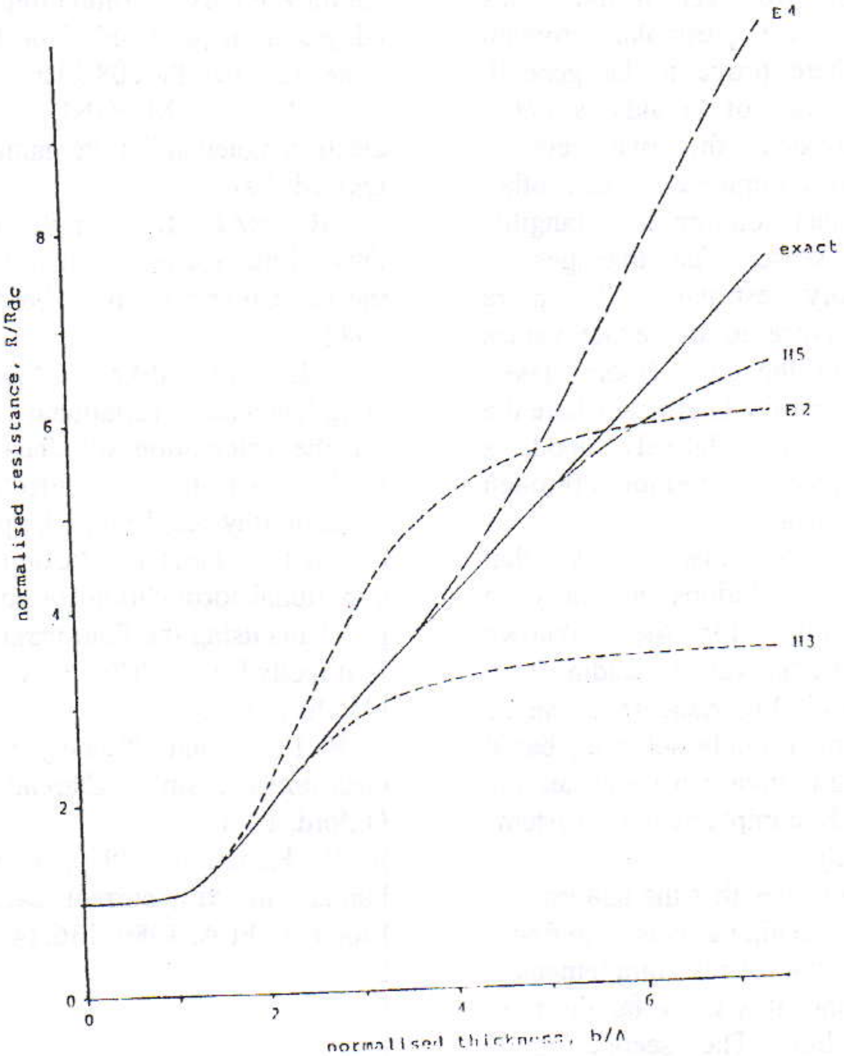

Figure 2

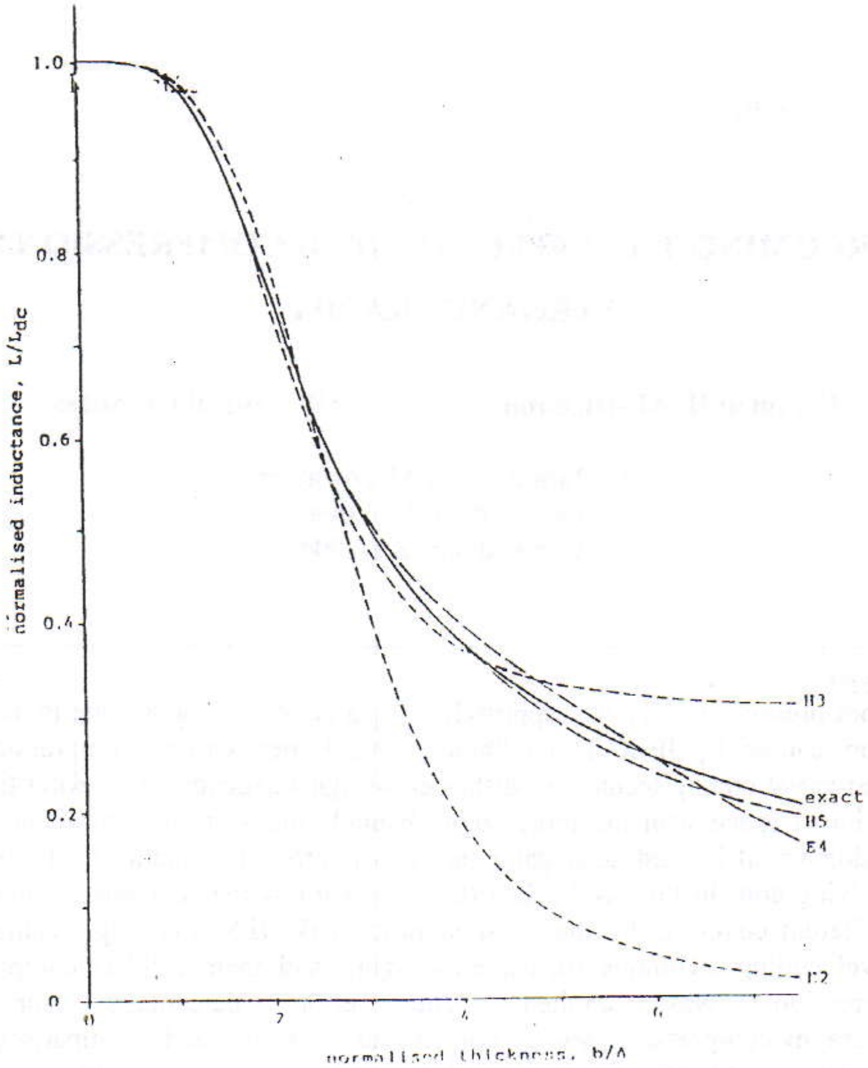

Figure 3

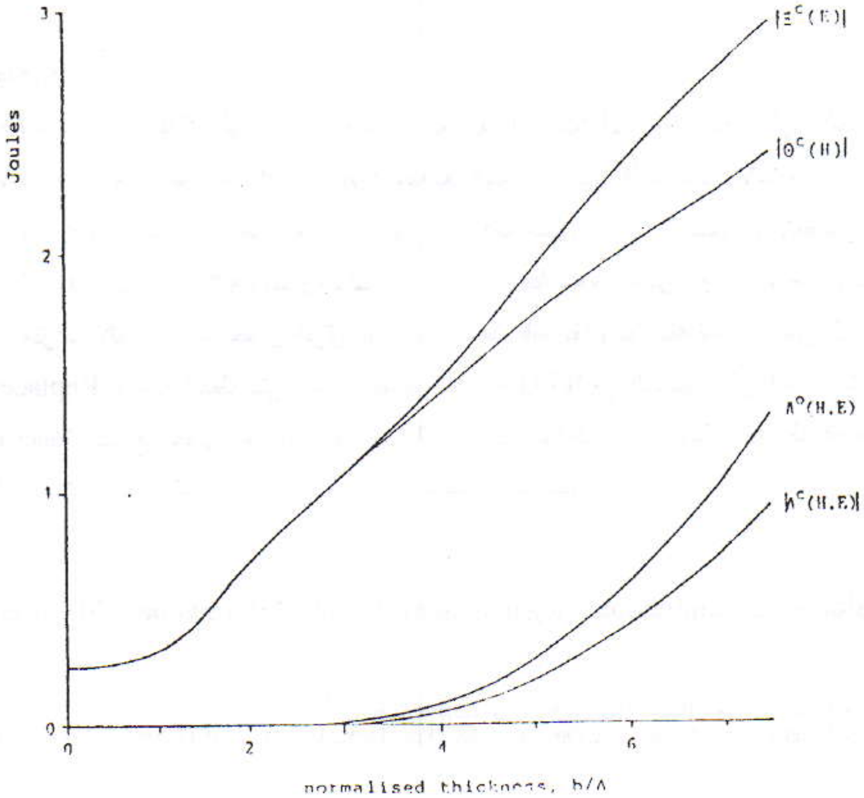

Figure 4