$\delta*$-Base for a tritopology

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Abstract :-

In this work , we introduce a new definitions for $\delta*$-Base for the $\delta*$-Neighbourhood system of a point ($\delta*$- Local Base) , $\delta*$-Base for a tritopology and $\delta*$-sub Base for a tritopology . And we gives the basic specifications for the new definitions .

Keywords :-

$\delta*$- Local Base , $\delta*$- Base , $\delta*$-sub Base .

الخلاصة :

في هذا العمل قدمت تعاريف جديدة في الفضاء الثلاثي التبولوجي و هي أساس $\delta*$- لنظام الجوارات لنقطة (أساس محلي $\delta*$-) و أساس $\delta*$- للفضاء الثلاثي التبولوجي و أساس $\delta*$- جزئي للفضاء الثلاثي التبولوجي . و كما قمت بإعطاء المواصفات الأساسية لهذه التعاريف في الفضاء الثلاثي التبولوجي .

Introduction :-

Throughout this paper we adopt the notations and terminology of [1] , [2] , [3] and [4] , X and Y are finite sets and the following conventions : (X,T), (X,T,Ω), (X,T,Ω,ρ) will always denot to Topological space , Bitopological space and Tritopological space respectively .

Let (X,T) be a topological space , and let A be asubset of X , then A is said to be a α -open set iff $A \subseteq T-\text{int}(T-cl(T-\text{int}(A)))$ [1] , and the family of all α -open sets is denoted by $\alpha.O(X)$. The complement of α -open set is called α -closed set .

Let (X,T,Ω) be a Bitopological space , and let A be asubset of X , then A is said to be a δ -open set iff $A \subseteq T-\text{int}(\Omega-cl(T-\text{int}(A)))$ [3] , and the family of all δ -open sets is denoted by $\delta.O(X)$. The complement of δ -open set is called δ -closed set .

And let (X,T,Ω,ρ) be a Tritopological space , a subset A of X is said to be $\delta*$-open set iff $A \subseteq T-\text{int}(\Omega-cl(\rho-\text{int}(A)))$ [2] , and the family of all $\delta*$-open sets is denoted by $\delta^*.O(X)$. The complement of $\delta*$-open set is called a $\delta*$-closed set .

And $\delta^{*}.\mathrm{O}(\mathrm{X})$ does not represent a topology (not always represent a topology) [2] , $(\mathrm{X},T,\Omega,\rho)$ is called discrete tritopological space with respect to $\delta*$- open if $\delta^{*}.\mathrm{O}(\mathrm{X})$ contains all subsets on X .

$(\mathrm{X},T,\Omega,\rho)$ is called indiscrete topological space with respect to $\delta*$-open if $\delta^{*}.\mathrm{O}(\mathrm{X})$ = { X , φ } .

Let $(\mathrm{X},T,\Omega,\rho)$ be a tritopological space , and let x ∈ X . A subset N of X is said to be $\delta*$-nhd of a point x iff there exists $\delta*$-open set U such that x ∈ U ⊂ N . The set of all $\delta*$-nhds of a point x is denoted by $\delta^{*}-\mathrm{N}(x)$.

1- $\delta*$- Local Base at a point :

1.1 Definition :

Let $(\mathrm{X},T,\Omega,\rho)$ be a tritopological space . A non empty collection $\delta*$-β(x) of $\delta*$-neighbourhoods of x is called a $\delta*$- base for the $\delta*$-neighbourhood system of x ($\delta*$- Local Base) iff for every $\delta*$-neighbourhood N of x there is a B∈ $\delta*$- β(x) such that B ⊂ N .

We then also say that $\delta*$- β(x) is a $\delta*$- Local Base at x or a fundamental system of $\delta*$-neighbourhoods of x .

If $\delta*$- β(x) is a $\delta*$- Local Base at x , then the members of $\delta*$- β(x) are called basic $\delta*$-neighbourhoods of x .

1.2 Remark :

If $\delta*$.O(X) represent a topology then every point x in X has a $\delta*$- local base in tritopology .

Proof :

Clearly because the intersection condition of the topology definition is hold .

1.3 Example :

Let X = {a,b,c,d} , T = { X , φ , {b,c,d} } .

, Ω = { X , φ , {a} }

ρ = {X,φ,{a},{c,d},{c},{b,c},{a,c,d}, {a,c},{a,b,c},{b,c,d}}

(X, T) , (X, Ω) , (X, ρ) are three topological spaces , then $(\mathrm{X},T,\Omega,\rho)$ is a tritopological space , such that :

$$\delta^{*}.\mathrm{O}(\mathrm{X})=\{X,\varphi,\{a\},\{c,d\},\{c\},\{b,c\},\{a,c,d\},\{a,b,c\},\{a,c\},\{a,b\},\{a,d\},\{a,b,d\},\{b,c,d\}\}$$

Clearly that $\delta*$.O(X) represent a topology . Then the $\delta*$- local base at each of the points a , b , c , d is given by $\delta*$- β(a) = { {a} } , $\delta*$- β(b) = { {b,c} } , $\delta*$- β(c) = { {c} } , $\delta*$- β(d) = { {c,d} } .

Observe that here a $\delta*$- Local Base at each point consists of a single $\delta*$-nhd of the point .

1.4 Remark :

If $\delta*$.O(X) does not represent a topology then not every point x in X has a $\delta*$- local base in tritopology .

Proof :

Clearly because the intersection condition of the topology definition is not hold .

1.5 Example :

Let X={a,b,c,d} , $T=\{X,\varphi,\{c,d\}\}$, $\Omega=\{X,\varphi,\{a,b,c\},\{a\}\}$ and $\rho=\{X,\varphi,\{d\},\{c,d\},\{a,d\}\}$

(X, T) , (X, Ω) , (X, ρ) are three topological spaces , then (X,T,Ω,ρ) is a tritopological space , such that :

$$\delta^{*}.O(X)=\{X,\varphi,\{a,d\},\{c,d\},\{a,b,d\},\{a,c,d\},\{b,c,d\}\}$$

Clearly that $\delta*$.O(X) does not represent a topology . Then the $\delta*$- Local Base at each of the points a , b , c , d is given by :

$\delta*$- β(a) = { {a,d} } , $\delta*$- β(c) = { {c,d} }

But there is no $\delta*$- β(b) because the intersection of {a,b,d} and {b,c,d} is {b,d} is not $\delta*$-open set (not exist) , i.e. there is no set B such that B is a subset of all $\delta*$-nhds of b .

Again , there is no $\delta*$- β(d) because the intersection of {a,d} and {c,d} is {d} is not $\delta*$- open set (not exist) , i.e. there is no set D such that D is a subset of all $\delta*$-nhds of d .

Hence not every point x in X has a $\delta*$- local base in tritopology .

1.6 Remark :

All the following theorems and remarks is true in tritopology if $\delta*$.O(X) represent a topology (i.e. if every x has $\delta*$- local base) .

Proof:

It is clear by the remarks and examples above .

1.7 Theorem :

Let (X,T,Ω,ρ) be a tritopological space . And let x∈X then the collection $\delta*$-β(x) of all $\delta*$-open subsets of X containing x is a $\delta*$- Local Base at x .

Proof :

Let N be any $\delta*$-nhd of x . Then there exists a $\delta*$-open set G such that x∈G ⊂ N , since G is a $\delta*$-open set containing x , G ∈ $\delta*$-β(x) .

This shows that $\delta*$-β(x) is a $\delta*$- Local Base at x .

1.8 Remark :

(I) In a discrete tritopological space the collection { {x} } consisting of a single member {x} forms a $\delta*$- Local Base at x∈X

Proof :

for {x} is a δ^*-open subset of X and so it is a δ^*-nhd of x and any other δ^*-nhd N of x must contain {x} .

(II) In a indiscrete tritopological space , the set X alone forms a δ^*- Local Base for every point $x \in X$.

Proof :

There is only δ^*-open X and φ in δ^*.O(X) . and must be X .

1.9 Theorem :

Let (X, T, Ω, ρ) be a tritopological space . And let δ^*-β(x) be a δ^*- Local Base for every point $x \in X$. Then δ^*-β(x) has the following properties :

(1) δ^*-β(x) $\neq$ φ for every $x \in X$.

(2) If $B \in \delta^*$-β(x) , then $x \in B$.

(3) If $A \in \delta^*$-β(x) and $B \in \delta^*$-β(x) , then there exists a set $C \in \delta^*$-β(x) such that $C \subset A \cap B$.

(4) If $A \in \delta^*$-β(x) , then there exists a set B such that $x \in B \subset A$ and such that for every point $y \in B$, there exists $C \in \delta^*$-β(y) satisfying $C \subset B$.

Proof :

(1) Since X is δ^*-open , it is δ^*-nhd of its point . Since δ^*-β(x) is δ^*- Local Base for every point $x \in X$ and X is a δ^*-nhd of x , it follows that there must exist a $B \in \delta^*$-β(x) such that $B \subset X$.

Hence δ^*-β(x) $\neq$ φ for every $x \in X$.

(2) If $B \in \delta^*$-β(x) , then B is a δ^*-nhd of x . So by definition of δ^*-nhd $x \in B$.

(3) If $A \in \delta^*$-β(x) , then A is a δ^*-nhd of x . Similarly B is a δ^*-nhd of x . It follows that $A \cap B$ is δ^*-nhd of x . Since δ^*-β(x) is δ^*- Local Base at x , it follows that there exists $C \in \delta^*$-β(x) such that $C \subset A \cap B$.

(4) Since $A \in \delta^*$-β(x) , A is a δ^*-nhd of x . Hence there exists a δ^*-open set B such that $x \in B \subset A$. Since B is δ^*-open set , it is a δ^*-nhd for every $y \in B$. Again since δ^*-β(y) is δ^*- Local Base at y and B is a δ^*-nhd for every $y \in B$, it follows that for every $y \in B$ there exists $C \in \delta^*$-β(y) such that $C \subset B$.

2- δ^*- Base for a tritopology :

2.1 Definition :

Let (X, T, Ω, ρ) be a tritopological space . A collection δ^*-β of subsets of X is said to form a δ^*- base for the tritopology (T, Ω, ρ) iff :

(I) δ^*-β $\subset$ δ^*.O(X) .

(II) for each point $x \in X$ and each δ^*-neighbourhood N of x there exists some $B \in \delta^*$-β such that $x \in B \subset N$.

2.2 Remark :

If δ^*.O(X) represent a topology then the tritopology (T, Ω, ρ) has a δ^*- base .

Proof :

Clearly because the intersection condition of the topology definition is hold .

2.3 Example :

Let $X = \{a,b,c\}$, $T = \{ X , \varphi , \{a,b\} \}$.
, $\Omega = \{ X , \varphi , \{c\} \}$
and $\rho = \{X, \varphi, \{a\}, \{b\}, \{a, b\}\}$

(X, T) , (X, Ω) , (X, ρ) are three topological spaces , then (X, T, Ω, ρ) is a tritopological space , such that :

$$\delta^{*}.O(X) = \{X, \varphi, \{a\}, \{b\}, \{a, b\}\}$$

Clearly that $\delta^*.O(X)$ represent a topology .

Then the collection $\delta^*\text{-}\beta = \{X, \{a\}, \{b\}\}$ is a δ^*- base for the tritopology (T, Ω, ρ) since :

(I) $\delta^*\text{-}\beta \subset \delta^*.O(X)$. And

(II) Each δ^*-neighbourhood of a contains {a} which is a member of $\delta^*\text{-}\beta$ containing a . Similarly each of δ^*-neighbourhood of b contains {b} which is a member of $\delta^*\text{-}\beta$ containing b . And X is a δ^*-neighbourhood of c which is a member of $\delta^*\text{-}\beta$ containing c .

2.4 Remark :

If $\delta^*.O(X)$ does not represent a topology then the tritopology (T, Ω, ρ) has not a δ^*- base .

Proof :

Clearly because the intersection condition of the topology definition is not hold .

2.5 Example :

Let $X=\{a,b,c,d\}$, $T = \{X, \varphi, \{c, d\}\}$
, $\Omega = \{X, \varphi, \{a, b, c\}, \{a\}\}$
and $\rho = \{X, \varphi, \{d\}, \{c, d\}, \{a, d\}\}$

(X, T) , (X, Ω) , (X, ρ) are three topological spaces , then (X, T, Ω, ρ) is a tritopological space , such that :

$$\delta^{*}.O(X) = \{X, \varphi, \{a,d\}, \{c,d\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\}$$

Clearly that $\delta^*.O(X)$ does not represent a topology .

Then this tritopology has not a δ^*- base because the second condition of the definition (2.1) is not hold . because the intersection of {a,b,d} and {b,c,d} is {b,d} is not a δ^*- open set (not exist) , i.e. each δ^*-nhds of b contains {b,d} , but {b,d} is not exist in $\delta^*.O(X)$ because $\delta^*.O(X)$ does not represent a topology . .

Again , because the intersection of {a,d} and {c,d} is {d} is not a δ^*- open set (not exist) , i.e. each δ^*-nhds of d contains {d} , but {d} is not exist in $\delta^*.O(X)$ because $\delta^*.O(X)$ does not represent a topology .

Hence there is no a δ^*- base .

2.6 Remark :

In the discrete tritopological space the collection δ^*-β ={ {x}: x∈X } consisting of all singleton subsets of X is a δ^*- Base for the tritopology (T,Ω,ρ) .

Proof:

Because for each singleton set is δ^*- open so that δ^*-β ⊂ δ^*.O(X) . Also for each x∈X and each δ^*-nhds N of x , {x}∈ δ^*-β is such that x ∈{x} ⊂ N .

2.7 Remark :

All the following theorems and remarks is true in tritopology if δ^*.O(X) represent a topology (i.e. if (T,Ω,ρ) has δ^*- base) .

Proof :

It is clear by the remarks (2.2) and (2.4) and examples (2.3) and (2.5) .

2.8 Theorem :

Let (X,T,Ω,ρ) be a tritopological space . A sub collection δ^*-β of δ^*.O(X) is a δ^*- Base for (T,Ω,ρ) iff every δ^*-open set can be expressed as the union of members of δ^*-β .

Proof :

Let δ^*-β is a δ^*- Base for (T,Ω,ρ) and let G ∈ δ^*.O(X) . Since G is δ^*-open set , it is a δ^*- nhd of each of its points . Hence by definition of δ^*- Base , to each x∈G ∃ a member B of δ^*-β s.t.

x∈ B ⊂ G

It follows that G = ∪ { B : B ∈ δ^*-β and B ⊂ G } .

Conversely , let δ^*-β ⊂ δ^*.O(X) and let every δ^*-open set G be the union of members of δ^*-β .

We have to show that δ^*-β is δ^*- Base for (T,Ω,ρ) .We have

(1) δ^*-β ⊂ δ^*.O(X) (given)

(2) let x ∈X and let N be any δ^*-nhd of x . Then there exists a δ^*-open set G such that x∈ G ⊂ N .

But G is the union of members of δ^*-β . Hence there exists B ∈ δ^*-β such that x∈ B ⊂ G ⊂ N . Thus δ^*-β is δ^*- Base for (T,Ω,ρ) .

2.9 Theorem :

Let (T_1,Ω_1,ρ_1) and (T_2,Ω_2,ρ_2) be a tritopologies for X which have a common δ^*- Base δ^*-β . Then δ^*_1.O(X) = δ^*_2.O(X) .

Proof :

Let G ∈ δ^*_1.O(X) and x∈ G . Since G is δ^*_1-open set , it is a δ^*_1- nhd of x and since δ^*-β is a δ^*- Base for (T_1,Ω_1,ρ_1) , there exists B ∈ δ^*-β such that x∈ B ⊂ G . Since δ^*-β is a δ^*- Base for (T_2,Ω_2,ρ_2) , and B ∈ δ^*-β it follows that B ∈ δ^*_2.O(X) . Hence G is δ^*_1- nhd of x . Since x is arbitrary .Then $G \in \delta^*_2.O(X)$ by theorem [$G \in \delta^*.O(X)$ iff G is a δ^*- nhd of each of its points] . Thus $\delta^*_1.O(X) \subset \delta^*_2.O(X)$ .By symmetry $\delta^*_2.O(X) \subset \delta^*_1.O(X)$. Hence $\delta^*_1.O(X) = \delta^*_2.O(X)$.

- **Properties of a δ^*- Base for a tritopology .**

2.10 Theorem :

Let (X, T, Ω, ρ) be a tritopological space . And Let δ^*-β is a δ^*- Base for (T, Ω, ρ) . Then δ^*-β has the following properties :

(1) for every $x \in X$, there exists a $B \in \delta^*$-β such that $x \in B$, that is , $X = \cup \{B : B \in \delta^*\text{-}\beta\}$

(2) for every $B_1 \in \delta^*$-β , $B_2 \in \delta^*$-β and every point $x \in B_1 \cap B_2$, there exists a $B \in \delta^*$-β such that $x \in B \subset B_1 \cap B_2$, that is , the intersection of any two members of δ^*-β is a union of members of δ^*-β .

Proof :

(1) Since X is a δ^*-open set , it is a δ^*- nhd of each of its points . Hence by definition of δ^*-Base , for every $x \in X$, there exists some $B \in \delta^*$-β such that $x \in B \subset X$ in other words ,
$X = \cup \{B : B \in \delta^*\text{-}\beta\}$.

(2) If $B_1 \in \delta^*$-β and $B_2 \in \delta^*$-β , then B_1 and B_2 are δ^*-open sets . Hence their intersection $B_1 \cap B_2$ is also δ^*-open and therefore $B_1 \cap B_2$ is a δ^*- nhd of each of its points . So by definition of δ^*- Base , to each $x \in B_1 \cap B_2$, there exists $B \in \delta^*$-β such that $x \in B \subset B_1 \cap B_2$, that is , $B_1 \cap B_2$ is the union of members of δ^*-β .

2.10 Theorem :

Let (X, T, Ω, ρ) be a discrete tritopological space and let δ^*-β be the collection of all singleton subsets of X , any class δ^*-β' of subsets of X is a δ^*- Base for the discrete tritopology (T, Ω, ρ) iff δ^*-β $\subset \delta^*$-β' .

Proof :

Let δ^*-β' be a δ^*- Base for the discrete tritopology (T, Ω, ρ) . Since each singleton {x} is δ^*-open set , {x} must be a union of members of δ^*-β' . But a singleton set can only be the union of itself or itself with the empty set φ . Hence {x} must belong to δ^*-β' . It follows that δ^*-β $\subset \delta^*$-β' .

Conversely , let δ^*-β $\subset \delta^*$-β' . Now δ^*-β ={ {x}: $x \in X$ } is a δ^*- Base for the discrete tritopology (T, Ω, ρ) by remark (2.6) . Also each member of δ^*-β' is δ^*-open so that δ^*-β' $\subset \delta^*.O(X)$.

Since δ^*-β $\subset \delta^*$-β' it follows that if any δ^*-open set G is the union of members of δ^*-β , it is also the union of members of δ^*-β'. Hence δ^*-β' is also a δ^*- Base for the discrete tritopology (T, Ω, ρ) .

3- δ^*- Sub base for a tritopology :

3.1 Definition :

Let (X, T, Ω, ρ) be a tritopological space . A collection δ^*-β_* of subsets of X is called a δ^*- sub base for the tritopology (T, Ω, ρ) iff : δ^*-$\beta_* \subset \delta^*$.O(X) . And finite intersections of members of δ^*-β_* form a δ^*- base for the tritopology (T, Ω, ρ) .

It follows that δ^*-β_* is a δ^*- sub base for the tritopology (T, Ω, ρ) iff every member of δ^*.O(X) is the union of finite intersections of members of δ^*-β_*

3.2 Remark :

If δ^*.O(X) represent a topology then the tritopology (T, Ω, ρ) has a δ^*- sub base .

Proof :

Clearly by using remark (2.2) .

3.3 Example :

Let X = {a,b,c,d} , T = { X , φ , {a,c,d} } .
, Ω = { X , φ , {b} }
and $\rho = \{X, \varphi, \{a\}\}$

(X, T) , (X, Ω) , (X, ρ) are three topological spaces , then (X, T, Ω, ρ) is a tritopological space , such that :

$$\delta^*.O(X) = \{X, \varphi, \{a\}, \{a,c\}, \{a,d\}, \{a,c,d\}\}$$

Clearly that δ^*.O(X) represent a topology .

Then the collection δ^*-$\beta_* = \{\{a, c\}, \{a, d\}\}$ is a δ^*- sub base for the tritopology (T, Ω, ρ) since the family δ^*-β of finite intersections of δ^*-β_* is given by :

$$\delta^*\text{-}\beta = \{\{a\}, \{a, c\}, \{a, d\}\}$$

Which is a δ^*- base for the tritopology (T, Ω, ρ) .

3.4 Remark :

If δ^*.O(X) does not represent a topology then the tritopology (T, Ω, ρ) has not a δ^*- sub base .

Proof :

Clearly by using remark (2.4) .

3.5 Example :

Let X={a,b,c,d} , $T = \{X, \varphi, \{c, d\}\}$
, $\Omega = \{X, \varphi, \{a, b, c\}, \{a\}\}$
and $\rho = \{X, \varphi, \{d\}, \{c, d\}, \{a, d\}\}$

(X, T) , (X, Ω) , (X, ρ) are three topological spaces , then (X, T, Ω, ρ) is a tritopological space , such that :

$$\delta^{*}.O(X) = \{X, \varphi, \{a,d\}, \{c,d\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\}$$

Clearly that δ^{*}.O(X) does not represent a topology .

Then this tritopology has not a δ^{*}- sub base because it has not δ^{*}- base because δ^{*}.O(X) does not represent a topology , then the second condition of the definition (2.1) is not hold .

Hence there is no a δ^{*}- sub base .

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