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Solving One-Dimensional Non-Linear Klein-Gordon Equations via Combining Adomian Polynomials and Rohit Transform Method

Rohit Gupta[1a*] and Rahul Gupta[2b]
[1]*Department of Physics (Applied Sciences), Yogananda College of Engineering & Technology, Jammu, India*
[2]*Department of Physics, G. D. Public School, Jammu, India*
[b]E-mail: guptara702@gmail.com
[a*]Corresponding author: guptartohit565@gmail.com

Abstract

In this study, one-dimensional non-linear Klein-Gordon equations are solved by applying the integral transform known as the Rohit transform method. The approximate solutions of one-dimensional non-linear Klein- Gordon equations are obtained by combining the Adomian polynomials with the Rohit transform. To show the effectiveness and performance of the Rohit transform method, five one-dimensional non-linear Klein- Gordon type equations are considered and solved. The graphs of the solutions obtained are plotted to indicate the generality and clarity of the proposed method. It can be easily verified that the proposed method yielded the results that satisfy their corresponding non-linear Klein-Gordon equations. The integral Rohit transform combined with Adomian polynomials brought progressive methodologies that offer new insights on the problems (i.e., one-dimensional non-linear Klein-Gordon) examined in the paper, distinguishing itself from existing methods and doubtlessly beginning up new research instructions. Moreover, it reduces the complexity that occurs when non-linear Klein-Gordon are solved by other methods available in the literature. It provided precise results for the specific problems discussed in the paper, surpassing the capabilities of different methods in terms of decision, constancy, or robustness to noise and disturbances.

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1. Introduction

The Klein-Gordon equation is very significant in quantum mechanics, condensed matter physics, chemical kinetics, fluid dynamics and solid-state physics, etc. Generally, it is written in the form [1, 2]:

$$\frac{\partial^2 v(x,t)}{\partial t^2} - k^2 \frac{\partial^2 v(x,t)}{\partial x^2} + g(v) = p(x,t) \quad (1)$$

with the initial conditions

$$v(x,0) = f(x) \text{ and } \frac{\partial v(x,0)}{\partial t} = s(x)$$

where: $p(x,t)$ is a source term, k is a constant, g is a non-linear function of v, and f and s are functions of x and t. Many advanced techniques have been used to obtain the approximate solutions of the one-dimensional non-linear Klein-Gordon equations such as Elzaki transform technique [3], homotopy perturbation technique [4, 5]. Adomian decomposition technique [6-8], Variation iterational technique [9], exponential function technique [10], the homotopy analysis technique [11], Sobolev Gradients [12], variational method and finite element approach [13], radial basis functions method [14], collocation method [15], reduced differential transform method [16, 17], homotopy perturbation Mohand transform method [18], local fractional derivative operators [19], Laplace decomposition method [20, 21]. In this study, one-dimensional non-linear Klein-Gordon equations were solved via combining the Adomian polynomials and integral transform known as Rohit transform method. The solutions were expressed in the form of a series that yielded the analytical solution with few iterations. To show the effectiveness and performance of the Rohit transform combined with Adomian polynomials, five one-dimensional non-linear Klein-Gordon type equations were considered and solved. It provided precise results for the specific problems discussed in the paper, surpassing the capabilities of other methods in terms of decision, constancy, or robustness to noise and disturbances. The study is organized as follows: Section 2 displays the basic properties of the Rohit transform, section 3 presents the theoretical approach of the method used on the considered equations, and section 4 shows the potency of the Rohit transform technique combined with the Adomian polynomials to solve some one-dimensional non-linear Klein-Gordon equations.

2. Properties of Rohit Transform Method

The Rohit Transform (RT) [22-24] is put into words for a function of exponential order by the integral equations as:

$$\mathrm{R}\{\mathrm{h(t)}\} = q^3 \int_0^\infty e^{-qt}\, h(t)dt,\ t \geq 0\ , q_1 \leq q \leq q_2.$$

The variable q is used to factor the variable t in the argument of the function h.

The Rohit transform (RT) of unidentified functions [25-28] is given by:

i. $\mathrm{R}\{t^n\} = q^3 \int_0^\infty e^{-qt}\, t^n\, dt = \int_0^\infty e^{-z} \left(\frac{z}{q}\right)^n \frac{dz}{q}\ , z = qt$

$$\mathrm{R}\{t^n\} = \frac{q^2}{q^n}\int_0^\infty e^{-z}\,(z)^n dz = \frac{q^2}{q^n}\lceil(n+1) = \frac{q^2}{q^n} n! = \frac{n!}{q^{n-2}}$$

Hence $R\{t^n\} = \frac{n!}{q^{n-2}}$

ii. $\mathrm{R}\{ sinbt\} = q^3 \int_0^\infty e^{-qt}\, sinbt\, dt = q^3 \int_0^\infty e^{-qt} \left(\frac{e^{ibt} - e^{-ibt}}{2i}\right) dt$

$$\mathrm{R}\{ sinbt\} = q^3 \int_0^\infty \left(\frac{e^{-(q-ib)t} - e^{-(q+ib)t}}{2i}\right) dt = -\frac{q^3}{2i(q-ib)}\,(e^{-\infty} - e^{-0}) + \frac{q^3}{2i(q+ib)}\,(e^{-\infty} - e^{-0})$$

$$\mathrm{R}\{ sinbt\} = \frac{q^3}{2i(q-ib)} - \frac{q^3}{2i(q+ib)} = \frac{b\, q^3}{q^2+b^2}$$

Hence $R\{sinbt\} = \frac{b\, q^3}{q^2+b^2}$

iii. $\mathrm{R}\{ cosbt\} = q^3 \int_0^\infty e^{-qt}\, cosbt\, dt = q^3 \int_0^\infty e^{-qt} \left(\frac{e^{ibt} + e^{-ibt}}{2}\right) dt$

$$\mathrm{R}\{\,cosbt\} = \; q^3 \int_0^\infty \left(\frac{e^{-(q-ib)t} + e^{-(q+ib)t}}{2}\right) dt$$

$\mathrm{R}\{\,cosbt\} = -\frac{q^3}{2(q-ib)}\,(\,e^{-\infty} - \,e^{-0}) \; - \frac{q^3}{2(q+ib)}\,(\,e^{-\infty} - \,e^{-0}) \; = \; \frac{q^3}{2(q-ib)} + \frac{q^3}{2(q+ib)} \; = \frac{q^4}{q^2+b^2}$

Hence $R\,\{cosbt\} = \frac{q^4}{q^2+b^2}$

iv. $\mathrm{R}\,\{e^{bt}\} = \; q^3 \int_0^\infty e^{-qt}\, e^{bt}\, dt = \; q^3 \int_0^\infty \left(e^{-(q-b)t}\right) dt = -\frac{q^3}{(q-b)}\,(\,e^{-\infty} - \,e^{-0}) \; = \frac{q^3}{(q-b)}$

Hence $R\,\{e^{bt}\} = \frac{q^3}{q-b}$

Let g(t) be a piecewise continuous function in some interval, then the Rohit Transform (RT) of $g'(t)$ is given by:

$$R\,\{g'(t)\} = \; q^3 \int_0^\infty e^{-qt}\, g'(t) dt$$

Integrating by parts and applying limits, we get:

$$\mathrm{R}\,\{\mathrm{g}'(\mathrm{t})\} = \; \mathrm{q}^3 \left[\mathrm{g}(0) - \int_0^\infty -\mathrm{q}\mathrm{e}^{-\mathrm{qt}}\, \mathrm{g}(\mathrm{t})\mathrm{dt}\right] = \; \mathrm{q}^3 \left[-\mathrm{g}(0) + \mathrm{q} \int_0^\infty \mathrm{e}^{-\mathrm{qt}}\, \mathrm{g}(\mathrm{t})\mathrm{dt}\right]$$

$\mathrm{R}\,\{\mathrm{g}'(\mathrm{t})\} = \mathrm{qR}\{\mathrm{g}(\mathrm{t})\} - \; \mathrm{q}^3\mathrm{g}(0)$

Hence $\mathrm{R}\,\{\mathrm{g}'(\mathrm{t})\} = \mathrm{qG}(\mathrm{q}) - \; \mathrm{q}^3\mathrm{g}(0)$

On replacing g(t) by$\mathrm{g}'(\mathrm{t})$and $\mathrm{g}'(\mathrm{t})$ by $\mathrm{g}''(\mathrm{t})$, we have

$\mathrm{R}\{\mathrm{g}''(\mathrm{t})\} = \mathrm{qR}\{\mathrm{g}'(\mathrm{t})\} - \; \mathrm{q}^3\mathrm{g}'(0) = \mathrm{q}\{\mathrm{qR}\{\mathrm{g}(\mathrm{t})\} - \; \mathrm{q}^3\mathrm{g}(0)\} - \; \mathrm{q}^3\mathrm{g}'(0)$

$\mathrm{R}\{\mathrm{g}''(\mathrm{t})\} = \mathrm{q}^2\mathrm{R}\{\mathrm{g}(\mathrm{t})\} - \; \mathrm{q}^4\mathrm{g}(0) - \; \mathrm{q}^3\mathrm{g}'(0) = \mathrm{q}^2\mathrm{G}(\mathrm{q}) - \; \mathrm{q}^4\mathrm{g}(0) - \; \mathrm{q}^3\mathrm{g}'(0)$

Hence $\mathrm{R}\{\mathrm{g}''(\mathrm{t})\} = \mathrm{q}^2\mathrm{G}(\mathrm{q}) - \; \mathrm{q}^4\mathrm{g}(0) - \; \mathrm{q}^3\mathrm{g}'(0)$

Similarly, $\mathrm{R}\{\mathrm{g}'''(\mathrm{t})\} = \mathrm{q}^3\mathrm{G}(\mathrm{q}) - \mathrm{q}^5\mathrm{g}(0) \; - \; \mathrm{q}^4\mathrm{g}'(0) - \; \mathrm{q}^3\mathrm{g}''(0)$.

In general, $\mathrm{R}\{\mathrm{g}^\mathrm{n}(\mathrm{t})\} = \mathrm{q}^\mathrm{n}\mathrm{R}\{\mathrm{g}(\mathrm{t})\} - \sum_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{q}^{\mathrm{n}-\mathrm{k}+3}\, \mathrm{g}^{\mathrm{k}-1}(0)$

$$\text{or } \mathrm{R}\{\mathrm{g}^\mathrm{n}(\mathrm{t})\} = \mathrm{q}^\mathrm{n}\mathrm{G}(\mathrm{q}) - \sum_{\mathrm{k}=0}^{\mathrm{n}-1} \mathrm{q}^{\mathrm{n}-\mathrm{k}+2}\, \mathrm{g}^\mathrm{k}(0)$$

3. Theoretical Approach of the Rohit Transform Method

The focus of this study is on solving the one-dimensional non-linear Klein-Gordon equations. The Adomian polynomial is combined with the Rohit transform method to obtain the exact solutions of the one-dimensional non-linear Klein-Gordon equations.

Considering an equation of the form:

$$\frac{\partial^n v(x,t)}{\partial t^n} + Lv(x,t) + Nv(x,t) = h(x,t) \quad (2)$$

where n is a non-zero positive integer and the initial conditions are: $\frac{\partial^{n-1} v(x,t)}{\partial t^n} = g_{n-1}(x)$ where: L indicates the linear differential equation, N is the non-linear terms of the differential equations and $h(x,t)$ are the source terms.

Applying the Rohit transform to Eq. (2), we get:

$$q^n V(x,q) - \sum_{k=0}^{n-1} q^{n-k+2}\, v^k(x,0) + R\{Lv(x,t)\} + R\{Nv(x,t)\} = R\{h(x,t)\}$$

$$q^n V(x,q) = R\{h(x,t)\} + \sum_{k=0}^{n-1} q^{n-k+2}\, v^k(x,0) - R\{Lv(x,t)\} - R\{Nv(x,t)\} \quad (3)$$

Thus, simplifying Eq. (3), we obtain:

$$V(x,q) = q^{-n} R\{h(x,t)\} + \sum_{k=0}^{n-1} q^{-k+2}\, v^k(x,0) - q^{-n}[R\{Lv(x,t)\} - R\{Nv(x,t)\} \quad (4)$$

Applying the inverse Rohit transform on Eq. (4) gives:

$$v(x,t) = R^{-1}\left\{\begin{array}{c} q^{-n}R\{h(x,t)\} \\ + \sum_{k=0}^{n-1} q^{-k+2}\, v^k(x,0) \end{array}\right\} - R^{-1}\left\{q^{-n}\begin{bmatrix} R\{Lv(x,t)\} + \\ R\{Nv(x,t)\} \end{bmatrix}\right\} \quad (5)$$

Eq. (5) can then be written as:

$$v(x,t) = f(x,t) - R^{-1}\{q^{-n}[R\{Lv(x,t)\} + R\{Nv(x,t)\}]\} \quad (6)$$

where $f(x,t)$ is the expression that arises from the initial conditions given and the source terms after it has been simplified. The solution is expressed as:

$$v(x,t) = \sum_{i=0}^{\infty} v_i(x,t) \quad (7)$$

The non-linear part is reduced to:

$$Nv(x,t) = \sum_{i=0}^{\infty} A_i \quad (8)$$

where A_i are the Adomian polynomials [29-31].

Substituting Eqs. (7) and (8) into Eq. (6) to obtain;

$$\sum_{i=0}^{\infty} v_i(x,t) = f(x,t) - R^{-1}\left\{q^{-n}\left[R\left\{L\sum_{i=0}^{\infty} v_i(x,t)\right\} + R\left\{\sum_{i=0}^{\infty} A_i\right\}\right]\right\} \quad (9)$$

Then from Eq. (9), for i = 0, we get:

$$v_0(x,t) = f(x,t) \quad (10)$$

The recursive relation is expressed as:

$$\mathrm{v_{i+1}(x,t) = -R^{-1}\{q^{-n}[R\{Lv_i(x,t)\} + R\{A_i\}]\}, where\ i\ \geq\ 0} \quad (11)$$

The analytical solution $v(x,t)$ can be approximated by the truncated series:

$$v(x,t) = \lim_{n\to\infty}\sum_{i=0}^{n} v_i(x,t)$$

4. Applications of Rohit Transform to the One-Dimensional Non-Linear Klein-Gordon Equations

4.1. Example 1

Considering the following non-homogenous non-linear Klein-Gordon equation:

$$\mathrm{\frac{\partial^2 v(x,t)}{\partial t^2} - \frac{\partial^2 v(x,t)}{\partial x^2} + v^2 = 6xt(x^2 - t^2) + x^6t^6, 0 \leq x \leq c\ (a\ constant)\ and\ t \geq 0} \quad (12)$$

with the initial conditions: $\mathrm{v(x,0) = 0}$ and $\mathrm{\frac{\partial v(x,0)}{\partial t} = 0}$
Applying the Rohit transform on Eq. (12), we obtain;

$$\mathrm{q^2V(x,q) - q^4v(x,0) - q^3v'(x,0) - R\left\{\frac{\partial^2 v(x,t)}{\partial x^2}\right\} + R\{v^2\} = R\{6xt(x^2 - t^2) + x^6t^6\}}$$

$$\mathrm{q^2V(x,q) + R\{v^2\} = R\{6xt(x^2 - t^2) + x^6t^6\} + R\left\{\frac{\partial^2 v(x,t)}{\partial x^2}\right\}}$$

$$\mathrm{q^2V(x,q) = R\{6xt(x^2 - t^2) + x^6t^6\} + R\left\{\frac{\partial^2 V(x,t)}{\partial x^2}\right\} - R\{v^2\}}$$

$$\mathrm{V(x,q) = q^{-2}R\{6xt(x^2 - t^2) + x^6t^6\} + q^{-2}R\left\{\frac{\partial^2 v(x,t)}{\partial x^2}\right\} - q^{-2}R\{v^2\}} \quad (13)$$

Taking the inverse Rohit transform of Eq. (13), we get:

$$\mathrm{v(x,t) = R^{-1}\{q^{-2}R\{6xt(x^2 - t^2) + x^6t^6\} + q^{-2}R\left\{\frac{\partial^2 v(x,t)}{\partial x^2}\right\} - q^{-2}R\{v^2\}\}}$$

$$\mathrm{v(x,t) = R^{-1}\{q^{-2}R\{6xt(x^2 - t^2) + x^6t^6\}\} + R^{-1}\left\{q^{-2}R\left\{\frac{\partial^2 v(x,t)}{\partial x^2}\right\} - q^{-2}R\{v^2\}\right\}}$$

$$\mathrm{v(x,t) = R^{-1}\{q^{-2}R\{6xt(x^2 - t^2) + x^6t^6\}\} + R^{-1}\left\{q^{-2}R\left\{\frac{\partial^2 v(x,t)}{\partial x^2}\right\} - q^{-2}R\{v^2\}\right\}} \quad (14)$$

From Eq. (14):

$$\mathrm{v_0 = R^{-1}\{q^{-2}R\{6xt(x^2 - t^2) + x^6t^6\}\}}$$

$$v_0 = R^{-1}\left\{q^{-2}\left\{6x^3q - 6x\frac{3!}{q} + x6\frac{6!}{q^4}\right\}\right\}$$

$$v_0 = R^{-1}\left\{\frac{6x^3}{q} - 6x\frac{3!}{q^3} + x6\frac{6!}{q^6}\right\}$$

$$v_0 = x^3t^3 - \frac{3xt^5}{10} + \frac{x^6t^8}{56} \quad (15)$$

Therefore, the recursive relation is expressed as:

$$v_{m+1} = R^{-1}\left\{q^{-2}\left(\frac{\partial^2}{\partial x^2}R[v_i] - R[A_i]\right)\right\} \quad (16)$$

For i = 0, Eq. (17) becomes:

$$v_1 = R^{-1}\left\{q^{-2}\left(\frac{\partial^2}{\partial x^2}R[v_0] - R[A_0]\right)\right\} \quad (17)$$

$$v_1 = \left(\frac{3xt^5}{10} + \frac{x^4t^{10}}{168}\right) - \left(\frac{x^6t^8}{56} + \frac{x^{12}t^{18}}{56.56.17.18} + \frac{9x^2t^{12}}{100.12.11} - \frac{3x^7t^{15}}{280.15.14} + \frac{x^9t^{13}}{28.13.12} - \frac{3x^4t^{10}}{5.10.9}\right) \quad (18)$$

Cancelling the noise terms $\frac{3xt^5}{10}$ and $\frac{x^6t^8}{56}$ from the components of v_0 and verifying that the remaining non-cancelled terms of v_0 satisfy Eq. (12), we find the approximate solution, which is given by:

$$v(x,t) = v_0 + v_1 \ldots$$

On substituting the values of $v_0, v_1, \ldots$, we get:

$$v(x,t) = x^3t^3$$

The numerical solution for Example 1 is shown in Fig. 1.

4.2. Example 2

Considering the given non-homogenous non-linear Klein-Gordon equation

$$\frac{\partial^2 v(x,t)}{\partial t^2} - \frac{\partial^2 v(x,t)}{\partial x^2} + v^2 = x^2\cos^2 t - x\cos t, \quad 0 \le x \le c(\text{a constant}) \text{ and } t \ge 0 \quad (20)$$

with the initial conditions:

$$v(x,0) = x \text{ and } \frac{\partial v(x,0)}{\partial t} = 0 \quad (21)$$

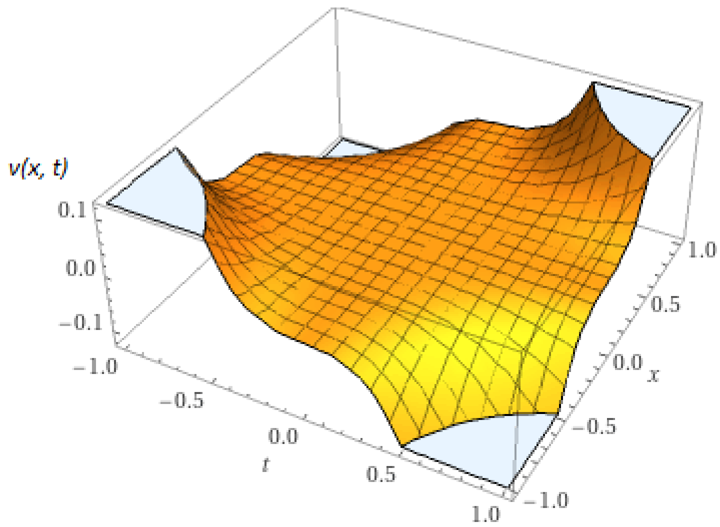

Figure 1: Numerical solution for example 1.

Applying the Rohit transform on Eq. (20), we obtain:

$$q^2V(x,q) - q^4v(x,0) - q^3v'(x,0) - R\left\{\frac{\partial^2 v(x,t)}{\partial x^2}\right\} + R\{v^2 = R\{x^2\cos^2 t - x\cos t\}$$

$$q^2V(x,q) - q^4x + R\{v^2\} = R\{x^2\cos^2 t - x\cos t\} + R\left\{\frac{\partial^2 v(x,t)}{\partial x^2}\right\}$$

$$q^2V(x,q) = R\{x^2\cos^2 t - x\cos t\} + R\left\{\frac{\partial^2 V(x,t)}{\partial x^2}\right\} + q^4x - R\{v^2\}$$

$$V(x,q) = q^{-2}R\{x^2\cos^2 t - x\cos t\} + q^{-2}R\left\{\frac{\partial^2 v(x,t)}{\partial x^2}\right\} + q^2x - q^{-2}R\{v^2\} \qquad \mathbf{(22)}$$

Taking the inverse Rohit transform of Eq. (22), we get:

$$v(x,t) = R^{-1}\{q^{-2}R\{x^2\cos^2 t - x\cos t\} + q^{-2}R\left\{\frac{\partial^2 v(x,t)}{\partial x^2}\right\} + q^2x - q^{-2}R\{v^2\}\}$$

$$v(x,t) = R^{-1}\{q^{-2}R\{x^2\cos^2 t - x\cos t\}\} + R^{-1}\left\{q^{-2}R\left\{\frac{\partial^2 v(x,t)}{\partial x^2}\right\} + q^2x - q^{-2}R\{v^2\}\right\}$$

$$v(x,t) = R^{-1}\{q^{-2}R\{x^2\cos^2 t - x\cos t\} + q^2x\} + R^{-1}\left\{q^{-2}R\left\{\frac{\partial^2 v(x,q)}{\partial x^2}\right\} - q^{-2}R\{u^2\}\right\} \qquad \mathbf{(23)}$$

From Eq. (23):

$$v_0 = R^{-1}\{q^{-2}R\{x^2\cos^2 t - x\cos t\} + q^2x\}$$

$$v_0 = R^{-1}\left\{q^{-2}R\left\{\frac{x^2}{2}(1+\cos 2t) - x\cos t\right\} + q^2x\right\}$$

$$v_0 = R^{-1}\left\{q^{-2}\left\{\frac{x^2}{2}\left(q^2+\frac{q^4}{q^2+4}\right) - x\,\frac{q^4}{q^2+1}\right\} + q^2x\right\}$$

$$v_0 = R^{-1}\left\{\frac{x^2}{2}\left(1+\frac{q^2}{q^2+4}\right) - x\,\frac{q^2}{q^2+1} + q^2x\right\}$$

$$v_0 = R^{-1}\left\{\frac{x^2}{2}\left(1+\frac{1}{4}\left(q^2-\frac{q^4}{q^2+4}\right)\right) + q^2x - x\left(q^2-\frac{q^4}{q^2+1}\right)\right\}$$

$$v_0 = R^{-1}\left\{\frac{x^2}{2}\left(1+\frac{1}{4}\left(q^2-\frac{q^4}{q^2+4}\right)\right) + x\,\frac{q^4}{q^2+1}\right\}$$

$$v_0 = \left\{\frac{x^2}{2}\left(\frac{t^2}{2}+\frac{1}{4}(1-\cos 2t)\right) + x\cos t\right\}$$

$$v_0 = x\cos t + \frac{x^2t^2}{4} + \frac{x^2}{8} - \frac{x^2}{8}\cos 2t \quad (24)$$

Therefore, the recursive relation is expressed as:

$$v_{m+1} = R^{-1}\left\{q^{-2}\left(\frac{\partial^2}{\partial x^2}R[v_i] - R[A_i]\right)\right\} \quad (25)$$

For i = 0, Eq. (25) becomes:

$$v_1 = R^{-1}\left\{q^{-2}\left(\frac{\partial^2}{\partial x^2}R[v_0] - R[A_0]\right)\right\} \quad (26)$$

Thus, simplifying Eq. (26) gives:

$$v_1 = \left(\frac{25t^4}{96} + \frac{1}{16}(1-\cos 2t)\right) - \left(\frac{x^2t^2}{4} + \frac{x^2}{8} - \frac{x^2}{8}\cos 2t \ldots\right) \quad (27)$$

On cancelling the noise terms $\frac{x^2t^2}{4}, \frac{x^2}{8}$ and $\frac{x^2}{8}\cos 2t$ from the components of v_0 and verifying that the remaining non-cancelled terms of v_0 satisfy Eq. (20), we find the approximate solution, which is given by:

$$v(x,t) = v_0 + v_1 \ldots$$

On substituting the values of $v_0, v_1, \ldots$, we get:

$$v(x,t) = x\cos t \quad (28)$$

The numerical solution for Example 2 is shown in Fig. 2.

4.3. Illustration Three

Considering the non-homogenous non-linear Klein Gordon equation:

$$\frac{\partial^2 v(x,t)}{\partial t^2} - \frac{\partial^2 v(x,t)}{\partial x^2} - v + v^2 = xt + x^2t^2, 0 \le x \le c \text{ (a constant) and } t \ge 0 \quad (29)$$

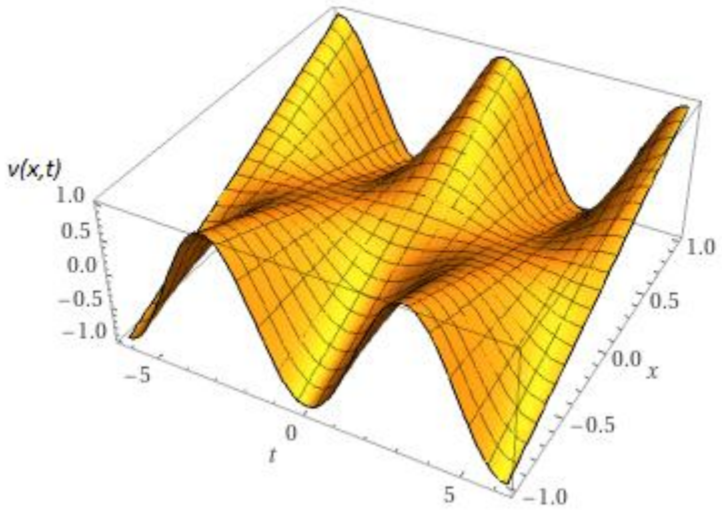

Figure 2: Numerical solution for example 2.

with the initial conditions:

$$v(x,0) = 1 \text{ and } \frac{\partial v(x,0)}{\partial t} = x \quad (30)$$

Applying the Rohit transform on Eq. (29), we obtain:

$$q^2 V(x,q) - q^4 v(x,0) - q^3 v'(x,0) - R\left\{\frac{\partial^2 v(x,t)}{\partial x^2}\right\} - R\{v\} + R\{v^2 = R\{xt + x^2 t^2\}$$

$$q^2 V(x,q) = q^4 + q^3 x + xq + 2x^2 + R\left\{\frac{\partial^2 v(x,t)}{\partial x^2}\right\} + R\{v\} - R\{v^2\}$$

$$V(x,q) = q^2 + q^1 x + x\,q^{-1} + 2x^2\,q^{-2} + q^{-2}\left(R\left\{\frac{\partial^2 v(x,t)}{\partial x^2}\right\} + R\{v\} - R\{v^2\}\right) \quad (31)$$

Taking the inverse Rohit transform of Eq. (31), we get:

$$v(x,t) = R^{-1}\{q^2 + q^1 x + x\,q^{-1} + 2x^2\,q^{-2}\} + R^{-1}\left\{q^{-2} R\left\{\frac{\partial^2 v(x,t)}{\partial x^2}\right\} + q^{-2} R\{v\} - q^{-2} R\{v^2\}\right\} \quad (32)$$

From Eq. (32):

$$v_0 = R^{-1}\{q^2 + q^1 x + x\,q^{-1} + 2x^2\,q^{-2}\}$$

$$v_0 = 1 + xt + \frac{xt^3}{6} + \frac{x^2\,t^4}{12} \quad (33)$$

The recursive relation is expressed as:

$$v_{m+1} = R^{-1}\left\{q^{-2} R\left\{\frac{\partial^2 v(x,t)}{\partial x^2}\right\} + q^{-2} R\{v\} - q^{-2} R\{v^2\}\right\} \quad (34)$$

For $i = 0$, Eq. (34) becomes:

$$v_1 = R^{-1}\left\{q^{-2}\left(\frac{\partial^2}{\partial x^2}R[v_0] + R[v_0] - R[A_0]\right)\right\} \tag{35}$$

where

$$A_0 = {v_0}^2 = \left(1 + xt + \frac{xt^3}{6} + \frac{x^2\,t^4}{12}\right)^2$$
$$= 1 + 2xt + \frac{xt^3}{3} + \frac{x^2\,t^4}{6} + x^2\,t^2 + \frac{xt^3}{3} + \frac{x^2\,t^4}{6} + \frac{x^2\,t^6}{36} + \frac{x^4\,t^8}{144} + \frac{x^3\,t^7}{36}$$
$$= 1 + 2xt + \frac{2xt^3}{3} + \frac{x^2\,t^4}{3} + x^2\,t^2 + \frac{x^2\,t^6}{36} + \frac{x^3\,t^7}{36} + \frac{x^4\,t^8}{144}$$

$$R[A_0] = q^2 + 2xq + \frac{4x}{q} + \frac{8x^2}{q^2} + 2x^2 + \frac{20x^2}{q^4} + \frac{140x^3}{q^5} + \frac{280x^4}{q^6}$$

Therefore, from Eq. (35):

$$v_1 = \frac{t^6}{360} + \frac{t^2}{2} + \frac{xt^3}{6} + \frac{x\,t^5}{120} + \frac{x^2\,t^6}{360} - \frac{t^2}{2} - \frac{xt^3}{3} - \frac{xt^5}{30} - \frac{x^2t^6}{90} - \frac{x^2t^4}{12} - \frac{20x^2t^8}{8!} - \frac{140x^3t^9}{9!} - \frac{280x^4t^{10}}{10!}$$

$$v_1 = \frac{t^6}{360} - \frac{xt^3}{6} - \frac{x\,t^5}{40} - \frac{x^2t^4}{12} - \frac{x^2\,t^6}{120} - \frac{20x^2t^8}{8!} - \frac{140x^3t^9}{9!} - \frac{280x^4t^{10}}{10!} \tag{36}$$

On canceling the noise terms $\frac{xt^3}{6}$ and $\frac{x^2t^4}{12}$ from the components of v_0 and verifying that the remaining non-canceled terms of v_0 satisfy Eq. (29), we find the approximate solution, which is given by:

$$v(x, t) = v_0 + v_1 \ldots$$

On substituting the values of $v_0, v_1, \ldots$, we get

$$v(x, t) = 1 + xt \tag{37}$$

The numerical solution for Example 3 is shown in Fig. 3.

4.4. Example 4

Considering the following non-homogenous non-linear Klein-Gordon equation:

$$\frac{\partial^2 v(x,t)}{\partial t^2} - \frac{\partial^2 v(x,t)}{\partial x^2} + v^2 = x^2t^2, \qquad 0 \le x \le c \text{ (a constant) and } t \ge 0 \tag{38}$$

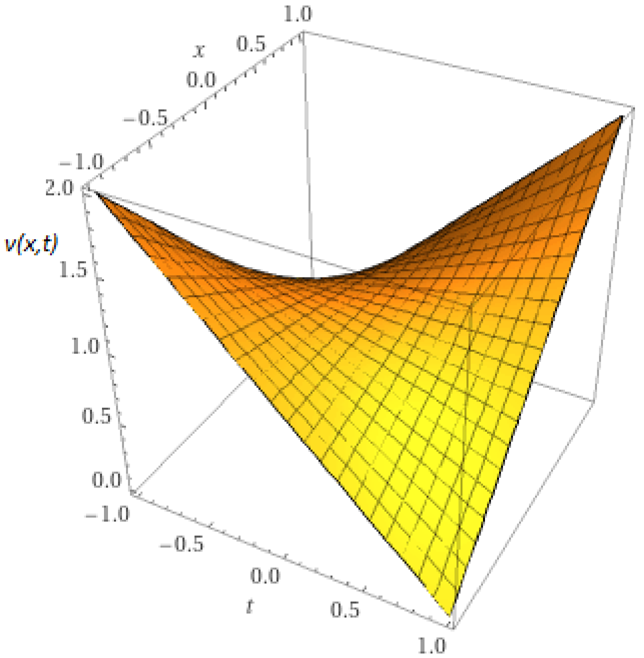

Figure 3: Numerical solution for example 3.

with the initial conditions:

$$v(x,0) = 0 \text{ and } \frac{\partial v(x,0)}{\partial t} = x \tag{39}$$

Applying the Rohit transform on Eq. (38), we obtain:

$$q^2V(x,q) - q^4v(x,0) - q^3v'(x,0) - R\left\{\frac{\partial^2 v(x,t)}{\partial x^2}\right\} + R\{v^2\} = R\{x^2t^2\}$$

$$q^2V(x,q) - q^3x + R\{v^2\} = R\{x^2t^2\} + R\left\{\frac{\partial^2 v(x,t)}{\partial x^2}\right\}$$

$$q^2V(x,q) = R\{x^2t^2\} + R\left\{\frac{\partial^2 V(x,t)}{\partial x^2}\right\} - R\{v^2\} + q^3x$$

$$V(x,q) = q^{-2}R\{x^2t^2\} + q^{-2}R\left\{\frac{\partial^2 v(x,t)}{\partial x^2}\right\} - q^{-2}R\{v^2\} + qx \tag{40}$$

Taking the inverse Rohit transform of Eq. (40), we get:

$$v(x,t) = R^{-1}\{q^{-2}R\{x^2t^2\} + q^{-2}R\left\{\frac{\partial^2 v(x,t)}{\partial x^2}\right\} - q^{-2}R\{v^2\} + qx\}$$

$$v(x,t) = R^{-1}\{q^{-2}R\{x^2t^2\} + qx\} + R^{-1}\left\{q^{-2}R\left\{\frac{\partial^2 v(x,t)}{\partial x^2}\right\} - q^{-2}R\{v^2\}\right\}$$

$$v(x,t) = R^{-1}\{q^{-2}R\{x^2t^2\} + qx\} + R^{-1}\left\{q^{-2}R\left\{\frac{\partial^2 v(x,t)}{\partial x^2}\right\} - q^{-2}R\{v^2\}\right\} \tag{41}$$

From Eq. (41):

$$v_0 = R^{-1}\{q^{-2}\, R\{\, x^2t^2\} + qx\}$$

$$v_0 = R^{-1}\{q^{-2}\, \{2x^2\} + qx\}$$

$$v_0 = \frac{x^2t^4}{12} + xt \qquad \mathbf{(42)}$$

Therefore, the recursive relation is expressed as:

$$v_{m+1} = R^{-1}\left\{q^{-2}\left(\frac{\partial^2}{\partial x^2} R[v_i] - R[A_i]\right)\right\} \qquad \mathbf{(43)}$$

For i = 0, Eq. (43) becomes:

$$v_1 = R^{-1}\left\{q^{-2}\left(\frac{\partial^2}{\partial x^2} R[v_0] - R[A_0]\right)\right\} \qquad \mathbf{(44)}$$

$$v_1 = \frac{t^6}{180} - \frac{x^2t^4}{12} - \frac{x^3t^7}{252} + \frac{x^4t^{10}}{12960} \qquad \mathbf{(45)}$$

On canceling the noise term $\frac{x^2t^4}{12}$ from the components of v_0 and verifying that the remaining non-canceled terms of v_0 satisfy Eq. (38), we find the approximate solution, which is given by:

$$v(x,t) = v_0 + v_1 \dots$$

On substituting the values of $v_0, v_1, \dots$, we get:

$$v(x,t) = xt \qquad \mathbf{(46)}$$

The numerical solution for Example 4 is shown in Fig. 4.

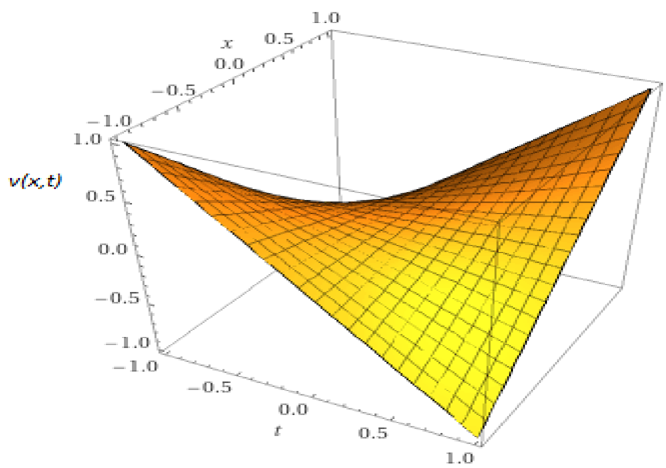

Figure 4: Numerical solution for example 4.

4.5. Example 5

Considering the following non-homogenous non-linear Klein-Gordon equation:

$$\frac{\partial^2 v(x,t)}{\partial t^2} - \frac{\partial^2 v(x,t)}{\partial x^2} + v^2 = 2\,(x^2 - t^2) + x^4 t^4,$$
$$0 \le x \le c \text{ (a constant) and } t \ge 0 \qquad (47)$$

with the initial conditions:

$$v(x,0) = 0 \text{ and } \frac{\partial v(x,0)}{\partial t} = 0 \qquad (48)$$

Applying the Rohit transform on Eq. (47), we obtain:

$$q^2 V(x,q) - q^4 v(x,0) - q^3 v'(x,0) - R\left\{\frac{\partial^2 v(x,t)}{\partial x^2}\right\} + R\{v^2\} = R\{2\,(x^2 - t^2) + x^4 t^4\}$$

$$q^2 V(x,q) + R\{v^2\} = R\{2\,(x^2 - t^2) + x^4 t^4\} + R\left\{\frac{\partial^2 v(x,t)}{\partial x^2}\right\}$$

$$q^2 V(x,q) = R\{2\,(x^2 - t^2) + x^4 t^4\} + R\left\{\frac{\partial^2 V(x,t)}{\partial x^2}\right\} - R\{v^2\}$$

$$V(x,q) = q^{-2} R\{2\,(x^2 - t^2) + x^4 t^4\} + q^{-2} R\left\{\frac{\partial^2 v(x,t)}{\partial x^2}\right\} - q^{-2} R\{v^2\} \qquad (49)$$

Taking the inverse Rohit transform of Eq. (49), we get:

$$v(x,t) = R^{-1}\left\{q^{-2} R\{2\,(x^2 - t^2) + x^4 t^4\} + q^{-2} R\left\{\frac{\partial^2 v(x,t)}{\partial x^2}\right\} - q^{-2} R\{v^2\}\right\}$$

$$v(x,t) = R^{-1}\{q^{-2} R\{2\,(x^2 - t^2) + x^4 t^4\}\} + R^{-1}\left\{q^{-2} R\left\{\frac{\partial^2 v(x,t)}{\partial x^2}\right\} - q^{-2} R\{v^2\}\right\}$$

$$v(x,t) = R^{-1}\{q^{-2} R\{2\,(x^2 - t^2) + x^4 t^4\}\} + R^{-1}\left\{q^{-2} R\left\{\frac{\partial^2 v(x,t)}{\partial x^2}\right\} - q^{-2} R\{v^2\}\right\} \qquad (50)$$

From Eq. (50):

$$v_0 = R^{-1}\{q^{-2} R\{2\,(x^2 - t^2) + x^4 t^4\}\}$$

$$v_0 = x^2 t^2 - \frac{t^4}{6} + \frac{x^4 t^6}{30} \qquad (51)$$

Therefore, the recursive relation is expressed as:

$$v_{m+1} = R^{-1}\left\{q^{-2}\left(\frac{\partial^2}{\partial x^2}R[v_i] - R[A_i]\right)\right\} \tag{52}$$

For i = 0, Eq. (52) becomes:

$$v_1 = R^{-1}\left\{q^{-2}\left(\frac{\partial^2}{\partial x^2}R[v_0] - R[A_0]\right)\right\} \tag{53}$$

$$v_1 = \frac{t^4}{6} - \frac{t^8}{2016} - \frac{x^4t^6}{30} + \frac{x^2t^7}{210} + \frac{x^2t^8}{140} - \frac{x^6t^{10}}{1350} + \frac{x^4t^{11}}{9900} - \frac{x^8t^{14}}{163800} \tag{54}$$

On canceling the noise terms $\frac{t^4}{6}$ and $\frac{x^4t^6}{30}$ from the components of v_0 and verifying that the remaining non-canceled terms of v_0 satisfy Eq. (47), we find the approximate solution, which is given by:

$$v(x,t) = v_0 + v_1 \ldots$$

On substituting the values of $v_0, v_1, \ldots.$, we get:

$$v(x,t) = x^2t^2 \tag{55}$$

The numerical solution for Example 5 is shown in Fig. 5.

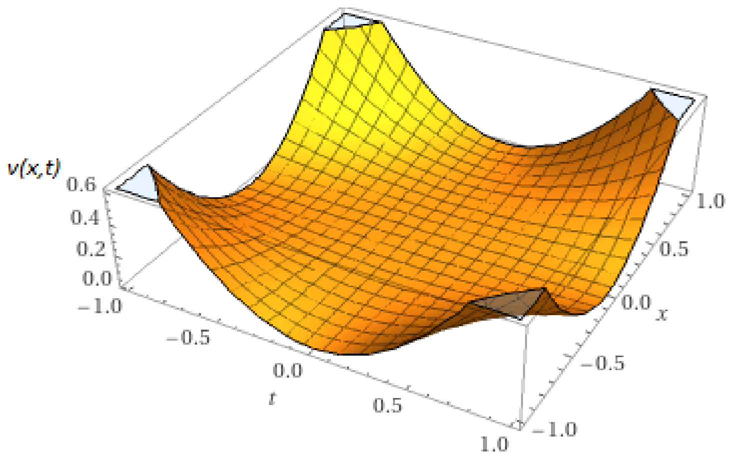

Figure 5: Numerical solution for example 5.

5. Results and Discussion

The Rohit transform method combined with the Adomian polynomials was used to solve the one-dimensional non-linear Klein-Gordon equations. Five examples of the one-dimensional non-linear Klein-Gordon equations were solved with this method. The method is effective in solving one-dimensional non-linear partial (Klein-Gordon) differential equations as the result procured satisfies their corresponding non-linear Klein-Gordon equations. Figs. 1, 2, 3, 4 and 5 show the three-dimensional graphs (i.e., numerical solutions) of the examples considered to give a detailed explanation of the behavior and shape of the one-dimensional non-linear Klein-Gordon equations. The graphs of the solutions obtained are plotted to indicate the generality and clarity of the proposed method. The integral Rohit transform combined with Adomian polynomials brought the progressive principles or methodologies that offer new insights or views on the problems examined in the paper, distinguishing itself from existing methods and doubtlessly beginning new research instructions. Moreover, it reduces the complexity that occurs when non-linear Klein-Gordon are solved by other methods available in the literature. This shows that the proposed technique is suitable and can be operated on other non-linear partial (Klein-Gordon) differential equations of higher integral order as well as fractional order.

6. Conclusions

In this work, the approximate solutions of the one-dimensional non-linear Klein-Gordon equations were obtained by combining the Rohit transform and the Adomian polynomials. The graphs of the obtained solutions were plotted to indicate the generality and clarity of the proposed method. It provided precise results for the specific problems discussed in the paper, surpassing the capabilities of other methods in terms of decision, constancy, or robustness to noise and disturbances. In the future, the Rohit transform may be utilized to solve integral equations of Volterra or Fredholm type and also linear and non-linear Klein-Gordan equations of higher integral order as well as fractional order.

Conflict of Interest

The author(s) declare that there is no conflict of interest.

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