2024, 21(9): 2984-2992
https://doi.org/10.21123/bsj.2024.9198
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
Baghdad Science Journal

Common Fixed-Point Theorems of Generalization of Kannan, Chatterjea, and Reich Contractive on b-Metric Space with an Application

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Received 07/06/2023, Revised 07/01/2023, Accepted 09/01/2024, Published Online First 20/02/2024, Published 01/09/2024

Abstract

This paper, established the existence and uniqueness of common fixed points for Kannan, Reich, and Chatterjea-type pairs of self-maps in complete b-metric space. In addition, an example and an application of the existence and uniqueness of common solutions for a system of functional equations arising in dynamic programming are discussed by using our results

Keywords: *b*-metric, Chatterjea Contraction, Common fixed point, Fixed point, Kannan Contraction, Reich Contraction.

Introduction

The Banach[1] fixed point theorem, also known as the Banach contraction theorem or the Banach contraction principle theorem, was first introduced in 1922. Banach's fixed point theorem guarantees that there is a unique fixed point of the mapping in a metric space and provides a method that can be used to obtain this fixed point. The theorem became the basis for the development of the fixed-point theory by several scientists, such as Kannan[2], Chatterjea[3], and Reich[4]. In 1989, the theory of b-metric spaces which is an extension of the metric spaces was introduced by Bakhtin[5] and popularized and developed by Czerwik[6] in 1993.

Debnath and Srivastava have presented a new existence of the Kannan and Reich fixed point theorem using Wardowski's technique[7] and some of the best proximity point results for Kannan contractive mapping pairs[8]. Debnath et al [9] presents some results of the fixed-point theorem for the Kannan, Reich, and Chatterjea contraction functions that only use metric spaces.

The main aim of this paper is to propose and prove fixed point theorems of functions using generalized Kannan, Reich, and Chatterjea contractions in b-metric spaces.

Materials and Methods

Definition 1[10]: Let X be a non-empty set and $d: X \times X \to \mathbb{R}^+$ is a real-valued function satisfying:

d_1) $d(v, w) = 0$ if and only if $v = w$;

d_2) $d(v, w) = d(w, v)$;

d_3) $d(v, w) \leq d(v, u) + d(u, w)$ for all $u, v, w \in X$.

Then d is called a metric in X, and (X, d) is called a metric space.

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Definition 2[11]: Let X be a non-empty set and let $k > 1$ be a given real number. Let $q: X \times X \to \mathbb{R}^+$ is a real-valued function satisfying:

q_1) $q(v, w) = 0$ if and only if $v = w$;

q_2) $q(v, w) = q(w, v)$;

q_3) $q(v, w) \le k[q(v, u) + q(u, w)]$ for all $u, v, w \in X$.

Then q is called a metric in X, and (X, q) is called b-metric space.

Example 1[12]: Let $X = [0,1]$ then $q(v, w) = (v - w)^2$ is a b-metric on R with $k = 2$.

Definition 3[13]: Let $\{v_n\}$ be a sequence in the metric space (X, d).

1. A sequence $\{v_n\}$ is called convergent if there is $v \in X$ such that $d(v_n, v) \to 0$ when $n \to +\infty$, or in other words for every $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $n \ge n_0$ implies that $d(v_n, v) < \varepsilon$.
2. A sequence $\{v_n\}$ is called convergent if there is $v \in X$ such that $d(v_n, v_m) \to 0$ when $n, m \to +\infty$, or in other words for every $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $n \ge n_0$ implies that $d(v_n, v_m) < \varepsilon$.

Definition 4[13]: Metric space (X, d) is said to be complete if and only if each Cauchy sequence in X is convergent in X.

Definition 3[14]: Let $\{v_n\}$ be a sequence in b-metric space (X, q).

1. A sequence $\{v_n\}$ is called convergent if there is $v \in X$ such that $q(v_n, v) \to 0$ when $n \to +\infty$, or in other words for every $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $n \ge n_0$ implies that $q(v_n, v) < \varepsilon$.
2. A sequence $\{v_n\}$ is called convergent if there is $v \in X$ such that $q(v_n, v_m) \to 0$ when $n, m \to +\infty$, or in other words for every $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $n \ge n_0$ implies that $q(v_n, v_m) < \varepsilon$.

Definition 4[14]: b-metric space (X, q) is said to be complete if and only if each Cauchy sequence in X is convergent in X.

Lemma 1[15]: Let (X, q) be a complete b-metric space with $k \ge 1$ and $\{v_n\} \subset X$ be a sequence in b-metric space satisfying the following equation

$$q(v_n, v_{n+1}) \le \alpha q(v_{n-1}, v_n),$$

For $n \in \mathbb{N}$ and $\alpha < \frac{1}{k}$ then $\{v_n\}$ is a Cauchy sequence in X.

Fixed Point Theorems of Kannan, Reich, and Chatterjea in Metric Space

Theorem 1[2]: Let (X, d) be a complete metric space, $T: X \to X$ be a self-map. Suppose there exists $\gamma \in \left[0, \frac{1}{2}\right)$ such that

$$d(Tv, Tw) \le \gamma[d(v, Tv) + d(w, Tw)],$$

for all $u, v \in X$ with $v \ne w$. Then T has a unique fixed point in X.

Theorem 2[3]: Let (X, d) be a complete metric space, $T: X \to X$ be a self-map. Suppose there exists $\gamma \in \left[0, \frac{1}{2}\right)$ such that

$$d(Tv, Tw) \le \gamma[d(v, Tw) + d(w, Tv)],$$

for all $u, v \in X$ with $v \ne w$. Then T has a unique fixed point in X.

Theorem 3[4]: Let (X, d) be a complete metric space, $T: X \to X$ be a self-map. Suppose there exists non-negative constants a, b, c satisfying $a + b + c < 1$ such that

$$d(Tv, Tw) \le \alpha d(v, Tv) + \beta d(w, Tw) + \gamma d(v, w),$$

for all $u, v \in X$ with $v \ne w$. Then T has a unique fixed point in X.

Fixed Point Theorems of Kannan, Reich, and Chatterjea in Metric Space

Theorem 4[9]: Let (X, d) be a complete metric space, $T_1, T_2: X \to X$ be a pair of self-maps. Suppose there exists $\gamma \in \left(0, \frac{1}{2}\right)$ such that

$$d(T_1 v, T_2 w) \le \gamma[d(v, T_1 v) + d(w, T_2 w)],$$

for all $u, v \in X$ with $v \ne w$. Then T_1 and T_2 have a unique common fixed point in X.

Theorem 5[9]: Let (X, d) be a complete metric space, $T_1, T_2: X \to X$ be a pair of self-maps. Suppose there exists $\gamma \in \left(0, \frac{1}{2}\right)$ such that

$$d(T_1 v, T_2 w) \le \gamma[d(v, T_2 w) + d(w, T_1 v)],$$

for all $u, v \in X$ with $v \ne w$. Then T_1 and T_2 have a unique common fixed point in X.

Theorem 6[9]: Let (X, d) be a complete metric space, $T_1, T_2: X \to X$ be a pair of self-maps. Suppose there exists $\gamma \in \left(0, \frac{1}{3}\right)$ such that

$$d(T_1 v, T_2 w) \le \gamma[d(v, Tv) + d(w, Tw) + d(v, w)],$$

for all $u, v \in X$ with $v \ne w$. Then T_1 and T_2 have a unique common fixed point in X.

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Results and Discussion

Theorem 7 uses a generalization of the Kannan contraction function (if $\beta = \alpha$, then Eq 1 becomes the Kannan contraction function).

Theorem 7: Let (X, q) be a complete b-metric space with $k > 1$ be a given real number and $T_1, T_2: X \to X$ be a pair of self-maps, such that

$$q(T_1 v, T_2 w) \leq \alpha q(v, T_1 v) + \beta q(w, T_2 w), \quad 1$$

for all $u, v \in X$ where $0 < \alpha + \beta < 1$, $k\alpha + \beta < 1$ and $0 < \alpha, \beta < \frac{1}{k}$. Then T_1 and T_2 have a unique common fixed point in X.

Proof. Since X is nonempty, let $v_0 \in X$ and define a sequence $\{v_n\}$ in X inductively by putting $v_{2n+1} = T_1 v_{2n}$ and $v_{2n+2} = T_2 v_{2n+1}$ for $n = 0,1,2,3, \ldots$. Will be shown $v_n \neq v_{n+1}$. Suppose that $v_{n_0} = v_{n_0+1}$ for some $n_0 \in \mathbb{N}$. From Eq 1, obtained

$$\begin{aligned} q(v_{n_0+1}, v_{n_0+2}) &= q(T_1 v_{n_0}, T_2 v_{n_0+1}) \\ &\leq \alpha q(v_{n_0}, T_1 v_{n_0}) \\ &\quad + \beta q(v_{n_0+1}, T_2 v_{n_0+1}) \\ &= \alpha q(v_{n_0}, v_{n_0+1}) \\ &\quad + \beta q(v_{n_0+1}, v_{n_0+2}) \\ &= \alpha q(v_{n_0+1}, v_{n_0+1}) + \\ &\quad \beta q(v_{n_0+1}, v_{n_0+2}) \\ &= \beta q(v_{n_0+1}, v_{n_0+2}). \end{aligned}$$

a contradiction, since $0 < \beta < \frac{1}{k} < 1$. Thus, it is permissible to assume that $v_n \neq v_{n+1}$ for all $n \in \mathbb{N}$. For $n \in \mathbb{N}$, the following cases are investigated.

Case 1: n is even. Here $n = 2i$ for some $i \in \{0,1,2,3 \ldots\}$. From Eq 1, obtained

$$\begin{aligned} q(v_{2i}, v_{2i+1}) &= q(T_1 v_{2i-1}, T_2 v_{2i}) \\ &\leq \alpha q(v_{2i-1}, T_1 v_{2i-1}) \\ &\quad + \beta q(v_{2i}, T_2 v_{2i}) \\ &= \alpha q(v_{2i-1}, v_{2i}) \\ &\quad + \beta q(v_{2i}, v_{2i+1}) \\ &\leq \frac{\alpha}{1-\beta} q(v_{2i-1}, v_{2i}). \end{aligned} \quad 2$$

Case 2: n is odd. Here $n = 2i + 1$ for some $i \in \{0,1,2,3 \ldots\}$. Using similar arguments as those given in Case 1,

$$q(v_{2i+1}, v_{2i+2}) \leq \frac{\alpha}{1-\beta} q(v_{2i}, v_{2i+1}). \quad 3$$

Combining Eq 2 and Eq 3 together, obtained

$$q(v_n, v_{n+1}) \leq \frac{\alpha}{1-\beta} q(v_{n-1}, v_n).$$

So,

$$q(v_n, v_{n+1}) \leq \lambda q(v_{n-1}, v_n),$$

with $\lambda = \frac{\alpha}{1-\beta}$,

$$q(v_n, v_{n+1}) \leq \lambda^n q(v_0, v_1). \quad 4$$

Since $+\beta < 1$, so we have

$$\begin{aligned} k\alpha + \beta &< 1 \\ k\alpha &< 1 - \beta \\ \frac{k\alpha}{1-\beta} &< 1 \\ \frac{\alpha}{1-\beta} &< \frac{1}{k}, \end{aligned}$$

there exist $0 < \lambda < 1$.

By using Lemma 1, thus $\{v_n\}$ is Cauchy in the complete b-metric space (X, q). So there exists $h \in X$ such that

$$q(v_n, h) \to 0 \text{ as } n \to \infty. \quad 5$$

Now, shown that h is the common fixed point of T_1 and T_2. So, by using triangular inequality and (1), obtained

$$\begin{aligned} q(h, T_1 h) &\leq k[q(h, v_{2n+2}) + q(v_{2n+2}, T_1 h)] \\ &= kq(h, v_{2n+2}) + kq(v_{2n+2}, T_1 h) \\ &= kq(h, v_{2n+2}) + kq(T_2 v_{2n+1}, T_1 h) \\ &\leq kq(h, v_{2n+2}) + k[\alpha q(h, T_1 h) + \beta q(v_{2n+1}, T_2 v_{2n+1})] \\ &= kq(h, v_{2n+2}) + k[\alpha q(h, T_1 h) + \beta q(v_{2n+1}, v_{2n+2})] \\ &\leq \frac{k}{1 - k\alpha} [q(h, v_{2n+2}) + \beta q(v_{2n+1}, v_{2n+2})]. \end{aligned}$$

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https://doi.org/10.21123/bsj.2024.9198
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
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Since $0 < \alpha < \frac{1}{k}$ then $1 - k\alpha > 0$. Taking the limit of Eq 4 and Eq 5 as $n \to \infty$, obtained

$$q(h, T_1 h) = 0.$$

So, $h = 0$. Hence $h = T_1 h$. Similarly, it can easily be proved that $h = T_2 h$. Therefore, h is the common fixed point of T_1 and T_2.

Now, the uniqueness of a common fixed point. Suppose $f \in X$ is another common fixed point of T_1 and T_2, then $T_1 f = T_2 f = f$ and $q(f, h) > 0$,

$$\begin{aligned} q(f,h) = q(T_1 f, T_2 h) &\leq \alpha q(f, T_1 f) + \beta q(h, T_2 h) \\ &= \alpha q(f,f) + \beta q(h,h) \\ &= 0. \end{aligned}$$

Which is a contradiction, since $q(f, h) > 0$, thus $f = h$. Hence, the common fixed point of T_1 and T_2 is unique.

Example 2: Let (X, q) be a complete b-metric space with $k = 2$ and define $q(v, w) = (v - w)^2$. Let $X = [0,1] \subset \mathbb{R}$ and define

$$T_1 v = \begin{cases} \frac{v}{3}, for\ v \in \left[0, \frac{1}{2}\right) \\ \frac{v}{5}, for\ v \in \left[\frac{1}{2}, 1\right] \end{cases},$$

and

$$T_2 v = \begin{cases} \frac{v}{4}, for\ v \in \left[0, \frac{1}{3}\right) \\ \frac{v}{6}, for\ v \in \left[\frac{1}{3}, 1\right] \end{cases},$$

Then the condition $q(T_1 v, T_2 w) \leq \alpha q(v, T_1 v) + \beta q(w, T_2 w)$ is satisfied for all $v, w \in X$ if taking $\alpha = \beta = \frac{1}{4}$. Thus, by Theorem 7, T_1 and T_2 have a unique common fixed point.

Theorem 8 uses a generalization of the Chatterjea contraction function (if $\beta = \alpha$, then Eq 6 becomes the Chatterjea contraction function).

Theorem 8: Let (X, q) be a complete b-metric space with $k > 1$ be a given real number and $T_1, T_2 : X \to X$ is a pair of self-maps, such that

$$q(T_1 v, T_2 w) \leq \alpha q(v, T_2 w) + \beta q(w, T_1 v), \tag{6}$$

for all $u, v \in X$ where $\alpha k(k + 1) < 1$ and $\alpha + \beta < 1$ with $\alpha, \beta < \frac{1}{k^2}$. Then T_1 and T_2 have a unique common fixed point in X.

Proof: Since X is nonempty, let $v_0 \in X$ and define a sequence $\{v_n\}$ in X inductively by putting $v_{2n+1} = T_1 v_{2n}$ and $v_{2n+2} = T_2 v_{2n+1}$ for $n = 0,1,2,3, \ldots$. Will be shown $v_n \neq v_{n+1}$. Suppose that $v_{n_0} = v_{n_0+1}$ for some $n_0 \in \mathbb{N}$. From Eq 6, obtained

$$\begin{aligned} q(v_{n_0+1}, v_{n_0+2}) &= q(T_1 v_{n_0}, T_2 v_{n_0+1}) \\ &\leq \alpha q(v_{n_0}, T_2 v_{n_0+1}) + \beta q(v_{n_0+1}, T_1 v_{n_0}) \\ &= \alpha q(v_{n_0}, v_{n_0+2}) + \beta q(v_{n_0+1}, v_{n_0+1}) \\ &= \alpha q(v_{n_0+1}, v_{n_0+2}), \end{aligned}$$

a contradiction, since $0 < \alpha < \frac{1}{k^2} < 1$. Thus, it is permissible to assume that $v_n \neq v_{n+1}$ for all $n \in \mathbb{N}$. For $n \in \mathbb{N}$, the following cases are investigated.

Case 1: n is even. Here $n = 2i$ for some $i \in \{0,1,2,3 \ldots\}$. From Eq 6, obtained

$$\begin{aligned} q(v_{2i}, v_{2i+1}) &= q(T_1 v_{2i-1}, T_2 v_{2i}) \\ &\leq \alpha q(v_{2i-1}, T_2 v_{2i}) + \beta q(v_{2i}, T_1 v_{2i-1}) \\ &= \alpha q(v_{2i-1}, v_{2i+1}) + \beta q(v_{2i}, v_{2i}) \\ &= \alpha q(v_{2i-1}, v_{2i+1}) \\ &\leq \alpha k[q(v_{2i-1}, v_{2i}) + q(v_{2i}, v_{2i+1}), \\ &\leq \frac{\alpha k}{1 - \alpha k} q(v_{2i-1}, v_{2i}). \end{aligned} \tag{7}$$

Case 2: n is odd. Here $n = 2i + 1$ for some $i \in \{0,1,2,3 \ldots\}$. Using similar arguments as those given in Case 1,

$$q(v_{2i+1}, v_{2i+2}) \leq \frac{\alpha k}{1 - \alpha k} q(v_{2i}, v_{2i+1}). \tag{8}$$

Combining Eq 7 and Eq 8 together, obtained

$$q(v_n, v_{n+1}) \leq \frac{\alpha k}{1 - \alpha k} q(v_{n-1}, v_n).$$

So,

$$q(v_n, v_{n+1}) \leq \lambda q(v_{n-1}, v_n),$$

with $\lambda = \frac{\alpha k}{1 - \alpha k}$,

$$q(v_n, v_{n+1}) \leq \lambda^n q(v_0, v_1), \tag{9}$$

Since $(k + 1) < 1$, so we have

$$\begin{aligned} \alpha k^2 + \alpha k &< 1 \\ \alpha k^2 &< 1 - \alpha k \\ \frac{\alpha k^2}{1 - \alpha k} &< 1 \\ \frac{\alpha k}{1 - \alpha k} &< \frac{1}{k}, \end{aligned}$$

there exist $0 < \lambda < 1$.

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https://doi.org/10.21123/bsj.2024.9198
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
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By using Lemma 1, thus $\{v_n\}$ is Cauchy in the complete b-metric space (X, q). So there exists $h \in X$ such that

$$q(v_n, h) \to 0 \text{ as } n \to \infty. \quad 10$$

Now, shown that h is the common fixed point of T_1 and T_2. So, by using triangular inequality and Eq 6, obtained

$$\begin{aligned} q(h, T_1 h) &\leq k[q(h, v_{2n+2}) + q(v_{2n+2}, T_1 h)] \\ &= kq(h, v_{2n+2}) + kq(v_{2n+2}, T_1 h) \\ &= kq(h, v_{2n+2}) \\ &\quad + kq(T_2 v_{2n+1}, T_1 h) \\ &\leq kq(h, v_{2n+2}) \\ &\quad + k[\alpha q(h, T_2 v_{2n+1}) \\ &\quad + \beta q(v_{2n+1}, T_1 h)] \\ &= kq(h, v_{2n+2}) \\ &\quad + k[\alpha q(h, v_{2n+2}) \\ &\quad + \beta q(v_{2n+1}, T_1 h)] \\ &\leq k[(1+\alpha) q(h, v_{2n+2}) \\ &\quad + \beta k\{q(v_{2n+1}, h) + q(h, T_1 h)\}] \\ &\leq \frac{k}{1-\beta k^2}[(1+\alpha) q(h, v_{2n+2}) \\ &\quad + \beta k q(v_{2n+1}, h)]. \end{aligned}$$

Since $0 < \beta < \frac{1}{k^2}$ then $1 - \beta k^2 > 0$. Taking the limit of Eq 10 as $n \to \infty$,

$$q(h, T_1 h) = 0.$$

So, $h = 0$. Hence $h = T_1 h$. Similarly, it can easily be proved that $h = T_2 h$. Using the same argument,

$$q(h, T_2 h) \leq \frac{k}{1-\alpha k^2}[(1+\beta) q(h, v_{2n+1}) + \alpha k q(v_{2n}, h)] .$$

Since $0 < \alpha < \frac{1}{k^2}$ then $1 - \alpha k^2 > 0$. Taking the limit of Eq 10 as $n \to \infty$,

$$q(h, T_2 h) = 0.$$

Therefore, h is the common fixed point of T_1 and T_2.

Now, the uniqueness of a common fixed point. Suppose $f \in X$ is another common fixed point of T_1 and T_2, then $T_1 f = \mathrm{T}_2 f = f$, and $q(f, h) > 0$,

$$\begin{aligned} q(f, h) = q(T_1 f, T_2 h) &\leq \alpha q(f, T_2 h) + \beta q(h, T_1 f) \\ &= \alpha q(f, h) + \beta q(h, f) \\ &\leq (\alpha + \beta) q(f, h). \end{aligned}$$

Which is a contradiction, since $\alpha + \beta < 1$ and $q(f, h) > 0$, thus $f = h$. Hence, the common fixed point of T_1 and T_2 is unique.

The next theorem uses a generalization of the Reich contraction function.

Theorem 9: Let (X, q) be a complete b-metric space with $k > 1$ be a given real number and $T_1, T_2 : X \to X$ be a pair of self-maps, such that

$$\begin{aligned} q(T_1 v, T_2 w) \leq \alpha q(v, w) &+ \beta q(v, T_1 v) \\ &+ \gamma q(w, T_2 w) \\ &+ \theta q(w, T_1 v), \end{aligned} \quad 11$$

for all $u, v \in X$ where $\alpha, \beta, \gamma, \theta > 0$, $k(\alpha + \beta) + \gamma < 1$ and $\beta + k\theta < \frac{1}{k}, \gamma < \frac{1}{k}$ with $\alpha + \theta < 1$. Then T_1 and T_2 have a unique common fixed point in X.

Proof: Since X is nonempty, let $v_0 \in X$ and define a sequence $\{v_n\}$ in X inductively by putting $v_{2n+1} = T_1 v_{2n}$ and $v_{2n+2} = T_2 v_{2n+1}$ for $n = 0,1,2,3, \ldots$. Will be shown $v_n \neq v_{n+1}$. Suppose that $v_{n_0} = v_{n_0+1}$ for some $n_0 \in \mathbb{N}$. From Eq 11, obtained

$$\begin{aligned} q(v_{n_0+1}, v_{n_0+2}) &= q(T_1 v_{n_0}, T_2 v_{n_0+1}) \\ &\leq \alpha q\left(v_{n_0}, v_{n_0+1}\right) \\ &\quad + \beta q\left(v_{n_0}, T_1 v_{n_0}\right) \\ &\quad + \gamma q\left(v_{n_0+1}, T_2 v_{n_0+1}\right) \\ &\quad + \theta q(v_{2i+1}, T_1 v_{2i}) \\ &= \alpha q\left(v_{n_0}, v_{n_0+1}\right) \\ &\quad + \beta q\left(v_{n_0}, v_{n_0+1}\right) \\ &\quad + \gamma q\left(v_{n_0+1}, v_{n_0+2}\right) \\ &\quad + \theta q(v_{n_0+1}, v_{n_0+1}) \\ &= \alpha q\left(v_{n_0+1}, v_{n_0+1}\right) \\ &\quad + \beta q\left(v_{n_0+1}, v_{n_0+1}\right) \\ &\quad + \gamma q\left(v_{n_0+1}, v_{n_0+2}\right) \\ &\quad + \theta q(v_{n_0+1}, v_{n_0+1}) \\ &= \gamma q\left(v_{n_0+1}, v_{n_0+2}\right). \end{aligned}$$

a contradiction, since $0 < \gamma < \frac{1}{k} < 1$. Thus, it is permissible to assume that $v_n \neq v_{n+1}$ for all $n \in \mathbb{N}$. For $n \in \mathbb{N}$, the following cases are investigated.

Case 1: n is even. Here $n = 2i$ for some $i \in \{0,1,2,3 \ldots\}$. From Eq 11, obtained

$$\begin{aligned} q(v_{2i}, v_{2i+1}) &= q(T_1 v_{2i-1}, T_2 v_{2i}) \\ &\leq \alpha q(v_{2i-1}, v_{2i}) \\ &\quad + \beta q(v_{2i-1}, T_1 v_{2i-1}) \\ &\quad + \gamma q(v_{2i}, T_2 v_{2i}) \\ &\quad + \theta q(v_{2i}, T_1 v_{2i-1}) \\ &= (\alpha + \beta) q(v_{2i-1}, v_{2i}) \\ &\quad + \gamma q(v_{2i}, v_{2i+1}) \\ &= \frac{\alpha + \beta}{1 - \gamma} q(v_{2i-1}, v_{2i}). \end{aligned} \quad 12$$

Case 2: n is odd. Here $n = 2i + 1$ for some $i \in \{0,1,2,3 \dots\}$. Using similar arguments as those given in Case 1,

$$q(v_{2i+1}, v_{2i+2}) \leq \frac{\alpha + \beta}{1 - \gamma} q(v_{2i}, v_{2i+1}). \quad 13$$

Combining Eq 12 and Eq 13 together, obtained

$$q(v_n, v_{n+1}) \leq \frac{\alpha + \beta}{1 - \gamma} q(v_{n-1}, v_n).$$

So,

$$q(v_n, v_{n+1}) \leq \lambda q(v_{n-1}, v_n),$$

with $\lambda = \frac{\alpha+\beta}{1-\gamma}$,

$$q(v_n, v_{n+1}) \leq \lambda^n q(v_0, v_1), \quad 14$$

Since $k(\alpha + \beta) + \gamma < 1$, then we have

$$\frac{\alpha + \beta}{1 - \gamma} < \frac{1}{k},$$

there exist $0 < \lambda < 1$.

By using Lemma 1, thus $\{v_n\}$ is Cauchy in the complete b-metric space (X, q). So there exists $h \in X$ such that

$$q(v_n, h) \to 0 \text{ as } n \to \infty. \quad 15$$

Now, shown that h is the common fixed point of T_1 and T_2. So, by using triangular inequality and Eq 11,

$$\begin{aligned} q(h, T_1 h) &\leq k[q(h, v_{2n+2}) + q(v_{2n+2}, T_1 h)] \\ &= kq(h, v_{2n+2}) + kq(v_{2n+2}, T_1 h) \\ &= kq(h, v_{2n+2}) + kq(T_2 v_{2n+1}, T_1 h) \\ &\leq kq(h, v_{2n+2}) + k[\alpha q(h, v_{2n+1}) + \beta q(h, T_1 h) + \gamma q(v_{2n+1}, T_2 v_{2n+1}) \\ &\qquad + \theta q(v_{2n+1}, T_1 h] \\ &= kq(h, v_{2n+2}) + k[\alpha q(h, v_{2n+1}) + \beta q(h, T_1 h) + \gamma q(v_{2n+1}, v_{2n+2}) \\ &\qquad + \theta q(v_{2n+1}, T_1 h)] \\ &\leq kq(h, v_{2n+2}) + k\big[\alpha q(h, v_{2n+1}) + \beta q(h, T_1 h) + \gamma q(v_{2n+1}, v_{2n+2}) \\ &\qquad + \theta k\big(q(v_{2n+1}, h) + q(h, T_1 h)\big)\big] \\ &\leq \frac{k}{1-k(\beta+k\theta)} [q(h, v_{2n+2}) + \alpha q(h, v_{2n+1}) + \gamma q(v_{2n+1}, v_{2n+2})]. \end{aligned}$$

Since $\beta + k\theta < \frac{1}{k}$ then $1 - k(\beta + k\theta) > 0$. Taking the limit of Eq 14 and Eq 15 as $n \to \infty$,

$$q(h, T_1 h) = 0.$$

So, $h = 0$. Hence $h = T_1 h$. Similarly, it can easily be proved that $h = T_2 h$. Using the same argument,

$$q(h, T_2 h) \leq \frac{k}{1 - k\gamma} [q(h, v_{2n+1}) + \alpha q(v_{2n}, h) + \beta q(v_{2n}, v_{2n+1}) + \theta q(h, v_{2n+1})].$$

Since $0 < \gamma < \frac{1}{k}$ then $1 - k\gamma > 0$. Taking the limit of Eq 14 and Eq 15 as $n \to \infty$,

$$q(h, T_2 h) = 0.$$

Therefore, h is the common fixed point of T_1 and T_2.

Now, the uniqueness of a common fixed point. Suppose $f \in X$ is another common fixed point of T_1 and T_2 then $T_1 f = \mathrm{T}_2 f = f$, and $q(f, h) > 0$,

$$\begin{aligned} q(T_1 f, T_2 h) &\leq \alpha q(f, h) + \beta q(f, T_1 f) + \gamma q(h, T_2 h) + \theta q(h, T_1 f) \\ &= \alpha q(f, h) + \beta q(f, f) + \gamma q(h, h) + \theta q(h, f) \\ &= (\alpha + \theta) q(f, h), \end{aligned}$$

Which is a contradiction, since $0 < \alpha + \theta < 1$ and $q(f, h)$, thus $f = h$. Hence, the common fixed point of T_1 and T_2 is unique.

An Application

In this section, an application of the main results related to Theorem 9 to dynamic programming is presented, which is to find the common solution of two functional equations.

Let Z be a decision space and Y be a state space. Let assume a problem of dynamic programming formulated in the form of functional equations as follows:

2024, 21(9): 2984-2992
https://doi.org/10.21123/bsj.2024.9198
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
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$$A_1(t) = \sup_{r\in Y}\left\{L(t,r) + b\left(t,r,A_1\big(K(t,r)\big)\right)\right\}, for\ t \in Z. \quad 16$$

$$A_2(t) = \sup_{r\in Y}\left\{L(t,r) + b\left(t,r,A_2\big(K(t,r)\big)\right)\right\}, for\ t \in Z. \quad 17$$

Let assume that D and P are Banach Spaces such that $Z \subseteq D, Y \subseteq P$, and $K: Z \times Y \to Z$, $L: Z \times Y \to \mathbb{R}$, $b: Z \times Y \times \mathbb{R} \to \mathbb{R}$. Shown that the functional Eqs 16 and 17 have a unique common solution.

Let $X(Z)$ be the set of all bounded real-valued mappings on Z. For all $v \in X(Z)$, define

$$\|v\| = \max_{u\in Z}|v(u)|.$$

Then $(X(Z), \|.\|)$ is a Banach space.

Define a function $q: X(Z) \times X(Z) \to R^+$ as follows:

$$q(v,w) = \max_{u\in Z}\big(v(u) - w(u)\big)^2,$$

then q is a complete b-metric space in $X(Z)$.

Let the following conditions hold:
L and b_i are bounded for $i = \{1,2\}$.
For $u \in Z$ and $v, w \in X(Z)$ define function $S_1, S_2: X(Z) \to X(Z)$ by

$$S_1 v(u) = \sup_{r\in Y}\{L(u,r) + b_1\big(u,r,v\big(K(u,r)\big)\big)\},$$

$$S_2 w(u) = \sup_{r\in Y}\{L(u,r) + b_2\big(u,r,w\big(K(u,r)\big)\big)\}.$$

Well defined.
For $(u,r) \in Z \times Y, v, w \in X(Z)$ and $x \in Z$,

$$|b_1(u,r,v(x) - b_2(u,r,w(x)|^2 \le \alpha q(v,w) + \beta q(v,S_1 v) + \gamma q(w,S_2 w) + \theta q(w,S_1 v),$$

for all $g, h \in X(Z)$ where $k(\alpha+\beta)+\gamma < 1$ with $\beta + k\theta, \gamma < \frac{1}{k}$ and $\alpha + \theta < 1$.

Theorem 10: If conditions 1–3 hold, then Eqs 16 and 17 have a unique common bounded solution.

Proof. Let $v, w \in X(Z)$ and $u \in Z$. For any $\varepsilon > 0$, there exist $g_1, g_2 \in Y$ such that

$$S_1 v(u) < L(u,g_1) + b_1\left(u,g_1,v\big(K(u,g_1)\big)\right) + \varepsilon \quad 18$$

$$S_2 w(u) < L(u,g_2) + b_2\left(u,g_2,w\big(K(u,g_2)\big)\right) + \varepsilon \quad 19$$

$$S_1 v(u) \ge L(u,g_2) + b_1\left(u,g_2,v\big(K(u,g_2)\big)\right) \quad 20$$

$$S_2 w(u) \ge L(u,g_1) + b_2\left(u,g_1,w\big(K(u,g_1)\big)\right) \quad 21$$

Then, using Eq 18 and Eq 21,

$$\begin{aligned} S_1 v(u) - S_2 w(u) &\le b_1\left(u,g_1,v\big(K(u,g_1)\big)\right) + \varepsilon - b_2\left(u,g_1,w\big(K(u,g_1)\big)\right) \\ &\le \left|b_1\left(u,g_1,v\big(K(u,g_1)\big)\right) - b_2\left(u,g_1,w\big(K(u,g_1)\big)\right)\right| + \varepsilon \\ &\le \big(\alpha q(v,w) + \beta q(v,S_1 v) + \gamma q(w,S_2 w) + \theta q(w,S_1 v)\big)^{\frac{1}{2}} + \varepsilon \end{aligned} \quad 22$$

Similarly, by Eq 19 and Eq 20,

$$S_2 w(u) - S_1 v(u) \le \big(\alpha q(v,w) + \beta q(v,S_1 v) + \gamma q(w,S_2 w) + \theta q(w,S_1 v)\big)^{\frac{1}{2}} + \varepsilon \quad 23$$

From Eq 22 and Eq 23,

$$|S_1 v(u) - S_2 w(u)| \le \big(\alpha q(v,w) + \beta q(v,S_1 v) + \gamma q(w,S_2 w) + \theta q(w,S_1 v)\big)^{\frac{1}{2}} + \varepsilon,$$

so,

$$\big(S_1 v(u) - S_2 w(u)\big)^2 \le \alpha q(v,w) + \beta q(v,S_1 v) + \gamma q(w,S_2 w) + \theta q(w,S_1 v) + \varepsilon.$$

Thus, for all $\varepsilon > 0$,

$$\big(S_1 v(u) - S_2 w(u)\big)^2 \le \alpha q(v,w) + \beta q(v,S_1 v) + \gamma q(w,S_2 w) + \theta q(w,S_1 v),$$

for all $u \in Z$. Then

$$q(S_1 v, S_2 w) \le \alpha q(v,w) + \beta q(v,S_1 v) + \gamma q(w,S_2 w) + \theta q(w,S_1 v).$$

Therefore, based on Theorem 9, then the functional Eqs 16 and 17 has a unique common solution.

Conclusion

In the b-metric space of the generalization of the contraction function Kannan, Reich, and Chatterjea obtain sufficient conditions for the occurrence of a single fixed point. A single fixed point in the generalization of the Reich contraction function can be used to determine the existence of a single solution to the system dynamics problem.

Acknowledgment

The author is very grateful to the anonymous referee and our colleagues who have provided very useful comments for completing this paper.

Authors' Declaration

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are ours. Furthermore, any Figures and images, that are not ours, have been included with the necessary permission for re-publication, which is attached to the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee at Hasanuddin University, Makassar, Indonesia.

Authors' Contribution Statement

N. I. I. contributed to conception, drafting manuscript, analysis, revision and proofreading, M. Z. contributed to analysis, and B. N. contributed to the design, implementation of the research, and to the analysis of the result.

References

1. Hashim AM, Kazem AA. Coincidence of Fixed Points with Mixed Monotone Property. Baghdad Sci J. 2023; 20(2): 292-105. http://dx.doi.org/10.21123/bsj.2022.6711
2. Ume JS. Fixed Point Theorems for Kannan-Type Maps. Fixed Point Theory Appl. 2015; 2015(1): 1-13. https://doi.org/10.1186/s13663-015-0286-5
3. Karapnar E, Nashine HK. Fixed Point Theorem for Cyclic Chatterjea Type Contractions. J Appl Math. 2012; 2012: 15 pages. https://doi.org/10.1155/2012/165698
4. Reich S. Some Remarks Concerning Contraction Mappings. Can Math Bull. 1971; 14(1): 121-124. https://doi.org/10.4153/cmb-1971-024-9
5. George R, Reshma KP, Padmavati A. Fixed Point Theorems for Cyclic Contractions in B-Metric Spaces. J Nonlinear Funct Anal. 2015; 5: 1-20. http://jnfa.scik.org
6. Nurwahyu B. Common fixed point theorems on generalized ratio contraction mapping in extended rectangular b-metric spaces. Int J Math Math Sci. 2019; 2019: 1-14. https://doi.org/10.1155/2019/2756870
7. Debnath P, Srivastava HM. New Extensions of Kannan's and Reich's Fixed Point Theorems for Multivalued Maps Using Wardowski's Technique with Application to Integral Equations. Symmetry (Basel). 2020; 12(7): 1090. https://doi.org/10.3390/SYM12071090
8. Debnath P, Srivastava HM. Global Optimization and Common Best Proximity Points for Some Multivalued Contractive Pairs of Mappings. Axioms. 2020; 9(3): 102. https://doi.org/10.3390/AXIOMS9030102
9. Debnath P, Mitrović ZD, Cho SY. Common Fixed Points of Kannan, Chatterjea and Reich Type Pairs of Self-Maps in a Complete Metric Space. Sao Paulo J Math Sci.. 2021; 15(1): 383-391. https://doi.org/10.1007/s40863-020-00196-y
10. Cieslik D. k-Steiner-minimal-trees in metric spaces. Discrete Math. 1999; 208: 119-124. www.elsevier.com/locate/disc
11. Nurwahyu B. Fixed Point Theorems for Cyclic Weakly Contraction Mappings in Dislocated Quasi Extended b -Metric Space. J Funct Spaces. 2019; 2019: 10 pages. https://doi.org/10.1155/2019/1367879
12. Aghajani A, Abbas M, Roshan JR. Common fixed point of generalized weak contractive mappings in partially ordered b-metric spaces. Math Slovaca. 2014; 64(4): 941-960. https://doi.org/10.2478/s12175-014-0250-6
13. Rudin W. Principles of Mathematical Analysis. 3rd edition. USA: McGraw-Hill, Inc.; 1976. 352 p.

2024, 21(9): 2984-2992
https://doi.org/10.21123/bsj.2024.9198
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
Baghdad Science Journal

14. Shatanawi W, Pitea A, Lazovi´c RL. Contraction Conditions Using Comparison Functions on b-Metric Spaces. Fixed Point Theory Algorithm Sci Eng. 2014; 1: 1-11. http://www.fixedpointtheoryandapplications.com/content/2014/1/135

15. Andy BAB, Aris N, Nurwahyu B. Eksistensi Dan Ketunggalan Titik Tetap pada Pemetaan Kontraksi Tergeneralisasi dalam Ruang b-Metrik. J Stat Industri Komputasi. 2019; 16(2): 208. https://doi.org/10.20956/jmsk.v16i2.7701

نظريات النقاط الثابتة الشائعة لتعميم كانان وتاترجيا وتقلص الرايخ في الفضاء المتري مع التطبيق

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الخلاصة

أثبت هذا البحث وجود وتفرد النقاط الثابتة المشتركة لأزواج الخرائط الذاتية من نوع كانان، ورايش، وشاترجيا في مساحة ب مترية كاملة. بالإضافة إلى ذلك، تمت مناقشة مثال وتطبيق لوجود وتفرد الحلول المشتركة لنظام المعادلات الوظيفية الناشئة في البرمجة الديناميكية باستخدام نتائجنا.

الكلمات المفتاحية: ب-متري، تقلص شاترجيا، النقطة الثابتة المشتركة، النقطة الثابتة، انكماش كانان، انكماش الرايخ.