M_τ- TIME AND TIME PROJECTION

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Abstract.

In this paper we discuss new random time which is called M_τ -time with some of its properties.In addition, we find the time projection associated with M_τ- time. Finally we compute the supremum of increasing family { M_τ^t : t∈[0,∞]} into two cases, the first case when $\vee q_t = I$, while the second case when $\vee q_t = q \neq$ I .

المستخلص

في هذا البحث نناقش زمن عشوائي جديد يدعى الزمن من النمط - M_τ^t مع بعض خواصه كذالك سوف نجد المسقط الزمني المرفق بهذا الزمن واخيرا سوف نحسب ادنى حد اعلى للعائلة { M_τ : t∈[0,∞] }وذالك في حالتين الاولى عندما $\vee q_t =$ I بينما الثانية في حالة $\vee q_t = q \neq$ I .

INTRODUCTION

In this paper we develop some of the concepts in [1], [2] and [5] within the non - commutative context. It was shown in [7] that one can define the general random time τ as a map from a subset [0, t] ⊆ [0,+∞] into proj A, such that $\tau(t) = q_t$, $\tau(0) = q_0 = 0$ and τ(s) is projection in A_s where s∈(0,t).

In [7] it was shown that for each general random time $\tau=(q_t)$ the orthogonal projection M_τ^t is called time projection associated with general random time, also we prove when t = o , t = ∞ this implies $M^t_{\tau(\theta)} = 0$, M_τ respectively. Therefore we can define new time that is M_τ-time as following:

An increasing family of projections $\hat{\tau} = (M_\tau^t)$ is called M_τ- time such that $\hat{\tau}$ (o)=0, $\hat{\tau}$ =(∞)= M_τ and $\hat{\tau}$ (t)= M_τ^t for each t ∈(0, ∞).

This paper divided into two sections:

The first section contains a brief review of notation non - commutative stochastic base, definitions of (random time, q-time, general random time) and time projection associated by general random time with some of its properties. The second section contains the definition of M_τ- time with some of its properties. Also we compute the supremum of increasing family of projections { M_τ : t∈[0,∞] } in two cases, the first case when τ is a random time while the second case when τ is q-time.

1. Notions And Preliminaries

Let B (H) be bounded linear operater on complex Hilbert space H, and let A ⊂ B(H) be a von Neumann algebra. For each non – negative real t, let A_t be von Neumann sub algebra of von Neumann algebra A. A non-commutative stochastic base which is a basic object of our considerations consists of the following elements: A von Neumann algebra A⊂ B(H) acting on Hilbert space H, a filtration $\{A_t: 0 \le t \le +\infty\}$which is an increasing (s≤ t implies $A_s \subseteq A_t$) family of von Neumann sub algebra of A such that:

$$A = A_\infty = \left(\bigcup_{t\ge 0} A_t\right)'' \text{ and } A_s = \bigcap_{t \ge s} A_t \text{ (right continuous)}$$

Also there is unite vector Ω belong to Hilbert space H and separating for A.Now if we denote the closure $A_t\Omega$ in Hilbert space H by H_t, we get that H_t is a closed subspace of H and hence H_t is a Hilbert space itself. Moreover for each $t\in R^+$, let P_t denote the orthogonal projection from H onto H_t. The family $\{ P_t: 0 \le t \le +\infty\}$ of orthogonal projection is an increasing and lies in the commutant of A_t.

Now we introduce the following definitions:

Definition (1.1) [7]
A random time τ, is a map $\tau :[0, \infty] \rightarrow$ proj A such that $\tau(0) = q_0 = 0$, $\tau(\infty) = q_\infty = I$ and τ(t) is projection in A_t, and $\tau(s) \le \tau(t)$, whenever $s \le t$.

Definition (1.2) [7]
By q – time we mean a map $\tau:[0, \infty] \rightarrow$ proj A such that $\tau(0) = q_0 = 0$, $\tau(\infty) = q$ and τ(t) is projection in A_t, and $\tau(s) \le \tau(t)$, where $s \le t$.

Note that in more general case we introduce the following definition:

Definition (1.3)[8]
A general random time on interval [0, t] we mean a map $\tau : [0, t] \rightarrow$ proj. A such that $\tau(0) = q_0 = 0$, $\tau(t) = q_t$ and τ(s) is projection in A_s, where $s\in(0,t)$.

Let now $\tau = (q_t)$ be general random time for each partition $\theta = \{0 = t_0 < t_1 < ... < t_n = t\}$, of interval [0,t], we define an operator $M^t_{\tau(\theta)}$ on H by the formula

$$M^t_{\tau(\theta)} = \sum_{i=1}^{n} (q_{t_i} - q_{t_{i-1}}) P_{t_i} = \sum_{i=1}^{n} \Delta q_{t_i} P_{t_i} \ .$$

Its turns out that $M^t_{\tau(\theta)}$ is projection, moreover, $M^t_{\tau(\theta)}$ decreases as θ refines. Thus there exist a unique orthogonal projection say M^t_τ which is called time projection defined as

$$M^t_\tau = \lim M^t_{\tau(\theta)} = \bigwedge_\theta M^t_{\tau(\theta)} \ .$$

The following propositions give some basic properties of linear operator $M^t_{\tau(\theta)}$.

Proposition (1.4)[8]
Let $\tau = (q_t)$ be a general random time .Then

1. $M^t_{\tau(\theta)}$ is an orthogonal projection.
2. For $\eta, \theta \in \Theta$ which is a partition of [0,t] with η finer than θ, then $M^t_{\tau(\theta)} \ge M^t_{\tau(\eta)}$.

Proposition (1.5)[8]

1. Let $\tau = (q_t)$ be general random time with $s \le t$, then $M^s_\tau = q_s M^t_\tau$.
2. Let $\tau = (q_t)$ be general random time then $M^s_\tau = q_s M_\tau$ when $t = \infty$, then $M^t_\tau = q_t M_\tau$ for all $s, t\in[0,\infty]$ **.**

2.M_τ-TIME

We begin this section by defined the following concepts.

Definition (2.1)

An increasing family of projections $\hat{\tau} = (M_\tau^t)$ is called M_τ-time such that $\hat{\tau}(0)=0$, $\hat{\tau}(\infty) = M_\tau$ and $\hat{\tau}(t)= M_\tau^t$ is projection in A_t.

Definition (2.2)

Let θ denote the set of all partitions of interval $[0,\infty]$. Then for each partitions θ in θ, say θ = {0 = $t_0<t_1<\ldots\ldots<t_n = +\infty$ },we define an operator $M\hat{\tau}_{(\theta)}$ on H as

$$M\hat{\tau}_{(\theta)} = \sum_{i=1}^{n}(M_\tau^{t_i} - M_\tau^{t_{i-1}})\, p_{t_i}$$

Proposition (2.3)

Let $\hat{\tau} =(M_\tau^t)$ be M_τ-time. Then

1. $M\hat{\tau}_{(\theta)}$ is bounded linear operator.
2. $M\hat{\tau}_{(\theta)}$ is self-adjoint projection on H for any θ in θ .

Proof 1. we have $M\hat{\tau}_{(\theta)} = \sum_{i=1}^{n}(M_\tau^{t_i} - M_\tau^{t_{i-1}})\, p_{t_i}$ .It is clear that $M\hat{\tau}_{(\theta)}$ equal to finite sum of bounded linear operators, therefore $M\hat{\tau}_{(\theta)}$ is bounded linear operator ■

2. we must prove that $M\hat{\tau}_{(\theta)}.\, M\hat{\tau}_{(\theta)}= M\hat{\tau}_{(\theta)}$

$$M\hat{\tau}_{(\theta)}.\, M\hat{\tau}_{(\theta)} = \sum_{i=1}^{n}(M_\tau^{t_i} - M_\tau^{t_{i-1}})\, p_{t_i} . \sum_{j=1}^{n}(M_\tau^{t_j} - M_\tau^{t_{j-1}})\, p_{t_j}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \Delta M_\tau^{t_i}\, P_{t_i} \Delta M_\tau^{t_j} P_{tj}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} P_{t_i} \Delta M_\tau^{t_i}\, \Delta M_\tau^{t_j} P_{tj} \quad [\text{since } M_\tau^{t_i} \in \mathcal{A}_{t_i}, P_{t_i} \in \mathcal{A}'_{t_i}].$$

There are two cases:

The first one if $i \neq j$ this implies $\Delta M_\tau^{t_i}\, \Delta M_\tau^{t_j} = 0$

The second case if $i = j$ this implies $\Delta M_\tau^{t_i}\, \Delta M_\tau^{t_j} = \Delta M_\tau^{t_i} = \Delta M_\tau^{t_j}$ and $P_{t_i} P_{t_j} = P_{t_i}$

$$M\hat{\tau}_{(\theta)}.\, M\hat{\tau}_{(\theta)} = \sum_{i=1}^{n}(M_\tau^{t_i} - M_\tau^{t_{i-1}})\, p_{t_i} = M\hat{\tau}_{(\theta)}.$$

Hence $M\hat{\tau}_{(\theta)}$ is projection.

Now to prove $M\hat{\tau}_{(\theta)}$ is a self a djoint, we must prove $M^*\hat{\tau}_{(\theta)}= M\hat{\tau}_{(\theta)}$

$$M^*\hat{\tau}_{(\theta)} = \Big(\sum_{i=1}^{n}(M_\tau^{t_i} - M_\tau^{t_{i-1}})\, p_{t_i}\Big)^* = \sum_{i=1}^{n} P_{t_i}^* (M_\tau^{t_i} - M_\tau^{t_{i-1}})^*$$

$$= \sum_{i=1}^{n} p_{t_i} (M_\tau^{t_i} - M_\tau^{t_{i-1}}) = \sum_{i=1}^{n} (p_{t_i} M_\tau^{t_i} - p_{t_i} M_\tau^{t_{i-1}})$$

$$= \sum_{i=1}^{n}(M_\tau^{t_i}\, p_{t_i} - M_\tau^{t_{i-1}}\, p_{t_i}) \quad [\text{since } M_\tau^{t_i}, M_\tau^{t_{i-1}} \in \mathcal{A}_{t_i}, P_{t_i} \in \mathcal{A}'_{t_i}].$$

$$= \sum_{i=1}^{n}(M_\tau^{t_i} - M_\tau^{t_{i-1}})\, p_{t_i} = M\hat{\tau}_{(\theta)}.$$

Hence $M_{\hat{\tau}(\theta)}$ is a self adjoint ■

Corollary 2.4.

Let $\hat{\tau} = (M_\tau^t)$ be M_τ-time. Then $M_{\hat{\tau}(\theta)} = M_\tau$ for all $\theta \in \theta$ and $M_{\hat{\tau}} = M_\tau$.

Proof : let $\theta = \{0 = t_0 < t_1 < \ldots\ldots < t_n = +\infty\}$ be a partition for $[0,+\infty]$.

We know that $M_{\hat{\tau}(\theta)} = \sum_{i=1}^{n} (M_\tau^{t_i} - M_\tau^{t_{i-1}})\, p_{t_i}$

But $M_\tau^{t_i} = q_{t_i} M_\tau$ and $M_\tau^{t_{i-1}} = q_{t_{i-1}} M_\tau$ proposition (1.5)

Thus $M_{\hat{\tau}(\theta)} = \sum_{i=1}^{n} (q_{t_i} M_\tau - q_{t_{i-1}} M_\tau)\, p_{t_i}$

$= \sum_{i=1}^{n} (q_{t_i} - q_{t_{i-1}}) M_\tau\, p_{t_i}$

$= \sum_{i=1}^{n} \Delta q_{t_i} p_{t_i} M_\tau$ [since $M_\tau p_{t_i} = p_{t_i} M_\tau$]

$= M_{\tau(\theta)}\, M_\tau$

$M_{\hat{\tau}(\theta)} = M_\tau$ ………(1)

By taking limit to both sides to relation (1), we obtain that

$M_{\hat{\tau}} = M_\tau$ ■

Remark (2.5)

Let $\sigma = (q_t)$ be q- time and let $\tau = (q_t)$ be random time . Then

$M_{\sigma(\theta)} = M_{\tau(\theta)} - (I - q)$ and $M_\sigma = M_\tau - (I - q)$.

Proof : let $\theta = \{ 0 = t_0 < t_1 < \ldots\ldots < t_n = +\infty \}$ be a partition for $[0,+\infty]$ we define

$$M_{\sigma(\theta)} = \sum_{i=1}^{n} (q_{t_i} - q_{t_{i-1}}) P_{t_i}$$

$$= \sum_{i=1}^{n-1} (q_{t_i} - q_{t_{i-1}}) P_{t_i} + (q - q_{t_{n-1}}) P_{t_n}$$

$$= \sum_{i=1}^{n-1} (q_{t_i} - q_{t_{i-1}}) P_{t_i} + (I - q_{t_{n-1}}) P_{t_n} - (I - q)$$

$$M_{\sigma(\theta)} = M_{\tau(\theta)} - (I - q)$$

By taking limit to both sides for previous relation, we obtain that

$$M_\sigma = M_\tau - (I - q)$$ ■

Proposition (2.6)

Let $\hat{\tau} = (M_\tau^t)$ be M_τ- time, where $t \in [0,+\infty]$. Then $\sup M_\tau^t = M_\tau$ where $\sup q_t = I$, and $\sup M_\tau^t = M_\sigma$, where $\sup q_t = q$, and $q \neq I$.

Proof : (1) If $\sup q_t = I$, we have

$\sup M_\tau^t = \bigvee_{t \geq 0} M_\tau^t$

$= \vee q_t M_\tau$ [since $M_\tau^t = q_t M_\tau$]

$= (\vee q_t) M_\tau = I M_\tau$ [since $\vee q_t = I$]

$= M_\tau$.

Thus $\sup M_\tau^t = M_\tau$.

(2) If $\sup q_t = q$, we have

$\sup M_\tau^t = \bigvee_{t\geq 0} M_\tau^t$

$= \vee q_t M_\tau$

$= (\vee q_t) M_\tau$ but $\sup q_t = q$, therefore $\sup M_\tau^t = q M_\tau$………(1)

Now we compute $q M_\tau$ as following:

Let $\theta = \{0 = t_0 < t_1 < \ldots\ldots < t_n = +\infty\}$ be partition for $[0,+\infty]$,then

$$q M_\tau = q \sum_{i=1}^{n} (q_{t_i} - q_{t_{i-1}}) P_{t_i}$$

$$q M_\tau = q \sum_{i=1}^{n-1} (q_{t_i} - q_{t_{i-1}}) P_{t_i} + q\,(I - q_{t_{n-1}})\,P_\infty$$

$$= \sum_{i=1}^{n-1} (q_{t_i} - q_{t_{i-1}}) P_{t_i} + (q - q_{t_{n-1}}) \quad \text{[since } P_\infty = I \text{]}$$

$$= \sum_{i=1}^{n} \Delta q_{t_i} P_{t_i} = M_{\sigma(\theta)} \quad \text{[by remark (2.5)]}$$

Thus $q M_\tau = M_{\sigma(\theta)}$………(2)

By the relations (1) and (2) we get that $\bigvee_{t\geq 0} M_\tau^t = M_\sigma$.

Therefore, from (1) and (2) we can say that

$$\text{Sup } M_\tau^t = \bigvee_{t\geq 0} M_\tau^t = \begin{cases} M_\tau & \text{if } \sup q_t = 1 \\ M_\sigma = M_\tau - (I - q) & \text{if } \sup q_t \neq 1 \end{cases} \quad \blacksquare$$

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