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Quotient Energy of Zero Divisor Graphs and Identity Graphs

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Abstract:
Consider the (p,q) simple connected graph $G = (V, E)$. The sum absolute values of the spectrum of quotient matrix of a graph G make up the graph's quotient energy. The objective of this study is to examine the quotient energy of identity graphs and zero-divisor graphs $\Gamma(R)$ of commutative rings using group theory, graph theory, and applications. In this study, the identity graphs $I(Z_p)$ derived from the group $(Z_p, +)$ and a few classes of zero-divisor graphs $\Gamma(R)$ of the commutative ring R are examined.

Keywords: Commutative ring, Identity Graphs, Quotient energy, Quotient matrix, Zero-Divisor Graphs.

Introduction:
In this study, connected, finite graphs are taken into consideration. Ivan Gutman provided the first definition of a graph's energy[1]. His definition was inspired by a concept that first surfaced much earlier, in 1930, when Erich Huckel put forth the well-known Huckel Molecular Orbital Theory. Chemists can estimate the energy associated with π-electron orbits using Huckel's technique.

In this article "The Energy of a Graph," Gutman proposed his definition of the energy of a generic simple graph in relation to the alleged total π-electron energy. As a result, the term "energy" was coined to mean the sum of the absolute values of the spectrum of the adjacency matrix of G. Some general ideas and graph parameters were taken into account[2-5]. A thorough investigation of the use of graph energy was conducted [6-9]. Beck[10] first raised the possibility of a graph relating to the zero divisors of a commutative ring in 1998. The zero divisor graph (R) is a simple graph with vertex set $Z(R)^*$, the set of all non-zero divisors in R, and any two different vertices x and y are adjacent if xy=0. It was modulated by Anderson D.F. and Livingson P.S.[11] after that. More studies on zero-divisor graphs also carried out[12,13].

Vasantha Kandasamy W.B[14] first proposed the concept of groups as graphs. Identity graphs are graphs $\mathcal{I}(Z_p)$ associated to group Z_p and $\mathcal{I}(Z_p)$ is defined as (V, E) where $\mathrm{V} = \mathrm{V}(Z_p)$, and two elements $x, y \in Z_p$ are adjacent iff $x.y = e$ or $x = e$ or $y = e$. In this study, the quotient energy of few zero-divisor graphs of commutative ring $\Gamma(R)$ and identity graphs of groups are to be examined.

Quotient energy of Zero-Divisor graph
Definition 1: Let be a (p, q) graph. The quotient matrix $\boldsymbol{Q(G)}$ is defined

$$Q = q_{ij} = \begin{cases} \dfrac{d(v_i)}{d(v_j)} & if\ v_i v_j \in E \\ 0 & otherwise \end{cases}$$

The characteristic polynomial of $Q(G)$ is $f_p(G, \lambda) = \det(Q(G) - \lambda I)$. The quotient spectrum of the graph G is the eigenvalues of the matrix Q. Let $\lambda_1, \lambda_2, \dots \lambda_p$ be the eigenvalues of $Q(G)$. Then the quotient energy is defined as $QE(G) = \sum_{i=1}^{p} |\lambda_i|$.

Theorem 1: $QE(\Gamma(Z_{2p})) = 2\sqrt{p-1}$ for $p > 2$ where p is a prime.

Proof: For any prime $p > 2, V\left(\Gamma(Z_{2p})\right) = \{v_1, v_2, \dots, v_p\} = \{2,4, \dots, 2(p-1), p\}$ and

$E\left(\Gamma(Z_{2p})\right) = \{v_i v_p / 1 \le i \le p-1\}$. Then the corresponding quotient matrix Q of $\Gamma(Z_{2p})$ is of the form $Q = \begin{pmatrix} 0 & (p-1)J_{1\times p-1} \\ \frac{1}{(p-1)}J_{p-1\times 1} & O_{p-1} \end{pmatrix}$, where J and O denotes the matrix of ones and zero matrix, respectively. Thus

$$f(\Gamma(Z_{2p}),\lambda) = (-1)^p \lambda^{p-2}(\lambda^2 - (p-1))$$

Therefore the quotient spectrum of $\Gamma(Z_{2p})$ is,

$$Spec\left(\Gamma(Z_{2p})\right) = \begin{Bmatrix} 0 & \sqrt{p-1} & -\sqrt{p-1} \\ p-2 & 1 & 1 \end{Bmatrix}$$

Hence,

$$QE\left(\Gamma(Z_{2p})\right) = 0_{(p-2\ times)} + \left|\sqrt{p-1}\right| + \left|-\sqrt{p-1}\right|$$

$$QE\left(\Gamma(Z_{2p})\right) = 2\sqrt{p-1}$$

Theorem 2: $QE\left(\Gamma(Z_{3p})\right) = 2\sqrt{2(p-1)}$ for $p > 3$ where p is a prime.

Proof: For any prime $p > 3, V\left(\Gamma(Z_{3p})\right) = \{u_1, u_2, v_1, v_2, \dots, v_{p-1}\}$where $u_1 = p, u_2 = 2p$ and $v_i = 3i\ for\ 1 \le i \le p-1$. And $E\left(\Gamma(Z_{3p})\right) = \{u_1 v_i, u_2 v_i, 1 \le i \le p-1\}$. Then $\left|V\left(\Gamma(Z_{3p})\right)\right| = p+1$ and $\left|E\left(\Gamma(Z_{3p})\right)\right| = 2(p-1)$. Then the corresponding quotient matrix Q is of the form $Q = \begin{pmatrix} O_2 & \left(\frac{p-1}{2}\right)J_{2\times p-1} \\ \left(\frac{2}{p-1}\right)J_{p-1\times 2} & O_{p-1} \end{pmatrix}$, where J and O denotes the, matrix of ones and zero matrix, respectively. Thus

$$f(\Gamma(Z_{3p}),\lambda) = \lambda^{p-1}(\lambda^2 - 2(p-1))$$

Therefore the quotient spectrum of $\Gamma(Z_{3p})$ is,

$$Spec\left(\Gamma(Z_{3p})\right) = \begin{Bmatrix} 0 & \sqrt{2(p-1)} & -\sqrt{2(p-1)} \\ p-1 & 1 & 1 \end{Bmatrix}$$

Hence,

$$QE\left(\Gamma(Z_{3p})\right) = 0_{(p-1\ times)} + \left|\sqrt{2(p-1)}\right| + \left|-\sqrt{2(p-1)}\right|$$

$$QE\left(\Gamma(Z_{3p})\right) = 2\sqrt{2(p-1)}$$

Theorem 3: $QE\left(\Gamma(Z_{5p})\right) = 2\sqrt{4(p-1)}$ for $p > 3$ where p is a prime.

Proof: For any prime $p > 3, V\left(\Gamma(Z_{5p})\right) = V_1 \cup V_2$, where $V_1 = \{u_1, u_2, u_3, u_4\} = \{p, 2p, 3p, 4p\}$ and $V_2 = \{v_1, v_2, \dots, v_{p-1}\} = \{5, 10, \dots, 5(p-1)\}$ and $E\left(\Gamma(Z_{5p})\right) = \{u_1 v_i, u_2 v_i, u_3 v_i, u_4 v_i, 1 \le i \le p-1\}$. Then $\left|V\left(\Gamma(Z_{5p})\right)\right| = p+3$ and $\left|E\left(\Gamma(Z_{5p})\right)\right| = 4(p-1)$. The corresponding quotient matrix Q is of the form $Q = \begin{pmatrix} O_4 & \left(\frac{p-1}{4}\right)J_{4\times p-1} \\ \left(\frac{4}{p-1}\right)J_{p-1\times 4} & O_{p-1} \end{pmatrix}$, where J and O denotes the matrix of ones and zero matrix, respectively. Thus

$$f(\Gamma(Z_{5p}),\lambda) = \lambda^{p+1}(\lambda^2 - 4(p-1))$$

Therefore the quotient spectrum of $\Gamma(Z_{5p})$ is,

$$Spec\left(\Gamma(Z_{5p})\right) = \begin{Bmatrix} 0 & \sqrt{4(p-1)} & -\sqrt{4(p-1)} \\ p+1 & 1 & 1 \end{Bmatrix}$$

Hence,

$$QE\left(\Gamma(Z_{5p})\right) = 0_{(p+1\ times)} + \left|\sqrt{4(p-1)}\right| + \left|-\sqrt{4(p-1)}\right|$$

$$QE\left(\Gamma(Z_{5p})\right) = 2\sqrt{4(p-1)}$$

Theorem 4: $QE\left(\Gamma(Z_{pq})\right) = 2\sqrt{(p-1)(q-1)}$ for $p, q > 2$ where p, q are distinct primes $p < q$.

Proof: For any distinct primes $p, q > 2$, with $p < q, V\left(\Gamma(Z_{pq})\right) = V_1 \cup V_2$, where $V_1 = \{u_1, u_2, \dots, u_{q-1}\} = \{p, 2p, 3p, \dots p(q-1)\}$; $V_2 = \{v_1, v_2, \dots, v_{p-1}\} = \{q, 2q, \dots, q(p-1)\}$; and $E\left(\Gamma(Z_{pq})\right) = \{u_i v_j; u_i \in V_1, v_j \in V_2; 1 \le i \le q-1; 1 \le i \le p-1\}$.Then

$\left|V\left(\Gamma(Z_{pq})\right)\right| = pq - p - q + 1$ and

$\left|E\left(\Gamma(Z_{pq})\right)\right| = 2(q-1)$.

The corresponding quotient matrix Q is of the form $Q = \begin{pmatrix} O_{p-1} & \left(\frac{q-1}{p-1}\right)J_{p-1\times q-1} \\ \left(\frac{p-1}{q-1}\right)J_{q-1\times p-1} & O_{q-1} \end{pmatrix}$, where J and O denotes the matrix of ones and zero matrix, respectively. Thus

$$f(\Gamma(Z_{pq}),\lambda) = \lambda^{p+q-4}\left(\lambda^2 - ((p-1)(q-1))\right)$$

Therefore the quotient spectrum of $\Gamma(Z_{pq})$ is,

$$Spec\left(\Gamma(Z_{pq})\right) = \begin{Bmatrix} 0 & \sqrt{(p-1)(q-1)} & -\sqrt{(p-1)(q-1)} \\ p+q-4 & 1 & 1 \end{Bmatrix}$$

Hence,

$$QE\left(\Gamma(Z_{pq})\right) = 0_{(p+q-4\ times)} + \left|\sqrt{(p-1)(q-1)}\right| + \left|-\sqrt{(p-1)(q-1)}\right|$$

$$QE\left(\Gamma(Z_{pq})\right) = 2\sqrt{(p-1)(q-1)}$$

Theorem 5: $QE\left(\Gamma(Z_{2p})+\Gamma(Z_4)\right)=1+\sqrt{8p-7}$ for $p>2$ where p is a prime.

Proof: For any prime $p>2, V\left(\Gamma(Z_{2p})+\Gamma(Z_4)\right)=\{u_1,u_2,\dots,u_{p-1},u_p,x\}$ $=\{2,4,\dots,2(p-1),p,x\}$, where $x=2\in Z_4$ and $E\left(\Gamma(Z_{2p})+\Gamma(Z_4)\right)=\{u_iu_p,u_ix,u_px/\ 1\le i\le p-1\}$. Then $\left|V\left(\Gamma(Z_{2p})+\Gamma(Z_4)\right)\right|=p+1$ and $\left|E\left(\Gamma(Z_{2p})+\Gamma(Z_4)\right)\right|=2p-1$. The corresponding quotient matrix Q is of the form $Q=\begin{pmatrix} J_2-I_2 & \left(\frac{p}{2}\right)J_{2\times p-1} \\ \left(\frac{2}{p}\right)J_{p-1\times 2} & O_{p-1}\end{pmatrix}$, where I, J and O denotes the identity matrix, matrix of ones and zero matrix, respectively. Thus

$$f\left(\Gamma(Z_{2p})+\Gamma(Z_4),\lambda\right)=\lambda^{p-2}(\lambda+1)(\lambda^2-\lambda-2(p-1))$$

Therefore the quotient spectrum of $\Gamma(Z_{2p})+\Gamma(Z_4)$ is,

$$Spec\left(\Gamma(Z_{2p})+\Gamma(Z_4)\right)=\begin{Bmatrix} 0 & -1 & \frac{1+\sqrt{8p-7}}{2} & \frac{1-\sqrt{8p-7}}{2} \\ p-2 & 1 & 1 & 1\end{Bmatrix}$$

Hence,

$$QE\left(\Gamma(Z_{2p})+\Gamma(Z_4)\right)=|-1|+0_{(p-2\ times)}+\left|\frac{1+\sqrt{8p-7}}{2}\right|+\left|\frac{1-\sqrt{8p-7}}{2}\right|$$

$$QE\left(\Gamma(Z_{2p})+\Gamma(Z_4)\right)=1+\sqrt{8p-7}$$

Theorem 6: $QE\left(\Gamma(Z_{2p})+\Gamma(Z_6)\right)=2(1+2\sqrt{p})$ for $p>2$ where p is a prime.

Proof: For any prime $p>2$, $V\left(\Gamma(Z_{2p})+\Gamma(Z_6)\right)=\{u_1,u_2,\dots,u_{p-1},u_p,x,y,z\}=\{2,4,\dots,2(p-1),p,x,y,z\}$, where $x=2, y=3$ *and* $z=4\in Z_6$ and $E\left(\Gamma(Z_{2p})+\Gamma(Z_6)\right)=\{u_iu_p,u_ix,u_iy,u_iz,u_px,u_py,u_pz,xy,yz/\ 1\le i\le p-1\}$. Then $\left|V\left(\Gamma(Z_{2p})+\Gamma(Z_6)\right)\right|=p+3$ and $\left|E\left(\Gamma(Z_{2p})+\Gamma(Z_6)\right)\right|=4p+1$. The corresponding quotient matrix Q is of the form $Q=\begin{pmatrix} A & B \\ C & D\end{pmatrix}$, where

$A=\begin{pmatrix} 0 & \left(\frac{p+2}{4}\right)J_{1\times p-1} \\ \left(\frac{4}{p+2}\right)J_{p-1\times 1} & O_{p-1}\end{pmatrix}$,

$B=\begin{pmatrix} 1 & \left(\frac{p+2}{p+1}\right)J_{1\times 2} \\ \left(\frac{4}{p+2}\right)J_{p-1\times 1} & \left(\frac{4}{p+1}\right)J_{p-1\times 2}\end{pmatrix}$,

$C=\begin{pmatrix} 1 & \left(\frac{p+2}{4}\right)J_{1\times p-1} \\ \left(\frac{p+1}{p+2}\right)J_{2\times 1} & \left(\frac{p+1}{4}\right)J_{2\times p-1}\end{pmatrix}$ and

$D=\begin{pmatrix} 0 & \left(\frac{p+2}{p+1}\right)J_{1\times 2} \\ \left(\frac{p+1}{p+2}\right)J_{2\times 1} & O_2\end{pmatrix}$. Thus

$$f\left(\Gamma(Z_{2p})+\Gamma(Z_6),\lambda\right)=\lambda^{p-1}(\lambda+1)(\lambda^3-\lambda^2-4p\lambda-6(p-1))$$
$$\le\lambda^{p-1}(\lambda+1)(\lambda^3-\lambda^2-4p\lambda+4p)$$
$$=\lambda^{p-1}(\lambda+1)(\lambda-1)(\lambda^2-4p)$$

Therefore the quotient spectrum of $\Gamma(Z_{2p})+\Gamma(Z_6)$ is,

$$Spec\left(\Gamma(Z_{2p})+\Gamma(Z_6)\right)=\begin{Bmatrix} 0 & -1 & 1 & \sqrt{4p} & -\sqrt{4p} \\ p-1 & 1 & 1 & 1 & 1\end{Bmatrix}$$

Hence,

$$QE\left(\Gamma(Z_{2p})+\Gamma(Z_6)\right)=0+|-1|+1+\left|\sqrt{4p}\right|+\left|-\sqrt{4p}\right|$$

$$QE\left(\Gamma(Z_{2p})+\Gamma(Z_6)\right)=2(1+2\sqrt{p})$$

Theorem 7: $QE\left(\Gamma(Z_{2p})+\Gamma(Z_9)\right)=2+2\sqrt{3p-2}$ for $p>2$ where p is a prime.

Proof: For any prime $p>2$, $V\left(\Gamma(Z_{2p})+\Gamma(Z_9)\right)=\{u_1,u_2,\dots,u_{p-1},u_p,x,y\}=\{2,4,\dots,2(p-1),p,x,y\}$, where $x=3$ and $y=6\in Z_9$ and $E\left(\Gamma(Z_{2p})+\Gamma(Z_9)\right)=\{u_iu_p,u_ix,u_iy,u_px,u_py,xy/\ 1\le i\le p-1\}$. Then $\left|V\left(\Gamma(Z_{2p})+\Gamma(Z_9)\right)\right|=p+2$ and $\left|E\left(\Gamma(Z_{2p})+\Gamma(Z_9)\right)\right|=3p$. Then the corresponding quotient matrix Q is of the form $Q=\begin{pmatrix} A & B \\ C & D\end{pmatrix}$,

$A=\begin{pmatrix} 0 & \left(\frac{p+1}{3}\right)J_{1\times p-1} \\ \left(\frac{3}{p+1}\right)J_{p-1\times 1} & O_{p-1}\end{pmatrix}$,

$B=\begin{pmatrix} J_{1\times 2} \\ \left(\frac{3}{p+1}\right)J_{p-1\times 1}\end{pmatrix}$, $C=\begin{pmatrix} J_{2\times 1} & \left(\frac{p+1}{3}\right)J_{2\times p-1}\end{pmatrix}$

and $D=J_2-I_2$. Thus,

$$f\left(\Gamma(Z_{2p})+\Gamma(Z_9),\lambda\right)=-\lambda^{p-2}(\lambda+1)^2\left(\lambda^2-2\lambda-3(p-1)\right)$$

Therefore the quotient spectrum of $\Gamma(Z_{2p})+\Gamma(Z_9)$ is,

$$Spec\left(\Gamma(Z_{2p})+\Gamma(Z_9)\right)=\begin{Bmatrix} 0 & -1 & 1+\sqrt{3p-2} & 1-\sqrt{3p-2} \\ p-2 & 2 & 1 & 1 \end{Bmatrix}$$

Hence,

$$QE\left(\Gamma(Z_{2p})+\Gamma(Z_9)\right)=0_{(p-2\ times)}+|-1|_{(2\ times)}+\left|1+\sqrt{3p-2}\right|+\left|1-\sqrt{3p-2}\right|$$

$$QE\left(\Gamma(Z_{2p})+\Gamma(Z_9)\right)=2+2\sqrt{3p-2}$$

Theorem 8: $QE\left(\Gamma(Z_{p^2})\right)=2(p-2)$ for $p>2$ where p is a prime.

Proof: Since $\Gamma(Z_{p^2})\cong K_{p-1}$, $QE\left(\Gamma(Z_{p^2})\right)=QE(K_{p-1})=2(p-2)$.

Theorem 9: $QE\left(\Gamma(\overline{Z_{p^2}})+\Gamma(Z_4)\right)=2\sqrt{p-1}$ for $p>2$ where p is a prime.

Proof: Since the resulting graph of $\Gamma(\overline{Z_{p^2}})+\Gamma(Z_4)\cong\Gamma(Z_{2p})$,

$$QE\left(\Gamma(\overline{Z_{p^2}})+\Gamma(Z_4)\right)=QE\left(\Gamma(Z_{2p})\right)=2\sqrt{p-1}$$

Theorem 10: $QE\left(\Gamma(\overline{Z_{p^2}})+\Gamma(Z_6)\right)=1+\sqrt{12p-7}$ for $p>2$ where p is a prime.

Proof: For any prime>2, $V\left(\Gamma(\overline{Z_{p^2}})+\Gamma(Z_6)\right)=\{u_1,u_2,\dots,u_{p-1},x,y,z\}=\{2,4,\dots,p(p-1),x,y,z\}$, where $x=2, y=3$ and $z=4\in Z_6$. And $E\left(\Gamma(\overline{Z_{p^2}})+\Gamma(Z_6)\right)=\{u_ix,u_iy,u_iz,\mathrm{xy},\mathrm{yz}/\ 1\le i\le p-1\}$. Then $\left|V\left(\Gamma(\overline{Z_{p^2}})+\Gamma(Z_6)\right)\right|=p+2$ and $\left|E\left(\Gamma(\overline{Z_{p^2}})+\Gamma(Z_6)\right)\right|=3p-1$. And the corresponding quotient matrix Q is of the form $Q=\begin{pmatrix} A & B \\ C & D\end{pmatrix}$, $A=\begin{pmatrix} 0 & \left(\frac{p+1}{p}\right)J_{1\times 2} \\ \left(\frac{p}{p+1}\right)J_{2\times 1} & O_2\end{pmatrix}$, $B=\begin{pmatrix} \left(\frac{p+1}{3}\right)J_{1\times p-1} \\ \left(\frac{p}{3}\right)J_{2\times p-1}\end{pmatrix}$,

$C=\left(\left(\frac{3}{p+1}\right)J_{p-1\times 1} \quad \left(\frac{3}{p}\right)J_{p-1\times 2}\right)$ and $D=O_{p-1}$

Thus

$$f\left(\Gamma(\overline{Z_{p^2}})+\Gamma(Z_6),\lambda\right)=-\lambda^{p-1}(\lambda^3-(3p-1)\lambda-4(p-1))\le-\lambda^{p-1}(\lambda^3-(3p-1)\lambda-(3p-2))=-\lambda^{p-1}(\lambda+1)(\lambda^2-\lambda-(3p-2))$$

Therefore the quotient spectrum of $\Gamma(\overline{Z_{p^2}})+\Gamma(Z_6)$ is,

$$Spec\left(\Gamma(\overline{Z_{p^2}})+\Gamma(Z_6)\right)=\begin{Bmatrix} 0 & -1 & \frac{1+\sqrt{12p-7}}{2} & \frac{1-\sqrt{12p-7}}{2} \\ p-1 & 1 & 1 & 1 \end{Bmatrix}$$

Hence,

$$QE\left(\Gamma(\overline{Z_{p^2}})+\Gamma(Z_6)\right)=|-1|+\left|\frac{1+\sqrt{12p-7}}{2}\right|+\left|\frac{1-\sqrt{12p-7}}{2}\right|$$

$$QE\left(\Gamma(\overline{Z_{p^2}})+\Gamma(Z_6)\right)=1+\frac{2(\sqrt{12p-7})}{2}$$

$$\Rightarrow QE\left(\Gamma(\overline{Z_{p^2}})+\Gamma(Z_6)\right)=1+\sqrt{12p-7}$$

Theorem 11: $QE\left(\Gamma(\overline{Z_{p^2}})+\Gamma(Z_9)\right)=1+\sqrt{8p-7}$ for $p>2$ where p is a prime.

Proof: For any prime $p>2$, $V\left(\Gamma(\overline{Z_{p^2}})+\Gamma(Z_9)\right)=\{u_1,u_2,\dots,u_{p-1},x,y\}=\{2,4,\dots,p(p-1),x,y\}$, where $x=3$ and $z=6\in Z_9$. And $E\left(\Gamma(\overline{Z_{p^2}})+\Gamma(Z_9)\right)=\{u_ix,u_iy,\mathrm{xy}/\ 1\le i\le p-1\}$. Then $\left|V\left(\Gamma(\overline{Z_{p^2}})+\Gamma(Z_9)\right)\right|=p+1$ and $\left|E\left(\Gamma(\overline{Z_{p^2}})+\Gamma(Z_9)\right)\right|=2p-1$. Then the corresponding quotient matrix Q is of the form $Q=\begin{pmatrix} A & B \\ C & D\end{pmatrix}$, where $A=J_2-I_2$, $B=\frac{p}{2}J_{2\times p-1}$, $C=\frac{2}{p}J_{p-1\times 2}$ and $D=O_{p-1}$. Thus

$$f\left(\Gamma(\overline{Z_{p^2}})+\Gamma(Z_9),\lambda\right)=\lambda^{p-2}(\lambda+1)(\lambda^2-\lambda-2(p-1))$$

Therefore the quotient spectrum of $\Gamma(\overline{Z_{p^2}})+\Gamma(Z_9)$ is,

$$Spec\left(\Gamma(\overline{Z_{p^2}})+\Gamma(Z_9)\right)=\begin{Bmatrix} 0 & -1 & \frac{1+\sqrt{8p-7}}{2} & \frac{1-\sqrt{8p-7}}{2} \\ p-2 & 1 & 1 & 1 \end{Bmatrix}$$

Hence,

$$QE\left(\Gamma(\overline{Z_{p^2}})+\Gamma(Z_9)\right)=0_{(p-2\ times)}+|-1|+\left|\frac{1+\sqrt{8p-7}}{2}\right|+\left|\frac{1-\sqrt{8p-7}}{2}\right|$$

$$QE\left(\Gamma(\overline{Z_{p^2}})+\Gamma(Z_9)\right)=1+\sqrt{8p-7}$$

Quotient energy of Identity graph $\mathcal{J}(Z_p)$ of the group $(Z_p,+)$

Theorem 12: For even $p > 2$, $QE(\mathcal{I}(Z_p,+)) = p - 2 + 2\sqrt{p-1}$.

Proof: For the Identity graph $\mathcal{I}(Z_p)$ of the group $(Z_p,+)$,
$V(\mathcal{I}(Z_p,+)) = \{0,1,\dots,p-1\} = \{u_1,u_2,\dots,u_p\}$.
Then the quotient matrix of $\mathcal{I}(Z_p,+)$ is of the form

$$Q = \begin{pmatrix} 0 & \left(\frac{p-1}{2}\right)J_{1\times p-2} & p-1 \\ \left(\frac{2}{p-1}\right)J_{p-2\times 1} & D & O_{p-2\times 1} \\ \frac{1}{p-1} & O_{1\times p-2} & 0 \end{pmatrix},$$

where D is a diagonal matrix with diagonal entry $J_2 - I_2$. Then the characteristic polynomial of $\mathcal{I}(Z_p,+)$ is,

$$f(\mathcal{I}(Z_p,+),\lambda) = (\lambda+1)^{\frac{p}{2}-1}(\lambda-1)^{\frac{p}{2}-2}(\lambda^3-\lambda^2-(p-1)\lambda+1)$$
$$\le (\lambda+1)^{\frac{p}{2}-1}(\lambda-1)^{\frac{p}{2}-2}(\lambda^3-\lambda^2-(p-1)\lambda+p-1)$$
$$= (\lambda+1)^{\frac{p}{2}-1}(\lambda-1)^{\frac{p}{2}-1}(\lambda^2-(p-1))$$

$$Spec(\mathcal{I}(Z_p,+)) = \begin{Bmatrix} -1 & 1 & \sqrt{p-1} & -\sqrt{p-1} \\ \frac{p}{2}-1 & \frac{p}{2}-1 & 1 & 1 \end{Bmatrix}$$

Now,

$$QE(\mathcal{I}(Z_p,+)) = |-1|_{\frac{p}{2}-1} + |1|_{\frac{p}{2}-1} + +|\sqrt{p-1}| + |-\sqrt{p-1}|$$
$$QE(\mathcal{I}(Z_p,+)) = \frac{p}{2}-1+\frac{p}{2}-1+2\sqrt{p-1}$$
$$QE(\mathcal{I}(Z_p,+)) = p-2+2\sqrt{p-1}$$

Theorem 13: For odd $p > 2$, $QE(\mathcal{I}(Z_p,+)) = p - 2 + \sqrt{4p-3}$.

Proof: For the Identity graph $\mathcal{I}(Z_p,+)$ of the group $(Z_p,+)$, $V(\mathcal{I}(Z_p,+)) = \{0,1,\dots,p-1\} = \{u_1,u_2,\dots,u_p\}$. the unique quotient matrix of $\mathcal{I}(Z_p,+)$ is of the form

$$Q = \begin{pmatrix} 0 & \left(\frac{p-1}{2}\right)J_{1\times p-1} \\ \left(\frac{2}{p-1}\right)J_{p-1\times 1} & D \end{pmatrix},$$

in which the $J_{2\times 2}$ repeated along the diagonal. Then the characteristic polynomial of $\mathcal{I}(Z_p,+)$ is,

$$f(\mathcal{I}(Z_p,+),\lambda) = -(\lambda-1)^{\lfloor p/2\rfloor-1}(\lambda+1)^{\lfloor p/2\rfloor}(\lambda^2-\lambda+(p-1))$$

$$Spec(\mathcal{I}(Z_p,+)) = \begin{Bmatrix} 1 & -1 & \frac{1+\sqrt{4p-3}}{2} & \frac{1-\sqrt{4p-3}}{2} \\ \lfloor p/2\rfloor - 1 & \lfloor p/2\rfloor & 1 & 1 \end{Bmatrix}$$

Now,

$$QE(\mathcal{I}(Z_p,+)) = 1_{(\lfloor p/2\rfloor-1\ times)} + |-1|_{(p/2\ times)} + \left|\frac{1+\sqrt{4p-3}}{2}\right| + \left|\frac{1+\sqrt{4p-3}}{2}\right|$$
$$QE(\mathcal{I}(Z_p,+)) = p-2+\sqrt{4p-3}$$

Conclusion:

Quotient energy of certain zero divisor graphs of ring and identity graphs of groups are evaluated in this present article. This method can be extended to any kind of graphs as well.

Authors' declaration:

- Conflicts of Interest: None.
- We hereby confirm that all the Figures in the manuscript are ours. Besides, the Figures and images, which are not ours, have been given the permission for re-publication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in A.P.C. Mahalaxmi College for Women.

Authors' contributions:

This work was carried out in collaboration between all authors. M. L K wrote and edited the manuscript with new ideas. K. P and L. P reviewed the results with suggestions for corrections. All authors read and approved the final manuscript.

References:

1. Gutman I. The Energy of a Graph: Old and New Results. Algebraic Combinatorics and Applications. Springer, Berlin, Heidelberg. 2001; 196–211. https://doi.org/10.1007/978-3-642-59448-9_13
2. Gao W, Asghar A, Nazeer W. Computing Degree-Based Topological Indices of Jahangir Graph. Eng Appl Sci. Lett. 2018; 1(1): 16–22. http://dx.doi.org/10.30538/psrp-easl2018.0003
3. Zangi S, Ghorbani M. Eslampour M. On the eigenvalues of some matrices based on vertex degree. Iranian J Math. Chem. 2018; 9: 149–156. 10.22052/ijmc.2017.93637.1303
4. Adirasari RP, Suprajitno H, Susilowati L. The Dominant Metric Dimension of Corona Product Graphs. Baghdad Sci J. 2021; 18(2): 0349. https://doi.org/10.21123/bsj.2021.18.2.0349
5. Hussein LH, Abed SS. Fixed Point Theorems in General Metric Space with an Application. Baghdad Sci J. 2021; 18(1): 0812. https://doi.org/10.21123/bsj.2021.18.1(Suppl.).0812
6. Basavanagoud B, Chitra E. Degree square sum energy of graphs. Int. j. math. appl. 2018; 6(2B): 193–205. http://ijmaa.in/index.php/ijmaa/article/view/716
7. Palani K, Kumari ML. Locating geo spectrum and geo energy of graphs. Adv Appl Math Sci. 2021; 21(4): 2157-

2167.https://www.mililink.com/issue_content.php?id=59&iId=395&vol=21&is=4&mon=February&yer=2022&pg=1615-2281
8. Palani K. Kumari ML. On Total Energy of a Graph. Adv Appl Math Sci. 2021; 21(2): 565-578. https://www.mililink.com/issue_content.php?id=59&iId=392&vol=21&is=2&mon=December&yer=2021&pg=519-1074
9. Palani K, Kumari ML, Pandiselvi L. Total Energy of Cycle and Some Cycle Related Graphs. J Phys.: Conf Ser. 2021; 1947: 012007. doi:10.1088/1742-6596/1947/1/012007
10. Beck I. Coloring of commutative rings. J Algebra. 1988; 116: 208−226. PII: 0021-8693(88)90202-5
11. Anderson DF, Livingston PS. The zero-divisor graph of a commutative ring. J Algebra. 1999; 217: 434−447. https://doi.org/10.1006/jabr.1998.7840
12. Rather BA, Ali F, Ullah A, Fatima N. Dad R. On Aγ eigenvalues of zero divisor graphs of integer modulo and Von Neumann regular rings. Symmetry. 2022; 14: 1710. https://doi.org/10.3390/sym14081710
13. Singh P, Bhat VK. Zero-divisor graphs of finite commutative rings: A survey. Surv. Math its Appl. 2020: 1-15. https://www.emis.de/journals/SMA/v15/p15_14.pdf
14. Kandasamy VWB, Smarandache F. Groups as Graphs. Romania: Editura CuArt; 2009. p. 170. https://ui.adsabs.harvard.edu/link_gateway/2009arXiv0906.5144V/doi:10.48550/arXiv.0906.5144

حاصل طاقة للبيانات المقسوم على الصفر ورسومات المطابقة

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الخلاصة:
إذا اعتبرنا ان (p,q) هورسم البياني المتصل البسيط $G = (V, E)$ مجموع القيم المطلقة لسلسلة مصفوفة الحاصل في الرسم البياني تشكل الرسم البياني لطاقة الحاصل. الهدف من هذه الدراسة هو فحص طاقة حاصل الرسوم البيانية للمطابقة والرسوم البيانية القاسم الصفري للحلقات التبادلية باستخدام نظرية المجموعة ونظرية الرسم البياني، والتطبيقات. في هذه الدراسة، يتم فحص الرسوم البيانية للمطابقة المشتقة من المجموعة وفئات قليلة من الرسوم البيانية لقاسم الصفر للحلقة التبادلية.R

الكلمات الرئيسية: الحلقة التبادلية، الرسوم المطابقة، طاقة الحاصل، مصفوفة الحاصل، الرسوم القاسم الصفري.