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On Pre-Door Space

Ghassan .A. Khtan
Department of Mathematics, College of Education, University of Kerbala,

Hussam A. Mohammed
Department of Mathematics College of Education, University of Kerbala,

Enaam H. Abd
Department of Computer, College of Sciences, University of Kerbala,

Abstract:

In this search obtain the results on pre-door space:

Door space is pre-door, space Submaximal pre-door space is pre-door space,Irreducible ubmaximal space is pre-door space , Quasi-precompact images of pre-door space are pre-door pace and A pre-Hausdorff pre-door space has at most one a pre-ccumulation point.

المستخلص:

في هذا البحث حصلنا على النتائج في الفضاءات التبولوجية Pre-door :

وهي كل فضاء (door) هو فضاء (pre-door) وتكافؤ الفضائين (pre-door) و (door) في الفضاء (submaximal) والفضائين (Irreducible) و (submaximal) يعطينا الفضاء (pre-door) ودراسة الخاصية التبولوجية للفضاء (pre-door) ويحتوي الفضاء pre-door و pre-Hausdorff على نقطة تجمع واحدة .

1. Preliminaries:

In this section we recall the basic definitions and results needs in this work, we study be recall the definition of door space and preopen set.

Definition 1.1 [4]: A topological space (X,τ) is called a door space if every subset of X is either open or closed.

Definition 1.2 [2]: A subset A of (X,τ) is called a preopen set if $A \subseteq Int(Cl\,A)$, $B \subseteq X$ is preopen set then B^c is called preclosed set. Where $Int(Cl\,A)$ is interior clouser of A.

Remarks 1.3 [2]:

1. Every open (closed) subset of (X,τ) is preopen (preclosed) set, but the converse is not true.

2. If $U \subseteq X$ is open set and V is preopen set in X, then $U \cap V$ is preopen in X, and if $V \subset Y \subset X$, then V is preopen in Y.

Definition 1.4 [1]: Let (X,τ) be a topological space and *A* subset of *X*. A point $x \in X$ is called a pre-ccumulation point of *A* if every preopen set *G* containing x contains a point of *A* other then *x*.

Definition 1.5 [1]: A topological space (X,τ) is called a pre-Hausdorff space if every two disjoint points can be separated by disjoint preopen sets.

Remark 1.6 [1]: Every Hausdorff space is pre-Hausdorff, but the converse is not necessarily true.

2. Pre-door space

Definition 2.1: A topological space (X,τ) is called a pre-door space if every subset of *X* is either preopen or preclosed.

Examples 2.2:

1. The discrete space (X,D) is a pre-door space.
2. Let $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ be a topology on $X = \{a, b, c\}$, then (X,τ) is pre-door space.

Proposition 2.3: Every door space is a pre-door space.

Proof: Let (X,τ) be a door space.
Since every open (closed) set is a preopen (preclosed) remark 1.3.1.
Thus every subset of *X* is preopen or preclosed set.
Therefore (X,τ) is a pre-door space.

Remark 2.4: The converse of proposition 2.3 is not true, for example:
Let $\tau = \{X, \phi, \{a\}\}$ be a topology on $X = \{a, b, c\}$. Then (X,τ) is pre-door space, but is not door space, since $\{b\}$, $\{c\}$, $\{a, b\}$ and $\{a, c\}$ are subsets of *X*, not either open or closed sets in *X*.

Recall that a topological space (X,τ) is called submaximal if every dense subset of *X* is open. Reilly and Vamanamurthy [3] have shown that (X,τ) is submaximal if and only if each preopen subset of (X,τ) is open.

Proposition 2.5: Every submaximal pre-door space is door space.

Proof: Let (X,τ) be a pre-door space, then every subset of *X* is a preopen or preclosed set. Let *S* subset of *X*, therefore *S* is preopen or preclosed set.
If *S* is a preopen, then *S* is open (in submaximal space), if *S* is preclosed, then S^c is preopen, where S^c is open, then *S* is closed, therefore *X* is door space.

Proposition 2.6: Every subspace *Y* of a pre-door space *X* is a pre-door space.

Proof: Let *A* subset of *Y* .Since *X* is pre-door space,
Then *A* is either preopen or preclosed in *X* and hence in *Y*.
Thus *Y* is also a pre-door space.

A non-void space X is irreducible [6] if it satisfies the following conditions:

a) Every two non-void open subset of X intersect.
b) X is not the union of a finite family of closed proper subsets.
c) Every non-void open subset of X is dense.
d) Every open subset of X is connected.

Proposition 2.7: Every irreducible submaximal space X is a pre-door space.

Proof: Let $A \subseteq X$, if A is dense in X, then by submaximality A is open.
If A is not dense, then we can find a non-void open set $B \subseteq A^c$.
Since X is irreducible, then B is dense and hence A^c is also dense.
Again by submaximality of X, A^c is open or equivalently A is closed.
Thus in any case A is either open or closed. By remark 1.3.1,
A is either preopen or preclosed. This shown that X is pre-door space.

Theorem 2.8: Let $(X_i)_{i\in I}$ be a family of topological spaces for the topological $sum\, X = \sum_{i\in I} X_i$ the following conditions are equivalent:

1- X is pre-door space.
2- Each X_i is a pre-door space and X_i is non-discrete for at most one index.

Proof: (1) $\Rightarrow$ (2) by proposition 2.6 each X_i is a pre-door space.
Assume next that for some index i and j, X_i and X_j are a non-discrete. Thus, there exists a non-preopen $A \subseteq X_i$ and a non-preclosed $B \subset X_j$. If $A \cup B$ is a preopen subset of X, then $(A \cup B) \cap X_i = A$ is a preopen subset of X_i, which is contradiction. If $A \cup B$ is preclosed in X, then B must be preclosed in X_j, which is again a contradiction.Thus condition (2) is proved.

(2) $\Rightarrow$ (1) we can assume that $I \neq \phi$, since otherwise $X \neq \phi$ and the claim would be trivial. By (2) for some $j \in I$, X_j is a pre-door space and X_i is discrete for every $i \neq j$. Let $A \subseteq X$. Then $A = \cup_{i\in I}(X_i \cap A)$ and moreover the set $X_i \cap A$ is preclopen in X_i for each hand $X_j \cap A$ is either preopen or preclosed in X_j.
Thus A is either preopen or preclosed in X or equivalently X is a pre-door space.

Recall that a function $f : X \to Y$ is called quasi-precompact if it satisfies.
The following condition: If $U \subset X$ is preopen such that $f^{-1}(f(U)) = U$, then $f(U)$ is preopen in Y. Sets satisfying the condition $f^{-1}(f(U)) = U$ are called the inverse sets of the function $f : X \to Y$.

Theorem 2.9: Quasi-precompact images of pre-door space are pre-door space.

Proof: Let $V \subset Y$. We need to show that V is either preopen or preclosed, since X is a pre-door space, and then $U = f^{-1}(V)$ is either preopen or preclosed in X. Let U be preopen.

Clearly $f(U) \subseteq V$ and thus $f^{-1}(f(U)) \subseteq f^{-1}(V) = U \subseteq f^{-1}(f(U))$ or equivalently $U = f^{-1}(f(U))$. By assumption $f(U) \subseteq f(f^{-1}(V)) = V \cap f(X) = V \cap Y = V$ is preopen in X. Hence $(Y - V) \cap f(X) = Y - V$ is preopen in Y and thus $Y - (Y - V) = V$ is preclosed in Y. This shows that Y is a pre-door space.

Corollary 2.10: preopen images as well as preclosed images of pre-door spaces are pre-door space.

Proof: Every preopen and every preclosed surjective function is quasi-precompact.

Proposition 2.11: A pre-Hausdorff pre-door space has at most one a pre-ccumulation point.

Proof: Let X be a pre-Hausdorff pre-door space, suppose for contradiction that there are two distinct a pre-ccumulation points x and y, since we are working in pre-door space it well be important to remember that any subset of X is preopen or precolsed set. If $\{x\}$ (similarly $\{y\}$) is preopen, then we have a problem in that since x is pre-ccumulation point, and $\{x\}$ is an preopen neighborhood of x, then by the definition, $\phi \neq \{x\} - \{x\} \cap X$, but clearly $\{x\} - \{x\} \cap X = \phi$, since X is pre-Hausdorff there are preopen neighborhoods U and V of x and y respectively, such that $U \cap V = \phi, U - \{x\} \cup \{y\}$ is preclosed if it where preopen, then we could say that $\{y\} = (U - \{x\} \cup \{y\}) \cap V$ is preopen, so we conclude that as $U - \{x\} \cup \{y\}$ is preclosed, thus $x - (U - \{x\} \cup \{y\})$ is preopen, and hence we have a contradiction, then X has at most one a pre-ccumulation point.

An ideal [5],[7] I on a topological space (X,τ) is a non-empty collection of subsets of X which satisfies the following two properties:

1) $A \in I$ and $B \subset A$ implies $B \in I$ (heredity),
2) $A \in I$ and $B \in I$ implies $A \cup B \in I$ (finite additivity).

Note that the following collections form important ideals in a space (X,τ):

I_f – the ideal of finite subset of X,
I_c – the ideal of countable subset of X,
I_{cd} – the ideal of closed discrete set in (X,τ),
I_n – the ideal of nowhere dense set in (X,τ),
I_k – the ideal of relatively compact set in (X,τ).

Definition 2.12: A topological space (X,τ,I) is called an I_{pre}-door space if every subset of X is either preopen or preclosed or belongs to I.

Clearly every pre door space is an I_{pre}-door space and in fact the two notions coincide in the case of the minimal ideal (i.e., when I=Ø). In what follows we try to find some classes of well-known topological spaces in which the concepts of I_{pre}-door space and pre-door spaces again coincide, but this time in the case when the ideal is non-trivial, considering some of the above mentioned collections of sets which form a topological ideal on any topological space.

Recall that a space is called pre-extremally disconnected if the closure of every preopen set is preopen.

Proposition 2.13: For a topological space (X,τ) the following conditions are equivalent:

1) (X,τ) is an pre-extremally disconnected pre-door space .
2) Every subset A of X is either preopen or both preclosed and discrete.

Proof: (1) $\Rightarrow$(2) Note first that $A \in I_{cd}$ if and only if *A* has no accumulation points. Assume now that *A* is not preopen and *N(A)* $N(A) \notin \phi$.

Since *X* is a pre-door space, then *A* is preclosed. Let $x \in N(A) \subset A$.
Set $S = (X \mid A) \bigcup \{x\}$. Since *x* is an accumulation point of *A*, then *S* can not be preopen, for in that case *S* would be an preopen pre-neighbourhood of x containing appoint of *A* different that x. Clearly this is impossible, since $S \bigcap A = \{x\}$. Since *S* is not preopen and since *X* is a pre-door space, then *S* is preclosed. thus $(X \mid A) \bigcup \{x\} = \overline{X \mid A} \bigcup \overline{\{x\}}$ and hence $X \mid A \subset \overline{X \mid A} \subset (X \mid A) \bigcup \{x\}$. By assumption, since *X* is pre-extremally disconnected. $\overline{(X \mid A)}$ is preopen . Since *S* is preclosed, then $X \mid A \subset \overline{X \mid A} \subset (X \mid A) \bigcup \{x\} = S$. Thus $X \mid A = \overline{X \mid A}$ or equivalently *A* is preopen. Thus contradicts with the assumption *A* is not preopen and *N(A)* $N(A) \notin \phi$.
(2) $\Rightarrow$(1) X is trivially a pre-door space. We need to show that *X* is pre-extremally disconnected. Let $A \subset X$ be preopen. Then $\overline{A} = A \bigcup N(A)$. If $N(A) = \phi$, then $\overline{A} = A$ is preopen. If $N(A) \neq \phi$, then $N(\overline{A}) \neq \phi$ and $\overline{A}$ is by (2) preopen.

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