A Decision – Theoretic Bayesian Approach For Selecting the Best of Gamma Populations With General Loss Function

Samira Faisal Hathot
Dept. of Mathematics
College of Education for Girls / University of Kufa
samirafaisal@yahoo.com

Abstract

Statistical selection procedures are used to select the best of a finite set of alternatives. This paper derives a procedure for selecting the best of two Gamma populations employing a decision-theoretic Bayesian framework with general loss function with Exponential prior .

The numerical result of this procedure are given with different loss functions constant , linear and quadratic , where in one equation we can obtain the Bayes risk for the three types of the loss functions : constant , linear and quadratic . in this paper the numerical results are given by using Math Works Matlab ver. 7.0.1 .

Keywords and phrases : selection procedure, general loss function , Bayesian decision theoretic , Exponential prior , Bayes risk .

الخلاصــة

تستخدم طرق الاختيار الاحصائية لاختيار الافضل من بين مجموعة محدودة من البدائل . هذا البحث يتضمن اشتقاق طريقة لاختيار الافضل من بين مجتمعيين يتوزعان توزيع كاما مستخدمين منهج القرار البيزي مع دالة خسارة مشتركة مع توزيع سابق للتجربة متمثل بالتوزيع الآسي .

النتائج العددية لهذا الاجراء تم إيجادها لدوال خسارة مختلفة ثابتة ، خطية وتربيعية حيث في معادلة واحدة بالامكان الحصول على الخطورة البيزية للأنواع الثلاثة من دوال الخسارة : الثابتة ، الخطية والتربيعية . وفي هذا البحث قدمنا نتائج عددية تم الحصول عليها باستخدام نظام الـ Matlab ver 7.0.1 .

1-Description of the Problem

The Gamma distribution has an important role for modeling the life time distribution of a variety of random phenomena . This distribution arises in many areas of application , including reliability , life – testing and survival analysis .

A common problem that arises in practice is the selection of the best of two Gamma populations with unknown parameters .

Formally , we can state the problem as follows : Consider two independent Gamma populations Π_1, Π_2 with known probability density function

$$h_i(y_i|\alpha_i,\theta_i) = \frac{\theta_i^{\alpha_i}}{\Gamma(\alpha_i)} y_i^{\alpha_i - 1} e^{-\theta_i y_i} \quad , y_i > 0 , \alpha_i > 0 \ , \theta_i > 0$$

With known shape parameter α_i and unknown scale parameter θ_i (i=1,2) . We consider the problem : how to find the best population (i.e. the one associated with the largest scale parameter

θ_i) . Let $\theta_{[1]} \leq \theta_{[2]}$ be the ordered values of the parameters θ_1, θ_2 . It is assumed that the exact pairing between the ordered and unordered parameters is unknown . The population Π_i with $\theta_i = \theta_{[2]}$(i=1,2) is called the best population . A correct selection is defined as the selection of the population associated with $\theta_{[2]}$.

Many researchers have considered this problem under different types of formulations .Shanti S. Gupta (1962) considered the problem of selecting a subset of k Gamma populations which includes the "best" population . P.Vellaisamy , D. Sharma (1988) considered classic procedure for selected Gamma population from two Gamma populations . Neeraj Misra , et.al. (2006) consider selected Gamma population under the scale invariant squared error loos function . Paul Van Der Laan & Constance Van Eden (1996) study the subset selection procedure for studied selecting the Best of Two Gamma population . Neeraj Misra (1994) considered subset selection procedure for selected Gamma populations . Dailami , N. ; Rao , M. Bhaskara ; Subramanyam , K. (1985) study the selection of the best Gamma population , determination of minimax sample size . Studied Nematollahi (2009) estimation of the scale parameter of the selected Gamma population under Entropy loss function .

The aim of this present paper is to derive approach for selecting the best of two Gamma populations , that is the one having the largest scale parameter $\theta_{[2]}$ by using Bayesian decision – theoretic framework with exponential prior and general loss function .

2-Basic Definitions and Concepts

2-1-Statistical Decision Theory

(i) Basic Ideas

Statistics may be consider as the science of decision making in the presence of uncertainty . The problems of statistical inferences can fit into the decision theory framework , for example , testing of a hypothesis H_o against a hypothesis H_1 may be regarded as a decision between two actions (i) accepting H_o or (ii) accepting H_1 .

In decision problems , the state of nature is unknown , but a decision maker must be made – a decision whose consequences depend on the unknown state of nature . Such a problem is a statistical decision problem when there are data that give partial information a bout the unknown state .

The basic elements of a statistical decision problem can be formalized mathematically as follows:

A set $\mathbb{A}$, the action space , consisting of all possible actions , $a \in \mathrm{A}$, available to the decision maker ;

a set Ω , the parameter space , consisting of all possible 'state of the nature' , $\theta \in \Omega$, one and only one of which obtains or will obtain (this 'true' state being unknown to the decision-maker) ;

a function L , the loss function , having domain $\Omega \times \mathrm{A}$ (the set of all ordered pairs of consequences $(\theta, a), \theta \in \Omega, a \in \mathrm{A}$) and codomain R ;

a set R_x , the space of X , consisting of all the possible realizations , $x \in R_x$ of a random variable X , having a distribution whose probability function (pf) belongs to a specified family $\{f(x;\theta); \theta \in \Omega\}$;

A set D , the decision space, consisting of all possible decisions , $d \in D$, each such decision function *d* having domain R_x and codomain $\mathbb{A}$.

(ii) The Risk Function

For given (θ, a) the loss function depends on the outcome x and thus a random variable . Its expected value , i.e. its average over all possible outcomes is called the risk function and is denoted by

$$R(\theta,d)=\int_{R_x} L(\theta,d(x))f(x;\theta)dx \qquad \text{(X continues)}$$

or

$$R(\theta,d)=\sum_{x\in R_x} L(\theta,d(x))f(x;\theta) \qquad \text{(X discrete)}$$

(iii) Minimax and Bayes Decision Functions

The decision function d^* that minimizes M(d)= max R(θ,d) is the minimax decision function . Similarly , the function d^{**} that minimizes the Bayes risk of a decision d is a Bayes decision function .

$$B(d)=E[R(\theta,d)]=\int_{\Omega} R(\theta,d)\pi(\theta)d\theta \qquad (\Theta \text{ continous})$$

or

$$B(d)=\sum_{\Omega} R(\theta,d)\pi(\theta) \qquad (\Theta \text{ discrete})$$

where $\pi(\theta)$ represents the distribution of degree of belief over Θ .

3- Solution of the Problem

We term our problem as a two-decision problem and represent it symbolically as

$$d_1 : \text{population } \Pi_1 \textit{ is said to be the best if } \theta_1 > \theta_2$$

and ………..…….(2-1)

$$d_2 : \text{population } \Pi_2 \textit{ is said to be the best if } \theta_1 \le \theta_2$$

For parameter $\underline{\theta}$ and action a the loos function is defined as :

$$L_i(\theta_i,a_i)=k_i(\theta_i-a_i)^r \quad ,\text{i}=1,2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2-2)$$

For r=0 , we have a constant loss function , for r=1 , we have a linear loss function and for r=2 , we have a quadratic loss function , k_1,k_2 give decision losses in units of costs .

Let us suppose that $\underline{y}_i=(y_{i1},y_{i2},\ldots,y_{in})$ be a random sample of size n arising from population Π_i . It follows that the likelihood function is

$$P_o(\underline{y}|\theta_i) = \left(\frac{\theta_i^{\alpha_i}}{\Gamma(\alpha_i)}\right)^n e^{-\theta_i \sum_{j=1}^{n} y_{ij}} \prod_{j=1}^{n} y_{ij}^{\alpha_i - 1} \quad , \quad \theta_i > 0 , y > 0 , \alpha_i > 0 \quad , \ldots\ldots\ldots\ldots\ldots(2\text{-}3)$$

Our first task in the Bayesian approach is the specification of a prior p.d.f g(λ) . we take the prior distribution to be a member of the conjugate class of Exponential priors $Exp(\lambda_i)$, where a member of this class has density function

$$g(\theta_i|\lambda_i) = \lambda_i e^{-\lambda_i\theta_i} \quad , \lambda_i > 0 \ , \theta_i > 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2\text{-}4)$$

By Baye's theorem the posterior probability function of θ is given by

$$g(\theta_i|\underline{y}_i) = \lambda_i' \, \mathrm{e}^{-\theta_i\lambda_i'} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2-5)$$

Where $\lambda' = \sum_{j=1}^{n} y_{ij} + \lambda_i \quad , \ i = 1,2$

We note , as a function of θ_i , $g(\theta_i|\underline{y}_i)$ has the form of Exponential probability density function with parameters λ_i' .

We derive the stopping (Baye's) risks of decision d_1 and d_2 for general loss function given in (2-2) and the stopping risk (the posterior expected looses) of making decision d_i denoted by $R_i(\theta_1, \theta_2; d_i)$

$$R_1(\theta_1,\theta_2;d_1) = k_1\left[\sum_{i=0}^{r}\frac{r!(-1)^{r-i}}{(\lambda_2')^{r-i}(\lambda_1')^{i}} - \sum_{i=0}^{r}\sum_{j=0}^{r-i}\frac{r!(-1)^{r-i}\lambda_1'(\lambda_2')^{-j}(r-j)!}{i!(r-i-j)!(\lambda_1'+\lambda_2')^{r-j+1}}\right]$$

$$R_2(\theta_1,\theta_2;d_2) = k_2\left[\sum_{i=0}^{r}\frac{r!(-1)^{r-i}}{(\lambda_1')^{r-i}(\lambda_2')^{i}} - \sum_{i=0}^{r}\sum_{j=0}^{r-i}\frac{r!(-1)^{r-i}\lambda_2'(\lambda_1')^{-j}(r-j)!}{i!(r-i-j)!(\lambda_1'+\lambda_2')^{r-j+1}}\right]$$

If we take r=0 we find from the above equations the posterior expected looses for constant loos function for the two decisions d_1 and d_2 , if we take r=1 we find from the above equations the posterior expected looses for linear loss function for the two decisions , if we take r=2 we find from the above equations the posterior expected looses for quadratic loos function for two decision d_1 and d_2 .

For the two – decision problem considered a above , the Bayesian selection procedure is given as follows :

Make decision d_1 that is selecting Π_1 as the best population if $R_1(\theta_1,\theta_2;d_1) < R_2(\theta_1,\theta_2;d_2)$ and

Make decision d_2 that is selecting Π_2 as the best population if $R_1(\theta_1,\theta_2;d_1) \geq R_2(\theta_1,\theta_2;d_2)$

4- Numerical Results and Discussions

This section contains some numerical result about this procedure , we take various sample size n and various priors . We write a program for this procedure from which we give three types of Risk for three types of loss functions (constant , linear and quadratic) . from this numerical result we note that :

1-the procedure is well defined , as we seen in table (1) .

2-as sample size n increase , the Bayes risk decreases .

3-The Bayes risk for quadratic loss function is less than the Bayes risk for linear and constant loss functions .

$\theta_1 = 10, \theta_2 = 2, \alpha_1 = \alpha_2 = 3$					
Prior Prob. (λ_1, λ_2)	n	Bayes Risk	Constant Loss	Linear Loss	Quadratic Loss
(4,7)	10	$R(d_1)$	0.4197	0.0016	2.0194e-005
		$R(d_2)$	1.5803	0.0183	8.3546e-004
	20	$R(d_1)$	0.3637	5.4201e-041	2.4397e-006
		$R(d_2)$	1.6363	0.0111	1.4006e-004
	30	$R(d_1)$	0.3343	3.2460e-004	6.4979e-007
		$R(d_2)$	1.6657	0.0086	1.0006e-004
	40	$R(d_1)$	0.3416	3.8191e-004	6.2400e-007
		$R(d_2)$	1.6584	0.0055	6.8677e-005
	50	$R(d_1)$	0.3281	2.0458e-004	4.3678e-007
		$R(d_2)$	1.6719	0.0048	2.3776e-005
(6,10)	10	$R(d_1)$	0.4522	9.5929e-004	1.7892e-005
		$R(d_2)$	1.5478	0.0222	3.9764e-004
	20	$R(d_1)$	0.3402	3.9715e-0051	4.4915e-006
		$R(d_2)$	1.6598	0.0137	1.4732e-004
	30	$R(d_1)$	0.3295	4.4710e-004	1.2465e-006
		$R(d_2)$	1.6705	0.0077	9.4003e-005
	40	$R(d_1)$	0.3388	2.6590e-004	4.6549e-007
		$R(d_2)$	1.6612	0.0071	8.2271e-005
	50	$R(d_1)$	0.2911	1.8176e-004	2.5639e-007
		$R(d_2)$	1.7089	0.0059	2.6276e-005
(8,12)	10	$R(d_1)$	0.4474	7.2249e-004	3.8048e-006
		$R(d_2)$	1.5526	0.0220	5.1106e-004
	20	$R(d_1)$	0.3379	4.2249e-0041	1.4942e-006
		$R(d_2)$	1.6621	0.0123	1.9099e-004
	30	$R(d_1)$	0.3890	2.9478e-004	9.1622e-007
		$R(d_2)$	0.3546	0.0089	6.5576e-005
	40	$R(d_1)$	0.3546	2.4135e-004	3.8832e-007
		$R(d_2)$	1.6454	0.0069	4.5099e-005

	50	R(d_1)	0.3458	1.6914e-004	2.1242e-007
		R(d_2)	1.6542	0.0053	2.9020e-005
(10,14)	10	R(d_1)	0.4164	0.0022	7.9105e-006
		R(d_2)	1.5836	0.0146	4.5275e-004
	20	R(d_1)	0.3864	5.0357e-004	2.5672e-006
		R(d_2)	1.6136	0.0155	1.7845e-004
	30	R(d_1)	0.3943	4.0063e-004	1.1416e-006
		R(d_2)	1.6057	0.0077	1.0671e-004
	40	R(d_1)	0.3683	3.0555e-004	7.3245e-007
		R(d_2)	1.6317	0.0061	5.0115e-005
	50	R(d_1)	0.3177	1.4337e-004	2.9984e-007
		R(d_2)	1.6823	0.0062	3.5540e-005

Table (1)

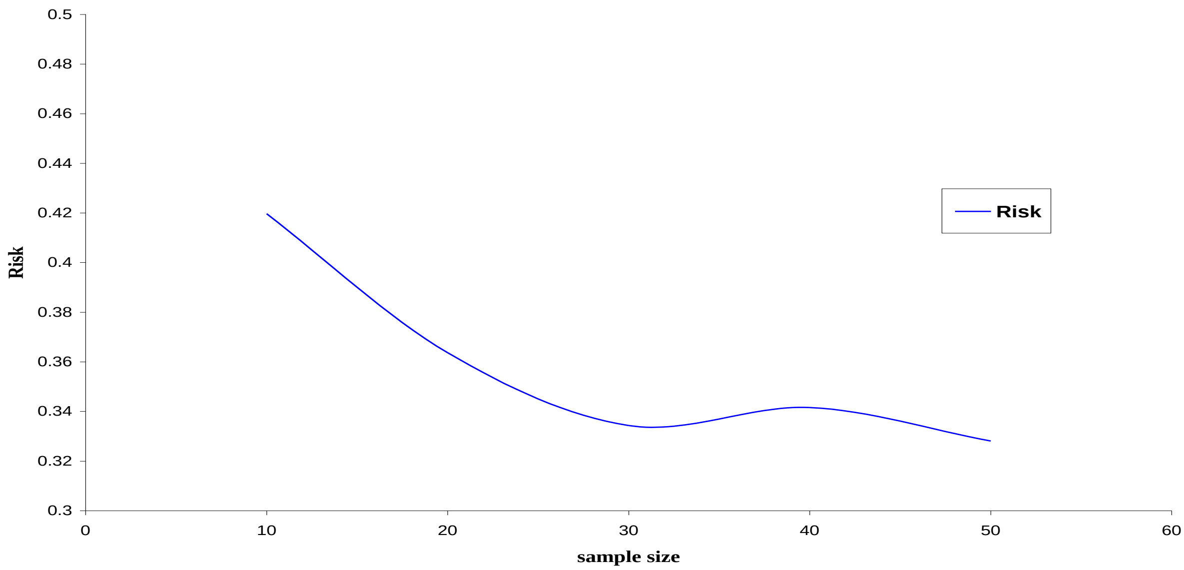

Figure (1) : the influence of the sample size on the posterior expected loss for constant loss function

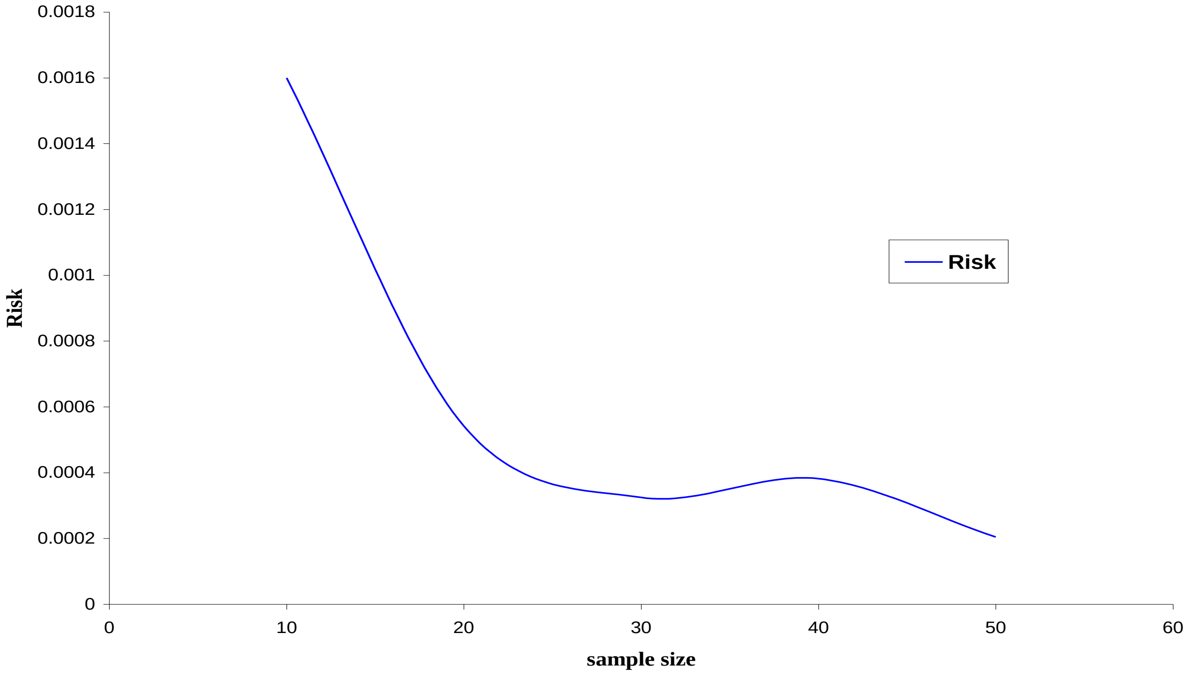

Figure (2) : the influence of the sample size on the posterior expected loss for linear loss function

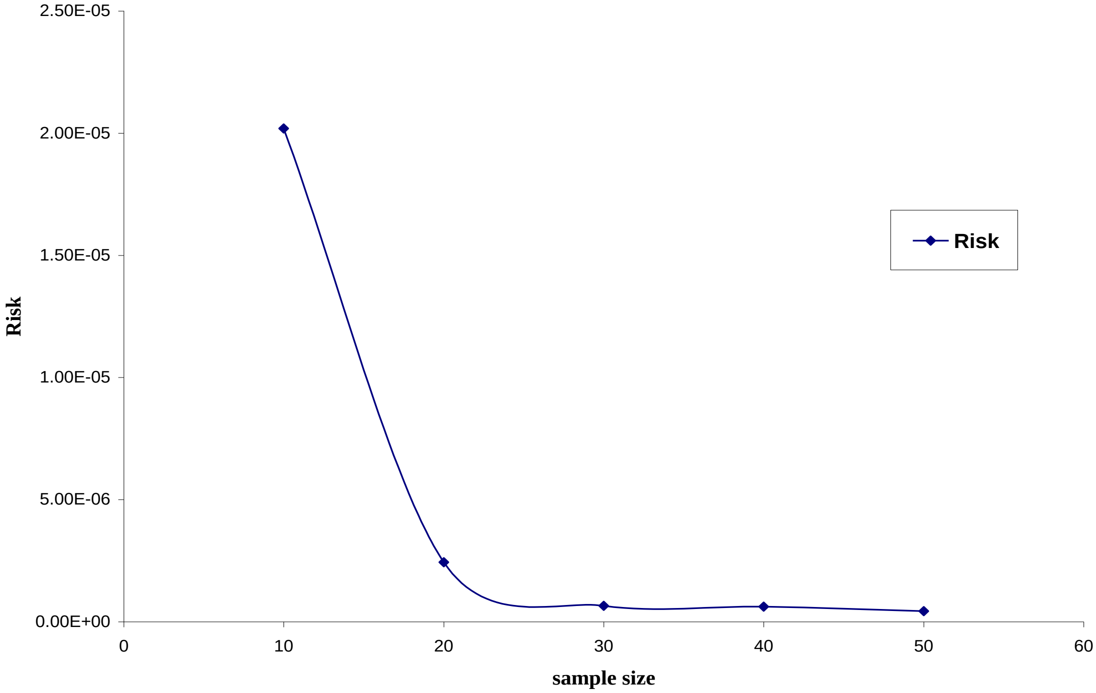

Figure (3) : the influence of the sample size on the posterior expected loss for quadratic loss function

Conclusions

In this paper we derives a procedure for selecting the best of two Gamma populations employing a decision – theoretic Bayesian frame work with general loss function with Exponential prior . From this paper we note that :

1- the procedure is well defined , as we seen in table (1)

2- as sample size n increase , the Bayes risk decreases with all loss functions .

3- the Bayes risk for quadratic loss function is less than the Bayes risk for linear and constant loss function .

References

1- Gupta , S.S. (1962) . On a selection and ranking procedure for gamma populations . Annals of mathematical statistics , vol. 14 , pp. 199-216 .

2- Vellaisamy P. and Sharma D. (1988) . Estimation of the mean of the selected Gamma population . Communications in statistics-theory and methods , vol. 117 , issue 8 . pp. 2797-2817 .

3- Misra , N. and van der M. and Edward , C. (2006) . On estimating the scale parameter of the selected Gamma population under the scale invariant squared error loss function . Journal of computational and applied mathematics , vol. 186 , pp. 268-282 .

4- Paual van Der laan and Contance van Eeden . (1996) . On using a loss function in selecting the best of two Gamma populations in terms of their scale parameters . vol. 28 , issue 4 , pp. 355-370 .

5- Misra , N. (1994) . Estimation of the average worth of the selected subset of gamma populations . The Indian journal of statistics , series B , vol. 56 , pt. 3 , pp. 344 -355 .

6- Dailami , N. and Rao , M. Bhaskara and Subramanyam , K. (1985) . On the selection of the best Gamma population , determination of minimax sample sizes .

7- N. Nematollahi ; F. Motamed-Shariati . (2009) . Estimation of the scale parameter of the selected Gamma population under the entropy loss function . Communications in statistics-theory and methods , vol. 38 , issue 2 , pp. 208-221 .