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أنعام محمد علي هادي ، شروق بهجت
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الخلاصة

لتكن R حلقة ابدالية بمحايد. في هذا البحث قدمنا ودرسنا المقاسات التوزيعية الضبابية والحلقات الحسابية الضبابية تعميمين (اعتياديين) للمقاسات التوزيعية والحلقات الحسابية. وأعطينا بعض الخواص الاساسية حول هذه المفاهيم.

Fuzzy Distributive Modules

I.M.A.Hadi, Sh. B.Semeein
Department of Mathematics, College of Education ,Ibn-Al-Haitham, University of Baghdad

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Abstract

Let R be a commutative ring with unity. In this paper we introduce and study fuzzy distributive modules and fuzzy arithmetical rings as generalizations of (ordinary) distributive modules and arithmetical ring. We give some basic properties about these concepts.

Introduction

In this paper we introduce and study fuzzy distributive modules as a generalization of the concept (distributive modules) in ordinary algebra.

In section one, we recall some basic definitions and results which we will be needed later.

In section two, we give some basic results about fuzzy distributive modules. Also we study the direct sum of fuzzy distributive modules.

In section three, we study the homomorphic image and inverse image of fuzzy distributive modules.

In section four, we introduce and study fuzzy arithmetical rings as a generalization of the concept (arithmetical rings) in ordinary algebra.

1. Preliminaries

In this section, some basic definitions and results are collected.

1.1 Definition [1]

Let S be a non-empty set and I be the closed interval [0,1] of the real line (real numbers). A fuzzy set A in S (a fuzzy subset of S) is a function from S into I.

1.2 Definition [2]

Let $x_t:S \longrightarrow [0,1]$ be a fuzzy set in S, where $x \in S$, $t \in [0,1]$ defined by:
$X_t(y)=t$ if $x=y$, and $x_t(y) = 0$ if $x \neq y$ $\forall y \in S$.
X_t is called a fuzzy singleton or fuzzy point in S.

1.3 Definition [3]

Let A and B be two fuzzy sets in S, then

1. $A=B$ if and only if $A(X)=B(X)$, for all $x \in S$.
2. $A \subseteq B$ if and only if $A(X) \leq B(X)$, for all $x \in S$.
3. $(A \cap B)(x)=\min\{A(x),B(x)\}$ for all $x \in S$.

1.4 Definition [4]

Let A be a fuzzy set in S, for all $t \in [0,1]$, the set $A_t=\{x \in S, A(x) \geq t\}$ is called level subset of A.

1.5 Remark [1]

The following properties of level subsets hold for each $t \in (0,1]$

1. $(A \cap B)_t=A_t \cap B_t$
2. $A=B$ if and only if $A_t=B_t$, for all $t \in (0,1]$.

1.6 Definition [5]

Let $(R,+,\cdot)$ be a ring and let X be a fuzzy set in R. Then X is called a fuzzy ring in ring $(R,+,\cdot)$ if and only if, for each x, $y \in R$

1. $X(x+y) \geq \min\{X(x), X(y)\}$

2. $X(x) = X(-x)$

3. $X(xy) \geq \min\{X(x), X(y)\}$.

1.7 Definition [6]

A fuzzy subset X of a ring R is called a fuzzy ideal of R, if for each x, y $\in$ R

1. $X(x-y) \geq \min\{X(x), X(y)\}$

2. $X(xy) \geq \max\{X(x), X(y)\}$.

1.8 Definition [2]

Let M be an R-module. A fuzzy set X of M is called a fuzzy module of M if

1. $X(x-y) \geq \min\{X(x), X(y)\}$, for all x, y $\in$M.

2. $X(rx) \geq X(x)$, for all $x\in M$ and $r\in R$.

3. $X(0)=1$.

1.9 Definition [4]

Let X and A be two fuzzy modules of an R-module M. A is called a fuzzy submodule of X if $A\subseteq X$.

1.10 proposition [7]

Let A be a fuzzy set of an R-module M. Then the level subset A_t, $t\in[0,1]$ is a submodule of M if and only if A is a fuzzy submodule of X where X is a fuzzy module of an R-module M.

1.11 Definition [8]

Let $X:R\longrightarrow[0,1]$ be a fuzzy ring, let $A:R\longrightarrow[0,1]$. A is called a fuzzy ideal of X if A satisfies the following:

1. $A\neq\varnothing$
2. $A(x-y) \geq \min\{A(x), A(y)\}$, for all x, y $\in$R.
3. $A(xy) \geq \min\{X(x), A(y)\}$, for all x, y $\in$R.
4. $A(x) \leq X(x)$, $\forall$ x $\in$ R.

1.12 Definition [9]

Let A, B be two fuzzy ideals of a fuzzy ring X. Then

1. The sum A+B of A and B is defined as:

$$(A+B)(x) = \sup_{a+b=x} \{\min\{A(a),B(b)\}, \forall x\in R.$$

2. The product AB of A and B is defined as

$$(AB)(x) = \sup_{x=\sum_{i=1}^{n} a_i b_i} \{\inf\{\min\{A(a_i),B(b_i)\}\}.$$

1.13 Proposition

Let A and B be two fuzzy submodules of a fuzzy module X .Then
$(AB)_t = A_tB_t$, $\forall$ $t\in(0,1]$.

Proof : by similar proof in [10,theorem 2.4] .

1.14 Proposition

Let A and B be two fuzzy submodules of fuzzy module. Then
$(A+B)_t = A_t+B_t$, $\forall$ $t\in(0,1]$.

Proof:- Let $x\in(A+B)_t$. Then
$(A+B)(x)=\sup\{\min\{A(a),B(b)\}, x=a+b\}\geq t$
But A+B has a suprimum property, so there exist a, b $\in$M such that
$\sup\{\min\{A(a),B(b)\}, x=a+b\}=\min\{A(a),B(b)\}\geq t$
consequently, $A(a)\geq t$, $B(b)\geq t$. Thus, $a\in A_t$ and $b\in B_t$, it follows that
$x=a+b\in A_t+B_t$
which means $(A+B)_t \subseteq A_t+B_t$

Now, let $x\in A_t+B_t$, then $\exists!$ $a\in A_t$ and $\exists!$ $b\in B_t$ such that x= a+b. Thus
$(A+B)(x)=\sup\{\min\{A(a),B(b)\},x=a+b\}$

$= \min\{A(a),B(b)\} \geq t$

Since the representation of any element of M is unique.

1.15 Definition [2]

Suppose that A and B be two fuzzy modules of R-modules M. We define (A:B) by:-

$(A:B)=\{r_1:r_1$ is a fuzzy singleton of R such that $r_1B \subseteq A\}$

and

$(A:B)(r)=\sup\{ t\in[0,1] \mid r_tB\subseteq A$, for all $r\in R\}$

If $B=(b_k)$, then:

$(A:(b_k))=\{r_t \mid r_tb_k \subseteq A$, r_t is a fuzzy singleton of R$\}$

1.16 Definition [11]

Let X and Y be two fuzzy modules of M_1, M_2 respectively. Define $X\oplus Y$: $M_1\oplus M_2 \longrightarrow [0,1]$ by

$(X\oplus Y)(a,b)=\min\{X(a),Y(b)\}$ for all $(a,b) \in M_1\oplus M_2$

$X\oplus Y$ is called a fuzzy external direct sum of X and Y.

1.17 Proposition [11]

Let X and Y are fuzzy modules of M_1 and M_2 respectively, then $X\oplus Y$ is a fuzzy module of $M_1\oplus M_2$.

1.18 Proposition [11]

Let A and B be two fuzzy submodules of a fuzzy module X, such that $X=A\oplus B$, then $X_s = A_s \oplus B_s$ for all $s\in(0,1]$.

2. Fuzzy Distributive Module

In this section we fuzzyify the concept of distributive modules into fuzzy distributive modules. Then we study some of their basic properties.

Recall that an R-module M is said to be distributive if for any R-submodules A, B and C of M,

$A\cap(B+C) = (A\cap B)+(A\cap C)$ [12].

2.1 Definition

Let M be an R-module, let X be a fuzzy module over M. X is called distributive if for any fuzzy submodules A, B and C of X,

$A\cap(B+C) = (A\cap B)+(A\cap C)$

The following result explains the relationship between fuzzy distributive modules and its level.

2.2 Theorem

A fuzzy module X of an R-module M is a fuzzy distributive if and only if X_t is a distributive module, $\forall t\in(0,1]$.

Proof: If X is fuzzy distributive module. To prove X_t is distributive module.

$\forall$ $t\in(0,1]$, let I, J, K be submodules of X_t. Define

$$A(x)=\begin{cases} t & x\in I \\ 0 & x\notin I \end{cases},\ B(x)=\begin{cases} t & x\in J \\ 0 & x\notin J \end{cases},\ C(x)=\begin{cases} t & x\in K \\ 0 & x\notin K \end{cases}$$

It is clear that A, B, C are fuzzy submodules of X and $A_t=I$, $B_t=J$, $C_t=K$. Since X is fuzzy distributive, $A\cap(B+C) = (A\cap B)+(A\cap C)$. Hence

$[A\cap(B+C)]_t = [(A\cap B)+(A\cap C)]_t$, $\forall$ $t\in(0,1]$.

$A_t\cap(B+C)_t = (A\cap B)_t+(A\cap C)_t$ (remark 1.5 and proposition 1.13)

$A_t\cap(B_t+C_t) = (A_t\cap B_t) +(A_t\cap C_t)$ (remark 1.5 and proposition 1.13)

This $I\cap(J+K) = (I\cap J)+(I\cap K)$

Conversely, if X_t is a distributive module, for all $t\in(0,1]$. To prove X is a fuzzy distributive module.

Let A, B and C fuzzy submodules in X. Then A_t, B_t, C_t are submodules in X_t, for all $t\in(0,1]$. Since X_t is a distributive R-module then

$A_t\cap(B_t+C_t) = (A_t\cap B_t) +(A_t\cap C_t)$

$A_t\cap(B+C)_t = (A\cap B)_t+(A\cap C)_t$ (remark 1.5 and proposition 1.13)

$[A\cap(B+C)]_t = [(A\cap B)+(A\cap C)]_t$ (remark 1.5 and proposition 1.13)

Then $A\cap(B+C) = (A\cap B)+(A\cap C)$. (remark 1.5 ,(2)) ■

2.3 Example

Let $M=R\oplus R$ where R is any ring, M is an R-module, let $X:M \longrightarrow[0,1]$ defined by $X(x)=1$, let

$$A(x,y)=\begin{cases}1 & (x,y)\in R(1,1)\\0 & \text{otherwise}\end{cases},B(x,y)=\begin{cases}1 & (x,y)\in R(0,1)\\0 & \text{otherwise}\end{cases},$$

$$C(x,y)=\begin{cases}1 & (x,y)\in R(1,0)\\0 & \text{otherwise}\end{cases}$$

$A_t=R(1,1)$, $B_t=R(0,1)$, $C_t=R(1,0)$, $\forall\ t\in(0,1]$.

$A_t\cap(B_t+C_t) =R(1,1)$,

$(A_t\cap B_t)+(A_t\cap C_t)=(R(1,1)\cap R(0,1))+(R(1,1)\cap R(1,0))=(0)+(0)=(0)$

Thus $A_t\cap(B+C)_t \neq (A\cap B)_t+(A\cap C)_t$, which implies X_t is not a distributive module. thus X is not a fuzzy distributive module.

2.4 Definition [13]

An R-module M is called chained if for each submodules A, B of M, either $A \subseteq B$ or $B\subseteq A$.

We fuzzified this concept as follows.

2.5 Definition

Let X be a fuzzy module of an R-module M then X is called a fuzzy chained module if for each fuzzy submodules A, B of X, either $A \subseteq B$ or $B \subseteq A$.

Now, we shall give a relationship between fuzzy distributive module and fuzzy chained module.

2.6 Proposition

Let X be a fuzzy chained module of an R-module M. Then X is a fuzzy distributive module.

Proof: Let A, B, C fuzzy submodules of X, we can assume that $A\subseteq B\subseteq C$. Hence $A\cap(B+C)=A\cap B=A$. But $(A\cap B)+(A\cap C)=A+A=A$.

Thus $A\cap(B+C) = (A\cap B)+(A\cap C)$. ■

2.7 Remark

The converse of proposition (2.6) is not true in general as the following example shows.

2.8 Example

Let $X(x)=1$ for all $x\in Z$, $X_t=Z$, $\forall t\in[0,1]$. But Z is distributive. Hence by theorem (2.2), X is a fuzzy distributive. However X is not chained since there exists fuzzy submodules A, B such that

$$A(x)=\begin{cases}1 & x\in 2Z\\0 & x\notin 2Z\end{cases},\ B(x)=\begin{cases}1 & x\in 3Z\\0 & x\notin 3Z\end{cases}$$ and $A\not\subseteq B$ and $B \not\subseteq A$.

Now, we can give the following.

2.9 Theorem

Let X be a fuzzy distributive module of an R-module M, then for all a_t, $b_k \in X$, $<1_j>=(a_t:b_k)+(b_k:a_t)$ for all $j\in(0,1]$.

Proof:

Let a_t, $b_k \in X$, then $a\in X_t$, $b\in X_k$. Assume $k\leq t$. Hence $X_k \supseteq X_t$ and so $a\in X_k$. Thus a, b $\in X_k$. But X_k is distributive R-module so $(a:b)+(b:a)=R$ (by [12,theorem (1.3),p.54). It follows that $1=r_1+r_2$ where $r_1\in(a:b)$, $r_2\in(b:a)$ for some r_1, r_2. Hence $1_j= (r_1)_j + (r_2)_j$ for all $j\in(0,1]$. But

$(r_1)_j \cdot b_k = (r_1 \cdot b)_s$, where s = min{j,k}

$= (r' \cdot a)_s$, since $r_1 \in (a:b)$

$\subseteq < a_s > \subseteq < a_k >$

Hence $(r_1)_j \in (a_t:b_k)$, $\forall$ j$\in$(0,1]

$(r_2)_j \cdot a_t = (r_2 \cdot a)_f$, where f = min{j,t}

$= (r'' \cdot b)_f$, since $r_2 \in (b:a)$

$\subseteq < b_t > \subseteq < b_k >$

Hence $(r_2)_j \in (b_k:a_t)$, $\forall$ j$\in$(0,1]

So $(r_1)_j + (r_2)_j \in (a_t:b_k) + (b_k:a_t)$ and hence $(r_1+r_2)_j \in (a_t:b_k) + (b_k:a_t)$.

Thus $1_j \in (a_t:b_k) + (b_k:a_t)$. ■

2.10 Remark

If Y≤X and X is fuzzy distributive module then Y is fuzzy distributive module.

Proof: Let A, B, C are fuzzy submodules of Y, then A, B, C are fuzzy submodules of X (since Y≤X). But X is fuzzy distributive module, so A∩(B+C) = (A∩B)+(A∩C) which implies Y is a fuzzy distributive. ■

Now, we study the direct sum of fuzzy distributive modules. But first we state and prove the following lemma.

2.11 Lemma

If M_1, M_2 are distributive R-modules such that $annM_1 + annM_2=R$, then $M_1 \oplus M_2=M$ is a distributive R-module.

Proof: Let A, B, C be submodules of M. Since $annM_1 + annM_2=R$, $A=A_1\oplus B_1$, $B= A_2\oplus B_2$, $C= A_3\oplus B_3$ for some submodules A_1, A_2, A_3 of M_1 and some submodules B_1, B_2, B_3 of M_2. To prove A∩(B+C) = (A∩B)+(A∩C)

$A\cap(B+C) = (A_1\oplus B_1)\cap[(A_2\oplus B_2)+(A_3\oplus B_3)]$

$= (A_1\oplus B_1)\cap[(A_2+A_3)+(B_2+B_3)]$

$= [A_1\cap(A_2+A_3)]\oplus[B_1\cap(B_2+B_3)]$

$= [(A_1\cap A_2)+(A_1\cap A_3)]\oplus[(B_1\cap B_2)+(B_1\cap B_3)]$ (M_1 and M_2 are distributive modules)

$= [(A_1\cap A_2)\oplus(B_1\cap B_2)] + [(A_1\cap A_3)\oplus(B_1\cap B_3)]$

$= [(A_1\oplus B_1)\cap(A_2\oplus B_2)]+[(A_1\oplus B_1)\cap(A_3\oplus B_3)]$

$= (A\cap B)+(A\cap C)$. ■

2.12 Proposition

Let X and Y be fuzzy distributive modules of R-modules M_1, M_2 respectively, then X⊕Y is a fuzzy distributive module of $M_1\oplus M_2$, provided $annM_1 + annM_2 = R$.

Proof:

By theorem (2.2), X_t and Y_t are distributive submodules of M_1 and M_2 respectively, for all t ∈(0,1]. Hence by lemma (2.11) $(X_t\oplus Y_t)$ is a distributive submodule of $M_1\oplus M_2$. But $(X\oplus Y)_t = (X_t\oplus Y_t)$ by ((11), lemma (2.2.4)). Thus X⊕Y is a fuzzy distributive module by theorem (2.2). ■

3. The Image and Inverse Image of Fuzzy Distributive Modules

In this section, we shall indicate the behaviour of fuzzy distributive modules under homomorphisms. To do this we need some definitions and propositions.

3.1 Definition (5)

Let f be a mapping from a set M into a set N, let A be a fuzzy set in M and B be a fuzzy set in N. The image of A denoted by f(A) is the set in N defined by

$$f(A)(y)=\begin{cases}\sup\{A(z)\mid z\in f^{-1}(y)\} & \text{if } f^{-1}(y)\neq\phi, \forall y\in A\\ 0 & \text{otherwise}\end{cases}$$

And the inverse image of f denoted by $f^{-1}(B)$, where $f^{-1}(B)(x)=B(f(x))$, for all x∈M.

Recall the following.

3.2 Definition (14)

Let f be a function from a set M into a set M'. A fuzzy subset A of M is called f-invariant if A(x)=A(y) whenever f(x)=f(y), where x, y $\in$M.

3.3 Definition (4)

Let X and Y be two fuzzy modules of R-modules M_1 and M_2 respectively. f:X$\longrightarrow$Y is called a fuzzy homomorphism if f: $M_1 \longrightarrow M_2$ is R-homomorphism and Y(f(x)) = X(x), for each x$\in M_1$.

3.4 Proposition

Let X and Y be two fuzzy modules of R-modules M_1 and M_2 respectively. f:X$\longrightarrow$Y be a fuzzy homomorphism if A and B are two fuzzy submodules of X and Y respectively, then

1. f(A) is a fuzzy submodule of Y, (14).
2. f^{-1}(A) is a fuzzy submodule of Y, (14).
3. f(A$\cap$B)=f(A)$\cap$f(B), whenever A, B are f-invariant, (15).
4. $f^{-1}(A\cap B)=f^{-1}(A)\cap f^{-1}(B)$, where f is monomorphism, (15).
5. f(A+B)=f(A)+f(B), (15).
6. $f(f^{-1}(A))=A$, (15).
7. $f^{-1}(f(A))=A$ whenever A is f-invariant, (15).

First we have the following result.

3.5 Proposition

Let X and Y be two fuzzy modules of R-modules M_1 and M_2 respectively. Let f:X$\longrightarrow$Y be a fuzzy epimorphism, and every fuzzy submodule of X is f-invariant. If X is a fuzzy distributive module, then Y is a fuzzy distributive module.

Proof: Let A, B, C be fuzzy submodules in Y. $f^{-1}(A)$, $f^{-1}(B)$, $f^{-1}(C)$ are fuzzy submodules in X by proposition 3.4, (2). Since X is a fuzzy distributive, then

$f^{-1}(A)\cap(f^{-1}(B)+ f^{-1}(C)) = (f^{-1}(A)\cap f^{-1}(B)) + (f^{-1}(A)\cap f^{-1}(C))$

$f^{-1}(A)\cap(f^{-1}(B)+ f^{-1}(C)) = (f^{-1}(A\cap B)) + (f^{-1}(A\cap C))$, proposition 3.4,(4))

$f[f^{-1}(A)\cap(f^{-1}(B)+ f^{-1}(C))]=f[(f^{-1}(A\cap B)) + (f^{-1}(A\cap C))]$

$f(f^{-1}(A))\cap f(f^{-1}(B)+ f^{-1}(C)) = f(f^{-1}(A\cap B)) + f(f^{-1}(A\cap C))$, proposition 3.4,(3),(4))

$A\cap(B+ C) = (A\cap B) + (A\cap C)$, proposition 3.4,(6)). ■

3.6 Proposition

Let X and Y be two fuzzy modules over R-modules M_1 and M_2 respectively. Let f:X$\longrightarrow$Y be a fuzzy homomorphism, and every fuzzy submodule of Y is f-invariant. If Y is a fuzzy distributive module, then X is a fuzzy distributive module.

Proof: Let A, B, C are fuzzy submodules in X. Hence f(A), f(B), f(C) are fuzzy submodules in Y, by proposition 3.4,(1). Since Y is a fuzzy distributive module, then

$f(A)\cap(f(B)+ f(C)) = (f(A)\cap f(B)) + (f(A)\cap f(C))$

$f(A)\cap(f(B)+ f(C)) = f(A\cap B) + f(A\cap C))$, proposition 3.4,(3),(5))

$f(A\cap(B+ C)) = f((A\cap B) + (A\cap C))$, proposition 3.4,(3),(5)).

$f^{-1}(f(A\cap(B+ C))) = f^{-1}(f((A\cap B) + (A\cap C)))$.

Then $A\cap(B+ C) = (A\cap B) + (A\cap C)$, proposition 3.4,(7)). ■

4. Fuzzy Arithmetical Rings

In this section, we introduce the notion of arithmetical fuzzy ring. First we have the following definition.

4.1 Definition[12]

A ring R is said to be an arithmetical ring if R, considered as R-module over it self, is distributive that is R is arithmetical if $I\cap(J+K)=(I\cap J)+(I\cap K)$ for all ideals I, J, K of R.

We fuzzify this definition as follows:

4.2 Definition

A fuzzy ring X of a ring R is called arithmetical if and only if $A\cap(B+ C) = (A\cap B) + (A\cap C)$ for all A, B, C fuzzy ideals of X.

4.3 Note

A fuzzy ring X is arithmetical if and only X_t is arithmetical ring $\forall t \in (0,1]$.

4.4 example

Let X(x)=1 for all $x \in Z_4$, $X_t = Z_4$, $\forall$ $x \in Z_4$ .But Z_4 is arithmetical ring. Hence By Note (4.3) , X is fuzzy arithmetical ring

Now, we can give the following theorem.

4.5 Theorem

A fuzzy ring X of a ring R is arithmetical if and only if

$A + (B \cap C) = (A+B) \cap (A+C)$

Proof: Let X be a fuzzy arithmetical ring, let A, B, C be fuzzy ideals of X. Hence A_t, B_t, C_t are ideals of X_t. Since X is fuzzy arithmetical ring then X_t is arithmetical ring (by note (4.3)). Hence,

$A_t + (B_t \cap C_t) = (A_t + B_t) \cap (A_t + C_t)$ ((16),Exc.18)

$A_t + (B \cap C)_t = (A+B)_t \cap (A+C)_t$, (remark (1.5), proposition (1.13))

$(A + (B \cap C))_t = ((A+B)_t \cap (A+C))_t$ (remark (1.5), proposition (1.13))

Thus $A + (B \cap C) = (A+B) \cap (A+C)$.

Conversely, to prove X is a fuzzy arithmetical ring.

We shall prove X_t is an arithmetical ring for all $t \in (0,1]$.

Let I, J, K be ideals in X_t. It follows that there exist A, B, C fuzzy ideals of X, where

$$A(x) = \begin{cases} t & x \in I \\ 0 & x \notin I \end{cases}, B(x) = \begin{cases} t & x \in J \\ 0 & x \notin J \end{cases}, c(x) = \begin{cases} t & x \in K \\ 0 & x \notin K \end{cases}$$

But by hypothesis, $A + (B \cap C) = (A+B) \cap (A+C)$. Hence

$[A + (B \cap C)]_t = [(A+B) \cap (A+C)]_t$ for all $t \in (0,1]$.

It follows that; $A_t + (B_t \cap C_t) = (A_t + B_t) \cap (A_t + C_t)$, (remark (1.5), proposition (1.13)). But $A_t = I$, $B_t = J$, $C_t = K$, hence $I \cap (J+K) = (I \cap J) + (I \cap K)$, which implies that X_t is an arithmetical ring, by ((16), Exc.18). Thus X is an arithmetical ring, by note (4.3). ■

4.6 Theorem

Let R be an integral domain, let X be a fuzzy ring such that X(a)=1 $\forall a \in R$. Then the following are equivalent

1. X is arithmetical
2. $A(B \cap C) = AB \cap AC$ for all fuzzy ideals A, B, C of X.
3. $(A+B)(A \cap B) = AB$ for all fuzzy ideals A, B of X.

Proof: (1) $\Rightarrow$ (2): to prove $A(B \cap C) = AB \cap AC$ for all fuzzy ideals A, B, C of X. Since X is a fuzzy arithmetical then X_t is an arithmetical ring for all $t \in (0,1]$ and since A_t, B_t, C_t are ideals of X_t, $t \in (0,1]$ we get $A_t(B_t \cap C_t) = A_tB_t \cap A_tC_t$, ([14],theorem (6.6)). Hence

$A_t(B \cap C)_t = (AB)_t \cap (AC)_t$, (proposition (1.12),(2), remark (1.5))

$(A(B \cap C))_t = (AB \cap AC)_t$, for all $t \in (0,1]$ (proposition 1.12,(2))

Thus $A(B \cap C) = AB \cap AC$.

(2) $\Rightarrow$ (3): If $A(B \cap C) = AB \cap AC$ for all fuzzy ideals A, B,C of X, let $t \in (0,1]$, let I, J, K be ideals of X_t. Then there exists fuzzy ideals A, B, C of X such that $A_t = I$, $B_t = J$, $C_t = K$, where

$$A(x) = \begin{cases} t & x \in I \\ 0 & x \notin I \end{cases}, B(x) = \begin{cases} t & x \in J \\ 0 & x \notin J \end{cases}, c(x) = \begin{cases} t & x \in K \\ 0 & x \notin K \end{cases}$$

By (2), $A(B \cap C) = (AB) \cap (AC)$, which implies that $(A(B \cap C))_t = (AB \cap AC)_t$, for all $t \in (0,1]$. Hence $A_t(B \cap C)_t = (AB)_t \cap (AC)_t$, (remark (1.5),proposition (1.12)), so that $A_t(B_t \cap C_t) = A_tB_t \cap A_tC_t$, (remark (1.5),proposition (1.12)).

$I (J \cap K) = (I \cap J) + (I \cap K)$. Then by ((16),Exc. 18) $(I+J)(I \cap J) = IJ$.

Thus $(A_t + B_t)(A_t \cap B_t) = A_tB_t$, $(A+B)_t (A \cap B)_t = (AB)_t$ which implies that $(A+B)(A \cap B) = AB$.

(3) $\Rightarrow$ (1):If $(A+B)(A \cap B) = AB$ for all fuzzy ideals A, B of X. Let $t \in (0,1]$, Let I, J, K be

ideals of X_t. Then there exists fuzzy ideals A, B, C of X such that A_t=I, B_t=J, C_t=K, where

$$A(x)=\begin{cases} t & x\in I \\ 0 & x\notin I \end{cases}, B(x)=\begin{cases} t & x\in J \\ 0 & x\notin J \end{cases}, c(x)=\begin{cases} t & x\in K \\ 0 & x\notin K \end{cases}$$

By (3), (A+B)(A∩B)=AB, which implies that $((A+B)(A\cap B))_t=(AB)_t$ for all t ∈(0,1]. Hence $(A+B)_t(A\cap B)_t=A_tB_t$, (proposition (1.12),(2)), so that
$(A_t+B_t)(A_t\cap B_t)=A_tB_t$, (remark (1.5), proposition (1.13)).
(J+K) (I∩J)=IJ. Then by ([14], theorem (6.6)) X_t is arithmetical ring for all t ∈(0,1]. Thus X is a fuzzy arithmetical ring (by note 4.3). ■

4.7 Theorem

Let R be a noetherian integral domain, let X be a fuzzy ring such that X(a)=1 ∀a∈R. Then the following are equivalent

1. X is arithmetical
2. A(B∩C)=AB ∩ AC for all fuzzy ideals A, B, C of X.
3. (A+B)(A∩B)=AB for all fuzzy ideals A, B of X.
4. If A, C are fuzzy ideals of X and if C⊆A, then there exists fuzzy ideal B such that A=BC.

Proof:

(1) ⇒ (2) and (2) ⇒ (3) follows directly by theorem (4.5).
(3) ⇒ (4), let A, C be fuzzy ideals of X such that C⊆A, implies $A_t \subseteq C_t$, then $A_t = B_t \cdot C_t$ ([14], theorem (6.26)), $A_t=(B\cdot C)_t$ then A = BC.
(4) ⇒ (1), let A, C are fuzzy ideals of X, if C⊆A, then there exists fuzzy ideal B of X such that A=BC, then $A_t=(B\cdot C)_t$ so w get $A_t=B_t\cdot C_t$, X_t is arithmetical ring ([14], theorem (6.26)). Thus X is a fuzzy arithmetical ring.

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