Journal of Kerbala University , Vol. 8 No.3 Scientific . 2010

THE CONTINUOUS CLASSICAL OPTIMAL CONTROL PROBLEM of A SEMILINEAR PARABOLIC EQUATION (COCP)

مسألة السيطرة الامثلية التقليدية من النمط المستمر لمعادلة تفاضلية جزئية شبه خطية من النمط المكافيء

إيون خريسوفيرغي [1] و جميل أمير علي الهوّاسي [2]
[1] قسم الرياضيات /كلية العلوم للرياضيات التطبيقية والفيزياء الطبيعية/الجامعة التكنلوجية الوطنية في اثينا
[2] جميل أمير علي الهواسي/ قسم الرياضيات/كلية العلوم/ الجامعة المستنصرية
hawasy20@yahoo.com

المستخلص:-

يتناول هذا البحث دراسة مسألة السيطرة الامثلية التقليدية من النمط المستمر لنظام من المعادلات التفاضلية الجزئية شبه الخطية من النوع المكافيء بوجود قيد التساوي وقيد اللامساوات . أولا" قمنا ببرهان وجود ووحدانية الحل لمعادلات تفاضلية جزئية شبه خطية لسيطرة تقليدية ثابتة ثم لسيطرات تقليدية مختلفة باستخدام طريقة كاليركن لتقريب الحلول المضبوطة . ثانيا" قمنا ببرهان نظرية الوجود لسيطرة أمثلية تقليدية تحت قيد التساوي وقيد اللامساوات. اخيرا" قمنا ببرهان الشوط الضرورية لوجود سيطرة امثلية تقليدية تحت القيدين اعلاه.

Abstract

We deal with in this work the continuous classical optimal control problem of a semilinear parabolic equation with equality and inequality constrains. First we prove for fixed classical control the existence and uniqueness for the solution of parabolic equation using the Galerkin method to approximate the exact solution and then for different classical controls. Second we prove the existence theory of a classical optimal control with equality and inequality constraints.
Finally we prove the necessary conditions for existence of a classical optimal control with the above constraints.

Introduction

During the last decades, many researchers interested to study relaxd optimal control problems for systems governed by differential equations as in [1], [4], [5], [6], [7] and many others. And since many applications in different fields of natural sciences lead to mathematical modules represent a classical optimal control governed by semilinear partial differential equations (heat equation)[2], so we interested in this work to study such optimal control problem with equality and inequality constraints. First, and for fixed classical control under some assumptions the existence and uniqueness of a solution (corresponding state) of a semilinear partial differential equations is proved by using the Galerkin method to approximate exact solution of this equation. With suitable assumptions, we state and prove the existence theorem for a classical optimal control with equality and inequality constrains, during this proof we show that for a convergence sequence of classical optimal control to a classical optimal control the corresponding sequence of states satisfying the states equations converges to a state satisfies the state equation. Finally we state and prove the necessary conditions for a classical optimal control problem under suitable assumptions.

0. Basic concepts:-

Definition 0.1[10] :- Let $Q \subset R^{d+1}$, a function $f(x,t,y,u): Q \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is said to be of *Caratheodory type* if it is continuous w.r.t. y and u for fixed $(x,t) \in Q$, and measurable w.r.t (x,t) for fixed (y,u).

Theorem 0.1(Alaoglu)[3]:- Let V be a Hilbert space , and $\{v_k\}$ be a bounded sequence of V , then there exists a subsequence of $\{v_k\}$ which converges weakly to some $v \in V$.

Lemma 0.1 [2]:- The Continuous Bellman-Gronwall (B-G) inequality:-
Let $I=[a,b]$, z , y and $\varphi: I \to \mathbb{R}$ are non-negative, $z \in C[I]$ and it is increasing on I, φ is integrable on I, $y \in C[I]$, if $y(t) \le z(t) + \int_a^t \varphi(\tau) y(\tau) d\tau$, $\forall t \in I$, then $y(t) \le z(t) e^{\psi(t)}$, $\forall t \in I$, where $\psi(t) = \int_a^t \varphi(\tau) d\tau$, $\psi'(t) = \varphi(t)$, $\psi(a) = 0$.

Lemma 0.2[2]:- Let f ,and $f_x : D \times \mathbb{R}^n \to \mathbb{R}^m$, are of the Caratheodory type, let $F: L^p(D,\mathbb{R}^n) \to \mathbb{R}^m$ be a functional such that $F(x) = \int_D f(v, x(v)) dv$, where $D \subset \mathbb{R}^d$ is measurable set, and $\|f_x(v,x(v))\| \le \zeta(v) + \eta(v)\|x\|$, $\forall (v,x) \in D \times \mathbb{R}^m$
Where $\zeta(v) \in L^2(D,\mathbb{R})$, $\eta(v) \in L^2(D,\mathbb{R})$, then the *Fréchet Derivative* of F exists for each $x \in L^2(D,\mathbb{R}^n)$, and is given by :- $\Phi'(x)h = \int_D f_x(v,x(v)) h(v) dx$.

Theorem 0.2 (Egorov's theorem)[10]:- Let $D \subset \mathbb{R}^d$ be a measurable subset , $\varphi: D \to \mathbb{R}$, and $\varphi \in L^1(D,\mathbb{R})$, if $\int_S \varphi(v) dv \ge 0$ (or ≤ 0 or $= 0$), for each measurable subset $S \subset D$, then $\varphi(v) \ge 0$ (or ≤ 0 or $= 0$) almost every where in D .

1. Description of the problem: - Let $I=(0,T)$, $T<\infty$, $\Omega \subset R^d$ be an open and bounded region with Lipschitz boundary $\Gamma = \partial\Omega$, $Q = \Omega \times I$, $\Sigma = \Gamma \times I$. We consider the following semilinear parabolic equations (in continuous form):

$$y_t + A(t)y = f(x,t,y,u) \text{ in } Q \qquad (1)$$

$$y(x,t) = 0 \text{ on } \Sigma, \qquad (2)$$

$$y(x,0) = y^0(x) \text{ in } \Omega, \qquad (3)$$

where $y = y_u$ is the *state*, u is the *classical control*, and $A(t)$ is the 2nd order elliptic differential operator, i.e.

$$A(t)y = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left[a_{ij}(x,t) \frac{\partial y}{\partial x_j} \right]$$

The *control constraints (The controls set) are*

$$u \in W,\ W \subset L^2(Q)$$

where W is defined by one of the following forms: -

$$W = W_U = \{ w \in L^2(Q) \mid w(x,t) \in U, \text{ a.e. in } Q \}, \text{ with } U \subset R$$

the *cost functional is*

$$G_0(u) = \int_Q g_0(x,t,y,u) dxdt,$$

the *constraints* on the state and the control variables are

$$G_1(u) = \int_Q g_1(x,t,y,u)dxdt = 0 ,$$

$$G_2(u) = \int_Q g_2(x,t,y,u)dxdt \le 0 ,$$

The set of *admissible controls* is

$$W_A = \{u \in W \mid G_1(u) = 0, G_2(u) \le 0\} .$$

The **Continuous Classical Optimal Control Problem (CCOCP)** is to minimize the cost functional subject to the above constraints, i.e., to find u such that

$$u \in W_A \quad \text{and} \quad G_0(u) = \underset{w \in W_A}{Min}\, G_0(w) ,$$

We denote by $|.|$ the Euclidean norm in $\mathbb{R}^n$, by $\|.\|_\infty$ the norm in $L^\infty(\Omega)$, by $(.,.)$ and $\|.\|_0$ the inner product and norm in $L^2(\Omega)$, by $(.,.)_1$ and $\|.\|_1$ the inner product and norm in Sobolev space $V = H_0^1(\Omega)$, by $< .,. >$ the duality between V and its dual V^*, and finally by $\|.\|_Q$ the norm in $L^2(Q)$.

2. The Solution of the State Equation:- in order to find the classical solution of problem (1-3), first we shall interpret these equations in the following weak form

$$< y_t, v > + a(t,y,v) = (f(x,t,y,u),v), \quad \forall v \in V \text{, a.e. on } I , \tag{4}$$

$$y(.,t) \in V \text{, a.e. on } I , \quad y(0) = y^0 , \tag{5}$$

where $a(t,y,v) = \sum_{i,j=1}^{d} \int_\Omega a_{ij}(x,t) \frac{\partial y}{\partial x_j} \frac{\partial v}{\partial x_i} dx$, and $y^0 \in L^2(\Omega)$.

We suppose the operator $A(t)$ satisfies the elliptic conditions

$$|a_{ij}(x,t)| \le \alpha_1 ,\ (x,t) \in Q \ \ \& \ \sum_{i,j=1}^{d} a_{ij}(x,t) v_j v_i \ge \alpha_2 \sum_{i=1}^{d} v_i^2 ,\ v \in \mathrm{R}^d$$

which imply that $a(t,v,w)$ is symmetric and for some α_1, α_2 , for each $v, w \in V$, and $t \in \bar{I}$, we obtain $|a(t,v,w)| \le \alpha_1 \|v\|_1 \|w\|_1$, and $a(t,v,v) \ge \alpha_2 \|v\|_1^2$.

We suppose also that f is of Caratheodory type on $Q \times (\mathrm{R} \times \mathrm{R})$ (e.g. continuous), it satisfies the following sublinearity condition w.r.t. y & u , and Lipschitz w.r.t. y , i.e.

$$|f(x,t,y,u)| \le \eta(x,t) + c_1|y| + c_1'|u| ,$$

where $(x,t) \in Q$, $y,u \in \mathrm{R}$, and $\eta \in L^2(Q,\mathbb{R})$ (Growth condition)

$$|f(x,t,y_1,u) - f(x,t,y_2,u)| \le L|y_1 - y_2| \text{, where } (x,t) \in Q ,\ y_1, y_2, u \in \mathrm{R} .$$

Theorem 2.1: For each control $u \in L^2(Q)$, the state equations (4-5) has a unique solution $y = y_u$, such that $y \in L^2(I,V)$, $y_t \in L^2(I,V^*)$

It can be proved also that $y(t) \equiv \tilde{y}(t)$ a.e. on I , where $\tilde{y} \in C(\bar{I}, L^2(\Omega))$.

Proof:- Let for each n, V_n be the set of continuous and piecewise affine functions in Ω. Let $\{V_n\}_{n=1}^{\infty}$ be a sequence of subspaces of V , such that $\forall v \in V$, there exists a sequence $\{v_n\}$, with $v_n \in V_n, \forall n$, and $v_n \xrightarrow[V]{S} v \Rightarrow v_n \xrightarrow[L^2(\Omega)]{S} v$.

Let $\{v_i, i = 1,2,...,M(n)\}$ be a finite basis of V_n, and we use the Galerkin method [8] to approximate the exact solution of y , to the approximation solution y_n , such that

$$y_n = y_n(t,x) = \sum_{i=1}^{n} c_i(t)v_i(x),$$

where $c_i(t)$ is unknown function of t, $\forall i = 1,2,...,n$.

We approximate the weak form of the state equation (4-5) w.r.t. x, using the Galerkin's method, i.e. for $i = 1,2,...,n$, we have

$$< y_{n_t}, v_i > + a(t, y_n, v_i) = (f(y_n,u), v_i),\ \forall v_i \in V_n \ , y_n \in L^2(I,V_n),\ \text{a.e in } I, \quad (6)$$

$$(y_n^0, v_i) = (y_0, v_i),\ \forall v_i \in V_n, \quad (7)$$

where $y_n^0 = y_n^0(x) = y_n(x,0) \in V_n \subset V \subset L^2(\Omega)$ is the projection of $y_0 \in L^2(\Omega)$ onto V_n w.r.t. the norm $\|.\|_0$, i.e. $y_n^0 \xrightarrow[L^2(\Omega)]{S} y_0$

Substituting the approximate solution y_n in equations (6&7), then these equations reduce to the following nonlinear system of ordinary differential equations with its initial condition, i.e.

$$AC'(t) + DC(t) = b(\bar{V}^T(x)C(t)), \quad (8)$$

$$AC(0) = b^0, \quad (9)$$

where $A = (a_{ij})_{n\times n}$, $a_{ij} = (v_j, v_i)$, $D = (d_{ij})_{n\times n}$, $d_{ij} = a(t, v_j, v_i)$, $C(t) = (c_j(t))_{n\times 1}$, $C'(t) = (c'_j(t))_{n\times 1}$, $C(0) = (c_j(0))_{n\times 1}$, $\bar{V} = (v_i)_{n\times 1}$, $b = (b_i)_{n\times 1}$, $b_i = (f(\bar{V}^T C(t), u), v_i)$, and $b^0 = (b_i^0)$, $b_i^0 = (y_0, v_i)$.

This system has a unique solution w.r.t. c_j, with c_j continuous on $\bar{I}$ [8]. in particular problem (6-7) has a unique solution y_n.

So we got that for each n, with $V_n \subset V$, problem (6-7) has a unique solution y_n, hence corresponding to the sequence $\{V_n\}_{n=1}^{\infty}$, we have the following sequence of approximation problems , for $n = 1,2,...$, i.e.

$$< y_{n_t}, v_n > + a(t, y_n, v_n) = (f(y_n,u), v_n),\ y_n \in L^2(I,V_n),\ \text{a.e in } I, \quad (10)$$

$$(y_n^0, v_n) = (y^0, v_n),\ \forall v_n \in V_n, \forall n, \quad (11)$$

which has a sequence of solutions $\{y_n\}_{n=1}^{\infty}$.

Replacing v_i in (6) by y_n and then taking the integral from 0 to T, i.e.

$$\int_o^T < y_{n_t}, y_n > + \int_0^T a(t, y_n, y_n)dt = \int_0^T (f(x,t,y_n,u), y_n)dt \le \int_0^T |(f(y_n,u), y_n)| dt, \quad (12)$$

Since $y_{n_t} \in L^2(I,V^*) = L^2(I,V)$, & $y_n \in L^2(I,V)$ in the 1st term of the left hand side (L.H.S.) of (12), hence for this term we can use Lemma1.2 in the ref. [9] and since that the 2nd term is positive, taking $T = t \in [0,T]$, for the upper bound of the integral in (12), we get

$$\frac{1}{2}\int_0^t \frac{d}{dt}\|y_n(t)\|_0^2 dt \le \int_0^t |(f(y_n,u), y_n)| dt \le \int_0^t \int_\Omega |f(y_n,u)||y_n| dx dt$$

using the assumptions on the function f in the R.H.S. (right hand side), we have

$\|y_n(t)\|_0^2 \le c_1 + \int_0^t c_2 \|y_n(t)\|_0^2 dt$, where c_1, c_2 denote to the various constants.

By using the Continuous Bellman-Gronwall's inequality, we get

$$\|y_n(t)\|_0^2 \le b(c),\ \forall t \in [0,T] \ \Rightarrow \|y_n(t)\|_{L^\infty(I,L^2(\Omega))} \le b(c) \ \Rightarrow \|y_n(t)\|_Q \le b_1(c)$$

Again by using Lemma1.2 (in the ref. [9]) for the 1st term in L.H.S. of (12), from the assumptions on $a(t,.,.)$ for the 2nd term in the same side, using the assumptions on the function f in its R.H.S., with $t=T$, and the above result we get

$$\|y_n(t)\|_0^2 - \|y_n^0\|_0^2 + 2\alpha_2\int_0^T \|y_n\|_1^2 dt \le \|\eta\|_Q^2 + c_1'\|u\|_Q^2 + c_2\|y_n\|_Q^2$$

$$\Rightarrow \|y_n\|_{L^2(I,V)}^2 \le b_2^2(c)$$

$\Rightarrow \|y_n\|_{L^2(I,V)} \le b_2(c) = b(c)$, where $b(c)$ denote for various constant.

From the above last steps we got that $\|y_n\|_{L^2(Q)} \le b(c)$, and $\|y_n\|_{L^2(I,V)} \le b(c)$, then by Alaoglu's theorem[3], there exists a subsequence of $\{y_n\}_{n\in\mathrm{N}}$, say again $\{y_n\}_{n\in\mathrm{N}}$, such that $y_n \xrightarrow[L^2(Q)]{W^*\Rightarrow W} y$, and $y_n \xrightarrow[L^2(I,V)]{W^*\Rightarrow W} y$.

Our aim now, is to prove that the sequence $\{y_n\}_{n=1}^{\infty}$ of the solutions of the problem (10-11) converges to the solution of the problem (4-5), by using Galerkin method, and then applying the compactness theorem involving partial fraction [9], to get that $y_n \xrightarrow[L^2(Q)]{S} y$ [see 2], and it remains that to passing the limit in the state equations (10-11), to get that y satisfies equations (4-5). To do this, again consider the weak state equations (10-11), and take any arbitrary $v\in V$, then there exists a sequence $\{v_n\}$, with $v_n \in V_n, \forall n$, such that $v_n \xrightarrow[V]{S} v$, which gives $v_n \xrightarrow[L^2(\Omega)]{S} v$.

Let $\varphi(t)\in C^1[0,T]$, such that $\varphi(T)=0$. Now by multiplying both sides of (10) by $\varphi(t)$, taking the integral from 0 to T, rewriting the 1st term in the L.H.S. of the obtained equation in another manner and then using integration by parts for this term we get that

$$-\int_0^T (y_n,v_n)\varphi'(t)dt + \int_0^T a(t,y_n,v_n)\varphi(t)dt = \int_0^T (f(y_n,u),v_n)\varphi(t)dt + (y_n^0,v_n)\varphi(0) \qquad (13)$$

Since $v_n \xrightarrow[L^2(\Omega)]{S} v$, then $v_n\varphi' \xrightarrow[L^2(Q)]{S} v\varphi'$, and since $y_n \xrightarrow[L^2(Q)]{W} y$, then

$$\int_0^T (y_n,v_n)\varphi'(t)dt \to \int_0^T (y,v)\varphi'(t)dt. \qquad (13a)$$

Since $v_n \xrightarrow[V]{S} v$, then $v_n\varphi \xrightarrow[L^2(I,V)]{S} v\varphi$, and since $y_n \xrightarrow[L^2(I,V)]{W} y$, then

$$\int_0^T a(t,y_n,v_n)\varphi(t)dt \to \int_0^T a(t,y,v)\varphi(t)dt. \qquad (13b)$$

Also since $y_n^0 \xrightarrow[L^2(\Omega)]{S} y_0$, and $v_n \xrightarrow[L^2(\Omega)]{s} v$, then

$$(y_n^0,v_n)\varphi(0) \to (y_0,v)\varphi(0). \qquad (13c)$$

From some of the above convergent and the assumption on f, we get

$$\int_0^T (f(y_n,u),v_n)\varphi(t)dt \to \int_0^T (f(y,u),v)\varphi(t)dt. \qquad (13d)$$

From (13a,b, c, &d), (13) becomes

$$-\int_0^T (y,v)\varphi'(t)dt + \int_0^T a(t,y,v)\varphi(t)dt = \int_0^T (f(y,u),v)\varphi(t)dt + (y_0,v)\varphi(0). \qquad (14)$$

Case1: - Choose $\varphi \in D[0,T]$, i.e. $\varphi(0)=\varphi(T)=0$, so (14) becomes

$$-\int_0^T (y,v)\varphi'(t)dt+\int_0^T a(t,y,v)\varphi(t)dt=\int_0^T (f(y,u),v)\varphi(t)dt$$

by using integration by parts for the 1st term in the L.H.S. of the above eq., we have

$$\int_0^T \frac{d}{dt}(y,v)\,\varphi(t)dt+\int_0^T a(t,y,v)\varphi(t)dt=\int_0^T (f(y,u),v)\varphi(t)dt \quad \Rightarrow$$

$$\int_0^T < y_t , v > \varphi(t)dt+\int_0^T a(t,y,v)\varphi(t)dt=\int_0^T (f(y,u),v)\varphi(t)dt \qquad (15)$$

i.e. y (the limit point) is a solution of state equation (4).

Case2: - Choose $\varphi \in C^1[0,T]$, such that $\varphi(T)=0$ & $\varphi(0)\neq 0$.

Let we rewrite (15) in the following form

$$\int_0^T \frac{d}{dt}(y,v)\,\varphi(t)dt+\int_0^T a(t,y,v)\varphi(t)dt=\int_0^T (f(y,u),v)\varphi(t)dt$$

By using integration by parts for the 1st term in L.H.S. of the above eq., we get

$$-\int_0^T (y,v)\varphi'(t)dt+\int_0^T a(t,y,v)\varphi(t)dt=\int_0^T (f(y,u),v)\varphi(t)dt + (y(0),v)\varphi(0) \qquad (16)$$

subtracting (14) from (16), we get that

$$(y_0,v)\varphi(0)=(y(0),v)\varphi(0) \quad ,\varphi(0)\neq 0, \forall \varphi\in[0,T],\ \Rightarrow (y_0,v)=(y(0),v)$$

i.e. the initial condition (5) holds.

We will prove here that $y_n \to y$ strongly in $L^2(I,V)$. We begin by substituting $v=y_n$ in weak form (6), and taking the integral from 0 to T, and by using Lemma1.2 (in the ref. [9]) we get that

$$\frac{1}{2}\int_0^T \frac{d}{dt}\|y_n\|_0^2 dt+\int_0^T a(t,y_n,y_n)dt=\int_0^T (f(t,y_n,u),y_n)\,dt \Rightarrow$$

$$\frac{1}{2}\|y_n(T)\|_0^2-\frac{1}{2}\|y_n(0)\|_0^2+\int_0^T a(t,y_n,y_n)dt=\int_0^T (f(t,y_n,u),y_n)\,dt\,. \qquad (16a)$$

Since y is solution of the weak form state eq. (4), then by substituting $v=y$ in (16a), and also by the same above way, we get that

$$\frac{1}{2}\|y(T)\|_0^2-\frac{1}{2}\|y(0)\|_0^2+\int_0^T a(t,y,y)dt=\int_0^T (f(t,y,u),y)\,dt \qquad (16b)$$

Since

$$\frac{1}{2}\|y_n(T)-y(T)\|_0^2-\frac{1}{2}\|y_n(0)-y(0)\|_0^2 +\int_0^T a(t,y_n-y,y_n-y)\,dt = \text{Eq}(17a)-\text{Eq}(17b)-\text{Eq}(17c) \qquad (17)$$

where

$$(17a)=\frac{1}{2}\|y_n(T)\|_0^2-\frac{1}{2}\|y_n(0)\|_0^2+\int_0^T a(t,y_n,y_n)\,dt\,,$$

$$(17b)=\frac{1}{2}(y_n(T),y(T))-\frac{1}{2}(y_n(0),y(0))+\int_0^T a(t,y_n,y)\,dt\,,$$

$$(17c)=\frac{1}{2}(y(T),y_n(T)-y(T))-\frac{1}{2}(y(0),y_n(0)-y(0))+\int_0^T a(t,y,y_n-y)\,dt\,.$$

Now, by substituting (16a) in (17a), the last one is equal to

$$\int_0^T \left(f(x,t,y_n,u), y_n \right) dt$$

From the assumptions on $a(t,.,.)$, and on f , and since $y_n \xrightarrow[L^2(Q)]{S} y$, we get that

$$\int_0^T \left(f(x,t,y_n,u), y_n \right) dt \to \int_0^T \left(f(x,t,y,u), y \right) dt \,. \qquad \text{(18a)}$$

By using the same way that we used in the beginning of the proof of this theorem to get that

$$y_n^0 = y_n(0) \xrightarrow[L^2(\Omega)]{S} y^0 = y(0), \qquad \text{(18b)}$$

Also, by the same way we can get that

$$y_n(T) \xrightarrow[L^2(\Omega)]{S} y(T) \,. \qquad \text{(18c)}$$

From (18b, & c), and that $y_n \xrightarrow[L^2(I,V)]{W} y$, we get that equation (17b) converges to the L.H.S. of (16b), and we get that

$$\frac{1}{2}\|y(T)\|_0^2 - \frac{1}{2}\|y(0)\|_0^2 + \int_0^T a(t,y,y)\,dt \underset{(16b)}{\overset{\text{from}}{=}} \int_0^T \left(f(t,y,u), y \right) dt \qquad \text{(18d)}$$

Also, from (18b, & c), we get that

$$\left(y(0), y_n(0) - y(0) \right) \to 0 \;\; \& \;\; \left(y(T), y_n(T) - y(T) \right) \to 0 \qquad \text{(18e)}$$

and $\|y_n(0) - y(0)\|_0^2 \to 0$ & $\|y_n(T) - y(T)\|_0^2 \to 0$ (18f)

again since $y_n \xrightarrow[L^2(I,V)]{W} y$ then

$$\int_0^T a\left(t, y, y_n - y\right) dt \to 0 \,. \qquad \text{(18g)}$$

Now, when $n \to \infty$ in both sides of (17), we get the following results: -

(1) The first two terms in the L.H.S. of (17) are tending to zero (from (18f)).

(2) Eq. (1.17a) $\underset{(16a)}{\overset{\text{from}}{=}} \int_0^T \left(f(x,t,y_n,u), y_n \right) dt \underset{(18a)}{\overset{\text{from}}{\to}} \int_0^T \left(f(x,t,y,u), y \right) dt$.

(3) Eq. (1.17b) $\to$ L.H.S. of (18d) $\underset{(16b)}{\overset{\text{from}}{=}} \int_0^T \left(f(t,y,u), y \right) dt$

(4) The 1st two terms in (17c) are tending to zero (from (18e), and the last one term also tends to zero (from (18g)).

From these results, the both sides (17) become

$$\int_0^T a\left(t, y_n - y, y_n - y\right) dt \xrightarrow[n\to\infty]{\text{where}} \int_0^T \left(f(x,t,y,u), y \right) dt - \int_0^T \left(f(t,y,u), y \right) dt - 0 = 0$$

From assumption on $a(t,.,.)$, we have that

$$\alpha_2 \int_0^T \|y_n - y\|_1^2 dt \le \int_0^T a\left(t, y_n - y, y_n - y\right) dt \xrightarrow[n\to\infty]{\text{where}} 0 \overset{\alpha_2 \neq 0}{\Rightarrow}$$

$$\int_0^T \|y_n - y\|_1^2 dt \to 0 \quad \Rightarrow y_n \xrightarrow[L^2(I,V)]{S} y$$

To prove the uniqueness of the solution, let $y_1 \& y_2$ be two solutions of state equation (4), subtracting the 2nd obtained equation from the 1st obtained equation we get

$$< (y_1 - y_2)_t, v > + a\left(t, y_1 - y_2, v\right) = \left(f(y_1,u) - f(y_2,u), v \right) \;,\; \forall v \in V \,.$$

Let we substitute $v = y_1 - y_2$, in the above equation, so we have

$$< (y_1 - y_2)_t, y_1 - y_2 > + a\left(t, y_1 - y_2, y_1 - y_2\right) = \left(f(y_1,u) - f(y_2,u), y_1 - y_2 \right) .$$

The 1st term in the L.H.S. of the above equation will written in another way (using Lemma1.2 in ref. [9]), and using assumption on $a(t,.,.)$, for the 2nd term, we get that

$$\frac{1}{2}\frac{d}{dt}\|y_1-y_2\|_0^2+\alpha_2\|y_1-y_2\|_1^2\le\left(f(y_1,u)-f(y_1,u),y_1-y_2\right) \qquad (19)$$

The L.H.S. of the above eq. is positive, taking the integral from 0 to t, for its both sides, then using the Lipschitiz condition, we get that

$$\int_0^t\frac{d}{dt}\|y_1-y_2\|_0^2\,dt\le 2\int_0^t\int_\Omega L|y_1-y_2|^2\,dxdt=2\int_0^t L\|y_1-y_2\|_0^2dt$$

$$\Rightarrow\quad \|(y_1-y_2)(t)\|_0^2\le 0+\int_0^t 2L\|y_1-y_2\|_0^2dt$$

By Bellman-Gronwall's inequality, we get

$$\|(y_1-y_2)(t)\|_0^2=0,\ \forall\ t\in I \qquad (19a)$$

Now, integrating both sides of (19) from $t=0$, to $t=T$, using the initial conditions and the property of norm (positive) for the 1st term in L.H.S. of the obtained equation, and the Lipschitz property for the function in the R.H.S and then using (19a), we get

$$\int_0^T\|y_1-y_2\|_1^2\,dt\overset{\alpha_2\ge 0}{\le}0\Rightarrow\|y_1-y_2\|_{L^2(I,V)}^2=0\Rightarrow y_1=y_2\,. \qquad ■$$

3. Existence of an Optimal Control:- In order to prove the existence of a classical optimal control, we suppose now in addition to the above assumptions the function g_l is of Caratheodory type on $Q\times(\mathrm{R}\times\mathrm{R})$ (e.g. continuous), and satisfies the following subquadratic condition w.r.t. $y\,\&\,u$, i.e. for each $l=0,1,2$, we have

$|g_l(x,t,u,y)|\le\eta_{2l}(x,t)+c_{2l}y^2+c'_{2l}u^2$, where, $y\,\&\,u\in\mathrm{R}$, with $\eta_{2l}\in L^1(Q)$.

Lemma3.1: - If the function f is Lipschitz w.r.t. y , and u, then the operator $u\mapsto y_u$, from $L^2(Q)$ into $L^\infty(I,L^2(\Omega))$, or $L^2(I,V)$, or $L^2(Q)$ is continuous.

Proof: - Let u and $u+\Delta u$ are two given bounded controls on $L^2(Q)$, so by Theorem2.1 $y=y_u$ and $y+\Delta y=y_u+\Delta y_u$ are the corresponding solutions and are satisfied the weak form (4-5), satisfying these solutions in (4-5) and then subtracting one of the other we get that

$$<\Delta y_t,v>+a(t,\Delta y,v)=\left(f(x,t,y+\Delta y,u+\Delta u)-f(x,t,y,u),v\right) \qquad (20)$$

$$\Delta y(0)=0\,. \qquad (21)$$

By substituting $v=2\Delta y$ in (20) using the same way that we used to get (19), we get also an equation like to (19) with Δy in position of y_n, and taking the integral from 0 to T , one gets that

$$\int_0^T\frac{d}{dt}\|\Delta y\|_0^2\,dt\le\int_0^T\left(f(x,t,y+\Delta y,u+\Delta u)-f(x,t,y,u),2\Delta y\right)dt\,,$$

by the assumptions on $a(t,.,.)$, and f is Lipschtitz w.r.t. y & u, and substituting $T=t\in[0,T]$, in the upper bound of the above integrals, we get

$$\int_0^t\frac{d}{dt}\|\Delta y\|_0^2\,dt\le 2\int_0^t\int_\Omega|f(x,t,y+\Delta y,u+\Delta u)-f(x,t,y,u)||\Delta y|\,dxdt\,.$$

$\Rightarrow\|\Delta y(t)\|_0^2\le L_2\|\Delta u\|_Q^2+\int_0^t L_3\|\Delta y(t)\|_0^2\,dt$, where L_2, L_3 denote various constant

By Bellman –Grownwall's inequality, one gets that

$$\Rightarrow\|\Delta y(t)\|_0\le M\|\Delta u\|_Q \qquad ,\ \forall t\in[0,T]$$

i.e. $\|\Delta y_u\|_{L^\infty(I,L^2(\Omega))} \le M\|\Delta u\|_Q \Rightarrow \|\Delta y_u\|_{L^2(Q)} \le M\|\Delta u\|_Q$.

Again by using the same above technique we can get that $\|\Delta y_u\|_{L^2(I,V)} \le M\|\Delta u\|_Q$

Now, let y_{u_1} and y_{u_2} are the correspond states to the controls u_1 and u_2, so $\Delta u = u_1 - u_2$, and $\Delta y = y_1 - y_2$. From the above steps we got that $\|\Delta y_u\|_{L^\infty(I,L^2(\Omega))} \le M\|\Delta u\|_Q$, i.e. the operator $u \mapsto y_u$ is Lipschitz continuous from $L^2(Q)$ into $L^\infty(I, L^2(\Omega))$. By the same way we can prove that this operator is also Lipschitz continuous from $L^2(Q)$ into $L^2(I,V)$, and into $L^2(Q)$.

Lemma 3.2:- Dropping the index l, the functional $G(u)$ is continuous on $L^2(Q)$.

Proof:- Since g is defined on $Q \times \mathbb{R} \times U$, measurable w.r.t. (x,t) for fixed y and u, and is continuous w.r.t. y and u for fixed (x,t) and satisfies the above conditions, then from proposition 1.2 in [6] follows that $G(u)$ is continous on $L^2(Q)$. ■

Theorem 3.1: In addition to the above assumptions, we assume that the set of controls W is of the form $W = W_U$, with U convex, and compact, $W_A \ne \emptyset$, where f has the form

$$f(x,t,y,u) = f_1(x,t,y) + f_2(x,t,y)u$$

where $|f_1(x,t,y)| \le \eta_1(x,t) + c_1|y|$, $|f_2(x,t,y)| \le \eta_2(x,t) + c_2|y|$,

with $\eta_1, \eta_2 \in L^2(Q)$, and $c_1 \ge 0, c_2 \ge 0$,

g_1 is independent of u, g_0 and g_2 are convex w.r.t. u, for fixed (x,t,y). Then there exists an optimal control.

Proof:- Since U is convex, then W is convex, and since U is compact, i.e. U is closed and bounded then by Egorov's theorem W is closed but $U \subset \mathbb{R}$, $L^\infty(Q) \subset L^2(Q)$ then W is bounded and then W is weakly compact.

Since $W_A \ne \emptyset$, then there exits a point $w \in W_A$, such that $G_1(w) = 0$, & $G_2(w) \le 0$, and then there exists a minimum sequence $\{u_k\}$, such that $u_k \in W_A$, $\forall k$, and

$$\lim_{k\to\infty} G_0(u_k) = \inf_{w \in W_A} G_0(w).$$

Since $u_k \in W_A$, $\forall k$, then $u_k \in W$, $\forall k$, but W is weakly compact, there exist a subsequences of $\{u_k\}$, say again $\{u_k\}$ which convergence weakly to some point u in W, i.e. $u_k \xrightarrow[L^2(Q)]{W} u$, and $\|u_k\|_Q \le c$, $\forall k$.

From Theorem 2.1, we got that for each control u_k, the state equation has a unique solution $y_k = y_{u_k}$, and we got that the norms $\|y_k\|_{L^\infty(I,L^2(\Omega))}$, $\|y_k\|_{L^2(Q)}$ and $\|y_k\|_{L^2(I,V)}$ are bounded, then by Alaoglu theorem [3]there exists a subsequence of $\{y_k\}$, say again $\{y_k\}$ which convergence to some point y w.r.t. the above norms, i.e.

$$y_k \xrightarrow[L^\infty(I,L^2(\Omega))]{W^* \Rightarrow W} y,\ y_k \xrightarrow[L^2(Q)]{W^* \Rightarrow W} y, \text{ and } y_k \xrightarrow[L^2(I,V)]{W^* \Rightarrow W} y$$

Now, we want to show that the norm $\|y_{k_t}\|_{L^2(I,V^*)}$ is bounded; to do this we rewrite the weak form state equation in the form

$$< y_{k_t}, v >= (f(t,x,y_k,u_k),v) - a(t,y_k,v),\ \forall v \in V$$

Integrating both sides of this equation from 0 to T, taking the absolute vales, using the Caschy – Schwarz inequality and then the assumptions on f to the R.H.S. for the obtained equation we get that

$$\left|\int_0^T < y_{k_t}, v > dt\right| \le \alpha_1 \left(\int_0^T \|y_k\|_1^2 dt\right)^{1/2} \left(\int_0^T \|v\|_1^2 dt\right)^{1/2}$$
$$+\left(\|\eta\|_Q + c_1\|y_k\|_Q + c_1'\|u_k\|_Q\right)\|v\|_Q ,$$
$$\le \alpha_1 \|y_k\|_{L^2(I,V)} \|v\|_{L^2(I,V)} + \bar{b}_3(c)\|v\|_{L^2(Q)}$$

where $\bar{b}_3(c)$ denotes for various constants,

Since $\|v\|_{L^2(Q)} \le c\|v\|_{L^2(I,V)}$ and $\|y_k\|_{L^2(I,V)} \le b_2(c)$, then

$$\Rightarrow \quad \frac{\left|\int_0^T < y_{k_t}, v > dt\right|}{\|v\|_{L^2(I,V)}} \le b_3(c) \ , \ \forall y_{k_t} \in V^*,$$

$$\Rightarrow \quad \sup_{(x,t)\in Q} \frac{\left|\int_0^T < y_{k_t}, v > dt\right|}{\|v\|_{L^2(I,V)}} \le b_3(c), \quad \Rightarrow \quad \left\|y_{k_t}\right\|_{L^2(I,V^*)} \le b_3(c).$$

Now, since

$$L^2(I,V) \subset L^2(Q) \cong (L^2(Q))^* \subset L^2(I,V^*),$$

then

$$\|y\|_{L^2(Q)} \le c\|y\|_{L^2(I,V)} \ \& \ \|y\|_{L^2(I,V^*)} \le c\|y\|_{\left(L^2(Q)\right)^*} = c\|y\|_{L^2(Q)}.$$

Which give that the injections of $L^2(I,V)$ into $L^2(Q)$, and of $(L^2(Q))^*$ into $L^2(I,V^*)$ are continuous, and since the injection of $L^2(I,V)$ into $L^2(Q)$ is compact,. So we get all the hypotheses of the first compactness theorem [9], we get that there exists a subsequence of $\{y_k\}$, say again $\{y_k\}$, such that $y_k(x,t) \xrightarrow[L^2(Q)]{S} y$.

Now we want to prove that the limit point y is y_u, since for each k, y_k is a solution of the sate equation (corresponding solution to the control u_k), then

$$< y_{k_t}, v > + a(t, y_k, v) = \left(f_1(x,t,y_k) + f_2(x,t,y_k)u_k, v\right).$$

Let $\varphi \in C^1[I]$, with $\varphi(T) = 0$, now by rewriting the 1st term in the L.H.S. of the above equation by another way, multiplying its both sides by $\varphi(t)$, and then taking the integral from 0 to T, for the both sides of the obtained above relation, we have

$$\int_0^T \frac{d}{dt}(y_k, v)\varphi(t)dt + \int_0^T a(t, y_k, v)\varphi(t)dt = \int_0^T \left(f_1(y_k) + f_2(y_k)u_k, v\varphi(t)\right)dt \quad (22)$$

To passage the limit in (22), for the L.H.S. we can use passage it using the same steps that we used in the proof of Theorem 2.1, while for the R.H.S., first from the assumptions on f_1, and $y_k(x,t) \xrightarrow[L^2(Q)]{S} y$, we have

$$\int_Q f_1(y_k) v\varphi(t) dxdt \to \int_Q f_1(y) v\varphi(t) dxdt , .$$

On the other hand, Let we choose $v \in C[\bar{\Omega}]$, set $w = v\varphi(t)$, $w \in C[\bar{Q}] \subset L^{\infty}(I,V) \subset L^2(I,V) \subset L^2(Q)$, then

$$\int_0^T \left(f_2(y_k)u_k, w \right) dt - \int_0^T \left(f_2(y)u, w \right) dt = \int_0^T \left(\left(f_2(y_k) - f_2(y) \right) u, w \right) dt$$
$$+ \int_0^T \left(f_2(y_k)(u_k - u), w \right) dt$$

since $u, u_k \in U \subset \mathbb{R}$, then also from the assumptions on f_2, and $y_k(x,t) \xrightarrow[L^2(Q)]{S} y$, we get that

$$\int_Q (f_2(y_k) - f_2(y)) uw dxdt \to 0, \ \& \int_0^T \left(f_2(y_k)(u_k - u), w \right) dt \to 0, \ \forall w \in C[\bar{Q}]$$

by substituting $w = v\varphi$, we get

$$\int_0^T (f_1(y_k), v)\varphi(t) dt + \int_0^T (f_2(y_k)u_k, v\varphi(t)) dt \to \int_0^T (f_1(y), v)\varphi(t) dt$$
$$+ \int_0^T (f_2(y)u, v\varphi(t)) dt, \forall v \in C[\bar{\Omega}]$$

since $C[\bar{\Omega}]$ is dense in V, then this holds also for every $v \in V$. Then we can passage the limit also in the R.h.S. of (22), and we get

$$< y_t, v > + a(t, y, v) = \left(f_1(x,t,y) + f_2(x,t,y)u, v \right), \ \forall v \in V, \text{ a.e.on } I.$$
$$(y_0, v) = (y^0(x), v).$$

which gives $y_k \to y = y_u$, is a solution of the state equation.

From Lemma 3.2, we get that $G_l(u)$ is continuous on $L^2(Q)$, for each $l = 0,1,2$.

Now, since g_1 is independent of u, and $y_k \xrightarrow[L^2(Q)]{S} y$, then

$$G_1(u) = \lim_{k\to\infty} G_1(u_k) = 0$$

From the assumptions on g_l, (for each $l = 0,2$), we get that

$$G_2(u) \le \underline{\lim}_{k\to\infty} G_2(u_k) = 0 \Rightarrow G_2(u) \le 0,$$

on the other hand we have that $G_0(u) \le \underline{\lim}_{k\to\infty} G_0(u_k) = \lim_{k\to\infty} G_0(u_k) = \inf_{w \in W_A} G_0(w)$

i.e. u is an optimal control of the considered problem. ■

4. The necessary conditions for optimality:- In order to state the necessary conditions for classical optimal control problem, we suppose in addition that the functions f_y, f_u, g_{ly}, g_{lu}, $l = 0,1,2$, are of the Caratheodory type (or continuous), on $Q \times (\mathrm{R} \times \mathrm{R})$ and satisfy

$$\left| f_y(x,t,y,u) \right| \le c_3 \ \& \ \left| f_u(x,t,y,u) \right| \le c_4, \ c_3 \ge 0, \ c_4 \ge 0$$
$$\left| g_{ly}(x,t,y,u) \right| \le \eta_{l5}(x,t) + c_{l5}|y| + c'_{l5}|u| \ , \ c_{l5} \ge 0, \ c'_{l5} \ge 0$$
$$\left| g_{lu}(x,t,y,u) \right| \le \eta_{l6}(x,t) + c_{l6}|y| + c'_{l6}|u| \ , \ c_{l6} \ge 0, \ c'_{l6} \ge 0$$

where $(x,t) \in Q$, $y, u \in \mathrm{R}$, $\eta_{l5} \in L^2(Q)$, and $\eta_{l6} \in L^2(Q)$.

Lemma 4.1: For simplicity we drop the index l in g_l & G_l, the *Hamiltonian* which is denoted by:-

$$H(x,t,y,z,u) = zf(x,t,y,u) + g(x,t,y,u)$$

and the *adjoint state* $z = z_u$ (where $y = y_u$) *equation* satisfies:-

$$-< z_t, v > + a(t, v, z) = (zf_y(x,t,y,u), v) + (g_y(x,t,y,u), v), \ \forall v \in V \qquad (23)$$
$$(z(x,T), v) = 0, \text{ in } \Omega \qquad (24)$$

Then the **Fréchet derivative** of G is given by

$$G'(u)\Delta u = \int_Q H_u(x,t,y,z,u)\Delta u dxdt \ ,$$

and the operators $u \mapsto z_u$, and $u \mapsto G'(u)$ are continuous.

Proof: - From the above assumptions the adjoint-state equations (23-24), has a unique solution $z = z_u$, for a given control $u \in W$ this can be proved by using the same way which used to prove the existence and uniqueness of the state equation (Theorem 2.1).
Now, let u is a given control, and $y = y_u$, is the corresponding solution of the state equation, and let $y + \Delta y = y_u + \Delta y_u$, be the correspond solution for the control $u + \Delta u$.
Now, from Lemma 3.1, we get that Δy is depend on Δu , hence $\Delta y \to 0$, when $\Delta u \to 0$, from the assumptions on g , the Fréchet derivative of g exists , we get that

$$G(u+\Delta u) - G(u) = \int_Q \left(g_y(y,u)\Delta y + g_u(y,u)\Delta u \right) dxdt + \varepsilon_1(\Delta u)\|\Delta u\|_Q , \qquad (25)$$

where $\varepsilon_1(\Delta u) \xrightarrow[\Delta u \to 0]{} 0$ & $\|\Delta u\|_Q \xrightarrow[\Delta u \to 0]{} 0$.

On the other hand, substituting the solutions y and $y + \Delta y$ in (4-5), taking the integral for the obtained equations from $t = 0$ to $t = T$, with $v = z$, and then subtracting the 1st obtained equation from the 2nd obtained equation and the same for the initial conditions , we get

$$\int_0^T < \Delta y_t, z > dt + \int_0^T a(t, \Delta y, z)\, dt = \int_0^T \left(f(x,t,y+\Delta y, u+\Delta u) - f(x,t,y,u), z \right) dt \qquad (26)$$

with $\Delta y(0) = 0$

Now, from the assumptions on f , the Fréchet derivative of f in the R.H.S. of (26) exists, and (26) becomes

$$\int_0^T < \Delta y_t, z > dt + \int_0^T a(t, \Delta y, z)\, dt = \int_0^T \left((f_y \Delta y + f_u \Delta u), z \right) dt + \varepsilon_2(\Delta u)\|\Delta u\|_Q \ (27)$$

where $\varepsilon_2(\Delta u) \xrightarrow[\Delta u \to 0]{} 0$ & $\|\Delta u\|_Q \xrightarrow[\Delta u \to 0]{} 0$

And by taking the integral from 0 to T , for both sides of the adjoint equation (23), with $v = \Delta y$, then integrating by parts the 1st term in the L.H.S. of the obtained equation we get

$$\int_0^T < \Delta y_t, z > dt + \int_0^T a(t, \Delta y, z)\, dt = \int_0^T \left(z f_y + g_u, \Delta y \right) dt \ , \qquad (28)$$

since $z(T) = \Delta y(0) = 0$.

Now, by subtracting (28) from (27), then substituting this result in (25), we get

$$G(u+\Delta u) - G(u) = \int_Q \left(z f_u + g_u \right) \Delta u dxdt + \tilde{\varepsilon}(\Delta u)\|\Delta u\|_Q \qquad (29)$$

where $\tilde{\varepsilon}(\Delta u) = \varepsilon_1(\Delta u) + \varepsilon_2(\Delta u) \xrightarrow[\Delta u \to 0]{} 0$, and $\|\Delta u\|_Q \xrightarrow[\Delta u \to 0]{} 0$.

Finally from the definition of the Fréchet derivative of G , we obtain (29) becomes

$$G'(u)\Delta u = \int_Q (z f_u + g_u)\Delta u dxdt \Rightarrow G'(u)\Delta u = \int_Q H_u(x,t,y,z,u)\Delta u dxdt \ . \ \blacksquare$$

Lemma 4.1:- the operator $u \mapsto z_u$ is continuous w.r.t. Lipschitz on $L^2(Q)$.
Proof: The proof follows by the same way which is used in Lemma 3.1.
Lemma4.2:- The operator $u \mapsto G'(u)$ is continuous on $L^2(Q)$.
Proof: The proof follows by the same way which used in proof of Lemma 3.2.

Theorem 4.1: Necessary Conditions for Optimality (Multipliers Theorem): -
If the control $u \in W_A$ is an optimal classical control with W convex, then u is (classical wakly) minimum , i.e. there exist multipliers $\lambda_l \in \mathrm{R}$, $l = 0,1,2$, with $\lambda_0 \geq 0$, $\lambda_2 \geq 0$, $\sum_{i=0}^{2} |\lambda_l| = 1$, such that

$$\int_Q H_u(x,t,y,z,u)\Delta u dx dt \geq 0 \,, \; \forall w \in W \,, \Delta u = w - u \qquad (30)$$

where $g = \sum_{l=0}^{2} \lambda_l g_l$ in the definition of H and z , and also

$$\lambda_2 G_2(u) = 0 \,, \text{ (Transversality condition)} \qquad (31)$$

The above relations are equivalent to the (weak) pointwise minimum principle

$$H_u(x,t,y,z,u)u(t) = \min_{v \in U} H_u(x,t,y,z,u)v \,, \text{ a.e. on } Q \qquad (32)$$

Proof: - From Lemma 3.2 and for each $l = 0,1,2$, the functional $G_l(u)$, is continuous of each $u \in W$, hence $G_l(u)$ is $\rho -$ local continuous at each $u \in W$, for each $l = 0,1,2$, for every ρ .
From Lemma 4.1 we get that the functional $G_l(u)$ has a continuous Fréchet derivative at each $u \in W$, hence $G_l(u)$ is $\rho -$ differentiable at each $u \in W$ for each ρ , and since $W \subset L^2(Q)$, $L^2(Q)$ is open, then

$$DG_l(u, w-u) = G'_l(u)(w-u), \; l = 0,1,2 \,.$$

Since the control $u \in W_A$ is optimal, therefore by using the Khuan-Tanger-Lagrange theorem there exists multipliers $\lambda_l \in \mathrm{R}$, $l = 0,1,2$, with $\lambda_0 \geq 0$, $\lambda_2 \geq 0$, $\sum_{i=0}^{2} |\lambda_l| = 1$, such that (30) & (31) are satisfied, from Theorem 3.1, with setting $\Delta u = w - u$ inequality (30) becomes

$$\int_Q \left[\lambda_0 \left(z_0 f_u + g_{0_u} \right) + \lambda_1 \left(z_1 f_u + g_{1_u} \right) + \lambda_2 \left(z_2 f_u + g_{2_u} \right) \right] \Delta u dx dt \geq 0 \,,$$

$$\Rightarrow \int_Q \left(z f_u + g_u \right) \Delta u dx dt \geq 0, \;\; \forall w \in W \,, \text{ where } g = \sum_{l=0}^{2} \lambda_l g_l \text{ and } z = \sum_{l=0}^{2} \lambda_l z_l$$

$$\Rightarrow \int_Q H_u(x,t,y,z,u)\Delta u dx dt \geq 0 \,, \forall w \in W \,.$$

Now, we prove that (30) is equivalent to the Minimum principle in pointwise weak form (32).
First, let $W_U = \{ w \in L^2(Q, \mathrm{R}) \,|\; w(x,t) \in U \subset \square \,,\text{ a.e. on } Q \}$, and $\{u_k\}$ be a dense sequence in W_U , and let $S \subset Q$, be a measurable set such that

$$w(x,t) = \begin{cases} w_k(x,t) \;, \text{ if } (x,t) \in S \\ u(x,t) \;\;\;, \text{ if } (x,t) \notin S \end{cases} .$$

Hence (31), becomes

$$\int_S H_u(x,t,y,u)(w_k - u) \geq 0 \,.$$

By using the Egorove's theorem, we get

$$H_u(x,t,y,z,u)(w_k - u) \geq 0 \,, \text{ a.e. on } Q \,,$$

i.e. it holds in a set $P_k = Q - Q_k$, with $\mu(Q_k) = 0$,

$$\Rightarrow \; H_u(x,t,y,z,u)(w_k - u) \geq 0 \;, \text{ in } P = \bigcap_k P_k \,,$$

and this is hold for each k , since P is independent of k , and we have $\mu(Q - P) = \mu(\bigcup_k Q_k) = 0$, since $\{u_k\}$ is dense in W_U , then

$H_u(x,t,y,z,u)(w-u) \geq 0$, $\forall w \in W$, in P, i.e. a.e. on Q, or

$H_u(x,t,y,z,u)(w-u) \geq 0$, a.e. on Q,

$\Rightarrow H_u(x,t,y,z,u)u = \min_{w \in U} H_u(x,t,y,z,u)w$, a.e. on Q.

And conversely, suppose that

$H_u(x,t,y,z,u)u = \min_{w \in U} H_u(x,t,y,z,u)w$, a.e. on Q

$\Rightarrow H_u(x,t,y,z,u)(w-u) \geq 0$, $\forall w \in W$, a.e. on Q

$\Rightarrow \int_Q H_u(x,t,y,z,u)\Delta u dx dt \geq 0$, $\forall w \in W$. ■

REFERENCES

[1] Alekseev, V. M., Tikhomirov, V. M. and Fomin, S.V., "Optimal theory ", "Translated from Russian by Volosov, V.M.", Consultant Bureau, New York, 1987.

[2] Al-hawasy Jamil,"Optimization and Approximation of Parabolic Boundary Value Problem ", Ph.D. thesis, School of App. Math. & Phy. Sci., Nathional Technical Univerity,Athens-Greece, 2004.

[3] Bacopoulos, A., and Chryssoverghi, I.," Numerical Solutions of Partial Differential Equations by Finite Elements Methods", Symeon Publishing Co., Athens, 1986.

[4] Borzi, A., "Multigrid Methods for Parabolic Distributed Optimal Control Problems", Journal of Computational and Applied Mathematics, 157(2003), pp. 365-382.

[5] *Bűskens* C. & Maurer H., "SQP-Methods for Solving Optimal Control Problems with Control and state Constraints: Adjoint Variables, Sensitivity and Real-time Control", J. Com. Appl. Math., Vol. 120, Issues 1-2, pp. 85-108,2000.

[6] Chryssoverghi, I.," Nonconvex Optimal Control of Nonlinear Monotone Parabolic Systems: Systems and Control Problems Letters",No. 8,pp.55-62(1986).

[7] Subasi, M., "An Optimal Control Problem Governed by the Potential of a Linear Schrodinger equation", Applied Mathematics and Computation, 131(2002), pp. 95-106.

[8] Thomee, V., " Galerkin Finite Element Methods for Parabolic Problems, Springer, Berlin, 1997.

[9] Temam, R., "Navier-Stokes Equationns", North-Holand Publishing Company, 1977.

[10] Warga, J., " Optimal Control of Differential and Functional Equations", Academic Press, New York and London, 1972.