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Detour Polynomials of Generalized Vertex Identified of Graphs

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Abstract:

The Detour distance is one of the most common distance types used in chemistry and computer networks today. Therefore, in this paper, the detour polynomials and detour indices of vertices identified of n-graphs which are connected to themselves and separated from each other with respect to the vertices for $n \geq 3$ will be obtained. Also, polynomials detour and detour indices will be found for another graphs which have important applications in Chemistry.

Keywords: Detour distance, Detour polynomial, Detour index, Special graphs, Vertices identified.

Introduction:

Let $\Phi = \Phi(V(\Phi), E(\Phi))$ be a connected graph without muliedges and loops (which write short Φ be a conn.graph) such that order of Φ is $|V(\Phi)| = p(\Phi)$ and size of Φ is $|E(\Phi)| = q(\Phi)$. The distance function σ from $V(\Phi) \times V(\Phi)$ in to non-negative integer is the number of edges between u and v in the shortest $u - v$ path in Φ , $u \neq v$; it is denoted by $\sigma(u, v)$,^{1,2} but there are other types of distance between any two distinct vertices u and v in a conn.graph. Φ is called detour distance and it has been defined by the length of a longest $u - v$ path in Φ ; it is denoted by $\mathfrak{D}(u, v)$,³. We note that $\mathfrak{D}(u, v) = 1$ iff the edge uv which removal resulted disconnected graph of Φ , and $\mathfrak{D}(u, v) = \sigma(u, v)$ iff the graph Φ is a connected without cycles. From clearly that $\mathfrak{D}(u, v) = p(\Phi) - 1$ iff Φ contains a "Hamiltonian $u - v$ path". The eccentricity of this distance of a vertex v which is denoted $e_{\mathfrak{D}}(v)$ and defined by $e_{\mathfrak{D}}(v) = \max_{u \in V(\Phi)} \{\mathfrak{D}(v, u)\}$ in a conn.graph Φ . The detour diameter $diam_{\mathfrak{D}}(\Phi) = \delta_{\mathfrak{D}}(\Phi)$ of a conn.graph Φ is defined by $\delta_{\mathfrak{D}}(\Phi) = \max_{v \in V(\Phi)} \{e_{\mathfrak{D}}(v)\}$ and the detour radius $rad_{\mathfrak{D}}(\Phi) = r_{\mathfrak{D}}(\Phi)$ is defined by $r_{\mathfrak{D}}(\Phi) = \min_{v \in V(\Phi)} \{e_{\mathfrak{D}}(v)\}$. We see that $e(v) \leq e_{\mathfrak{D}}(v)$, $\forall v \in V(\Phi)$, since $d(u, v) \leq \mathfrak{D}(u, v)$, $\forall u, v \in V(\Phi)$. This requires us to be $diam(\Phi) \leq diam_{\mathfrak{D}}(\Phi)$ and $rad(\Phi) \leq rad_{\mathfrak{D}}(\Phi)$,^{4,5}.

There is another concept related to the detour distance which is called the connected detour numbers,^{6,7}. Also, there is another concept of numbers in graph theory which is called domination and chromatic number,⁸⁻¹².

"The **detour polynomial** $\mathfrak{D}[\Phi, x]$ of a conn.graph Φ , is defined" by

$$\mathfrak{D}[\Phi, x] = \sum_{\{u,v\} \in V(G)} x^{\mathfrak{D}(u,v)} \text{ or } \mathfrak{D}[\Phi, x] = \sum_{k \geq 1} \mathcal{C}_{\mathfrak{D}}(\Phi, k) x^k, \text{ where } \mathcal{C}_{\mathfrak{D}}(\Phi, k) \text{ be the number of unordered pairs } u \text{ and } v \text{ such that } \mathfrak{D}(u, v) = k.$$

In 1995, the authors introduced definition the **detour index** of a conn.graph $\mathfrak{D}(\Phi)$ which defined as $\mathfrak{D}(\Phi) = \sum_{i=1}^{p(\Phi)-1} \sum_{j=i+1}^{p(\Phi)} dij$ where $[dij]$ is a matrix of detour distance. The detour index is horribly important in applied sciences,¹³. After that Lukovits tested the detour index on the correlation between some chemical compositions,¹⁴. Also, the detour index can be obtained from another method by:

$$\begin{aligned} \mathfrak{D}(\Phi) &= \sum_{\{u,v\} \in V(\Phi)} \mathfrak{D}(u, v) \\ &= \mathfrak{D}'(\Phi; x)|_{x=1} = \sum_{k \geq 1} k \mathcal{C}_{\mathfrak{D}}(\Phi, k). \end{aligned}$$

Let $\mathcal{C}_{\mathfrak{D}}(v, \Phi, k)$ be the number of vertices u , ($u \neq v$) such that $\mathfrak{D}(u, v) = k$.

The detour polynomial of a vertex v in Φ is define as:

$$\mathfrak{D}[v, \Phi, x] = \sum_{k \geq m_{\mathfrak{D}}(v)}^{e_{\mathfrak{D}}(v)} \mathcal{C}_{\mathfrak{D}}(v, \Phi, k) x^k,$$

where $m_{\mathfrak{D}}(v) = \min_{u \in V(\Phi)} \{\mathfrak{D}(v, u)\}$. We note that:

$$\mathfrak{D}[\Phi, x] = \frac{1}{2} \sum_{v \in V(\Phi)} \mathfrak{D}[v, \Phi, x].$$

The detour index of a vertex v which denoted by $\mathfrak{D}(v, \Phi)$ and defined as:

$$\mathfrak{D}(v, \Phi) = \sum_{u \in V(\Phi)} \mathfrak{D}(v, u).$$

Theorem1:¹⁵

1. $\mathfrak{D}[\mathcal{K}_p, x] = \binom{p}{2} x^{p-1}$.
2. $\mathfrak{D}[\mathcal{W}_p, x] = \binom{p}{2} x^{p-1}$.
3. $\mathfrak{D}[\mathcal{G}_p, x] = px^{\frac{p}{2}} \left(\binom{p}{2} - \frac{1}{2} \right)$, where p is even.
4. $\mathfrak{D}[\mathcal{G}_p, x] = px^{\frac{p+1}{2}} \left[\binom{p}{2} \right]$, where p is odd.

A number of writers had obtained detour polynomials, restricted detour polynomials and indices of detour for many graphs, applications in graph theory and graphs resulting from the synthesis of other graphs under operations defined in graph theory,¹⁶⁻²². In 1993, Gutman,²³ constructed new graph from two graphs not connected to each other. The two vertices u and v of any two conn.graphs Φ_1 and Φ_2 with disjoint vertex sets respectively, then the vertex identified graph $\Phi_1 \cdot \Phi_2$ is obtain from Φ_1 and Φ_2 by two vertices

identically u and v . Mohammed Saleh,²⁴ obtained $\mathfrak{D}(\Phi_1 \cdot \Phi_2; x)$, as specified in the following theorem.

Theorem2: Let $u, (v)$ be any vertex of $\Phi_1, (\Phi_2)$, then

$$\mathfrak{D}[\Phi_1 \cdot \Phi_2, x] = \mathfrak{D}[\Phi_1, x] + \mathfrak{D}[\Phi_2, x] + \mathfrak{D}[u, \Phi_1, x] \mathfrak{D}[v, \Phi_2, x]. \#$$

See the books,^{1,2}, there are certain classes of graphs that occur for their definitions. The detour polynomials and detour indices will be determined of a conn. graphs resulting from others graphs using the idea of vertex identification.

Main Results:

In the following sections, polynomials of detour distance are found from the graphs by identified their vertices with examples of the important special graphs.

Detour Polynomial of Vertex Identified Graphs

Let $\langle \Phi_1, \Phi_2, \dots, \Phi_n \rangle$ be the set of pairwise disjoint graphs to vertices $u_i, v_i \in V(\Phi_i), i = 1, 2, \dots, n, n \geq 3$, then the vertex identified graph $\prod_{i=1}^n VI_v(\Phi_i) = (\Phi_1 \cdot \Phi_2 \cdot \dots \cdot \Phi_n) = \langle \Phi_1, \Phi_2, \dots, \Phi_n; v_1 \cdot u_2; v_2 \cdot u_3; \dots; v_{n-1} \cdot u_n \rangle$ of $\{\Phi_i\}_{i=1}^n$ with respect to the vertices $\{v_i, u_{i+1}\}_{i=1}^{n-1}$ is the graph obtained from the graphs $\Phi_1, \Phi_2, \dots, \Phi_n$ by identifying the vertex v_i with the vertex u_{i+1} for all $i = 1, 2, \dots, n - 1$. (See Fig. 1).

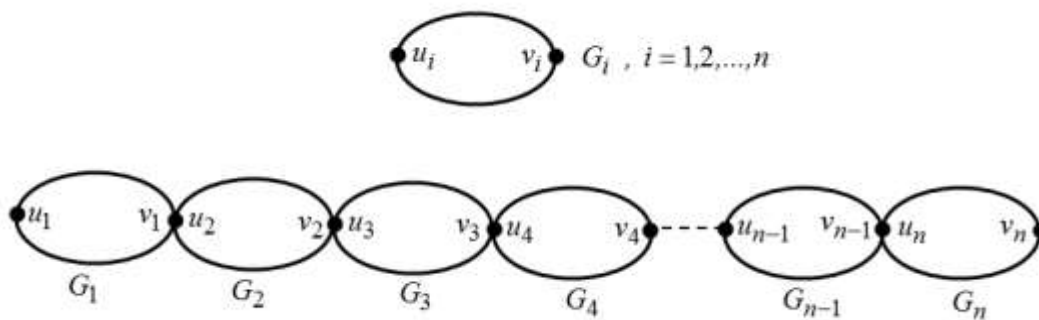


Figure 1. Graph $\prod_{i=1}^n VI_v(\Phi_i), \Phi_i = G_i$.

From clearly that :

$$p(\prod_{i=1}^n VI_v(\Phi_i)) = \sum_{i=1}^n p(\Phi_i) - n + 1,$$

$$q(\prod_{i=1}^n VI_v(\Phi_i)) = \sum_{i=1}^n q(\Phi_i).$$

Proposition: $diam_{\mathfrak{D}}(\prod_{i=1}^n VI_v(\Phi_i)) = \max_{i < j} \{e_{\mathfrak{D}}(v_i) + e_{\mathfrak{D}}(u_j) + \sum_{r=i+1}^{j-1} \mathfrak{D}(v_r, u_r)\}$.

Proof: Obviously. #

Now, the detour polynomial will be given of $\prod_{i=1}^n VI_v(\Phi_i)$ for all $n \geq 2$ in the following theorem.

Theorem 3: $\forall n \in \mathbb{N} - \{0,1\}$, there will be:

$$\mathfrak{D}[\prod_{i=1}^n VI_v(\Phi_i), x] = \sum_{i=1}^n \mathfrak{D}[\Phi_i, x] + \sum_{h=1}^{n-1} \mathfrak{D}[v_h, \prod_{i=1}^h VI_v(\Phi_i), x] \mathfrak{D}[u_{h+1}, \Phi_{h+1}, x].$$

Proof: The theorem is proved using mathematical induction on, $n \geq 2$.

For $n = 2$, then the formula in this theorem is true .

The theorem is assumed that is true for each value less than n , $n \geq 3$ and prove it for all n .

Then, using Theorem 2, will be got the following:

$$\begin{aligned} \mathfrak{D}[\prod_{i=1}^n VI_v(\Phi_i), x] &= \mathfrak{D}[(\prod_{i=1}^{n-1} VI_v(\Phi_i)) \cdot \Phi_n, x] \\ &= \mathfrak{D}[\prod_{i=1}^{n-1} VI_v(\Phi_i), x] + \mathfrak{D}[\Phi_n, x] \\ &+ \mathfrak{D}[v_{n-1}, \prod_{i=1}^{n-1} VI_v(\Phi_i), x] \mathfrak{D}[u_n, \Phi_n, x]. \end{aligned}$$

Using the mathematical induction hypothesis, the following is obtained:

$$\begin{aligned} \mathfrak{D}[\prod_{i=1}^n VI_v(\Phi_i), x] &= \sum_{i=1}^{n-1} \mathfrak{D}[\Phi_i, x] \\ &+ \sum_{h=1}^{n-2} \mathfrak{D}[v_h, \prod_{i=1}^h VI_v(\Phi_i), x] \mathfrak{D}[u_{h+1}, \Phi_{h+1}, x] \\ &+ \mathfrak{D}[\Phi_n, x] \\ &+ \mathfrak{D}[v_{n-1}, \prod_{i=1}^{n-1} VI_v(\Phi_i), x] \mathfrak{D}[u_n, \Phi_n, x]. \end{aligned}$$

Thus ,

$$\begin{aligned} \mathfrak{D}[\prod_{i=1}^n VI_v(\Phi_i), x] &= \sum_{i=1}^n \mathfrak{D}[\Phi_i, x] \\ &+ \sum_{h=1}^{n-1} \mathfrak{D}[v_h, \prod_{i=1}^h VI_v(\Phi_i), x] \mathfrak{D}[u_{h+1}, \Phi_{h+1}, x]. \end{aligned}$$

Lemma4: For all $u_i, v_i \in V(\Phi_i), i = 1, 2, \dots, t$, $t \geq 2$, there will be:

$$\begin{aligned} \mathfrak{D}[v_t, \prod_{i=1}^t VI_v(\Phi_i), x] &= \mathfrak{D}[v_t, \Phi_t, x] \\ &+ \sum_{i=1}^{t-1} \mathfrak{D}[v_i, \Phi_i, x] x^{\sum_{h=i+1}^t \mathfrak{D}(v_h, u_h)}. \end{aligned}$$

Proof: The mathematical induction will be used on t , $t \geq 2$. From Fig.1, the following is noticed that, when $t = 2$.

$$\begin{aligned} \mathfrak{D}[v_2, \prod_{i=1}^2 VI_v(\Phi_i), x] &= \mathfrak{D}[v_2, \Phi_2, x] \\ &+ \mathfrak{D}[v_1, \Phi_1, x] x^{\mathfrak{D}(v_2, u_2)}. \end{aligned}$$

Thus , the lemma is true for $t = 2$.

Now, the theorem assumed that is true for all value less than t , $t \geq 3$. For all t , there will be

$$\begin{aligned} \mathfrak{D}[v_t, \prod_{i=1}^t VI_v(\Phi_i), x] &= \mathfrak{D}[v_t, \Phi_t, x] \\ &+ \mathfrak{D}[v_{t-1}, \prod_{i=1}^{t-1} VI_v(\Phi_i), x] x^{\mathfrak{D}(v_t, u_t)} \\ &= \mathfrak{D}[v_t, \Phi_t, x] + \{ \mathfrak{D}[v_{t-1}, \Phi_{t-1}, x] \\ &+ \sum_{i=1}^{t-2} \mathfrak{D}[v_i, \Phi_i, x] x^{\sum_{h=i+1}^{t-1} \mathfrak{D}(v_h, u_h)} \} x^{\mathfrak{D}(v_t, u_t)} \\ &= \mathfrak{D}[v_t, \Phi_t, x] \\ &+ \sum_{i=1}^{t-1} \mathfrak{D}[v_i, \Phi_i, x] x^{\sum_{h=i+1}^t \mathfrak{D}(v_h, u_h)}. \end{aligned}$$

From Theorem 3 and Lemma 4, the following theorem will be obtained.

Theorem 5: For all $n \in \mathbb{N}$, $n \geq 3$, there will be:

$$\begin{aligned} \mathfrak{D}[\prod_{i=1}^n VI_v(\Phi_i), x] &= \sum_{i=1}^n \mathfrak{D}[\Phi_i, x] \\ &+ \sum_{i=1}^{n-1} \mathfrak{D}[v_i, \Phi_i, x] \mathfrak{D}[u_{i+1}, \Phi_{i+1}, x] \\ &+ \sum_{i=2}^{n-1} \sum_{h=1}^{i-1} \mathfrak{D}[v_h, \Phi_h, x] \mathfrak{D}[u_{i+1}, \Phi_{i+1}, x] x^{\sum_{t=h+1}^i \mathfrak{D}(v_t, u_t)}. \end{aligned}$$

Proof: From Theorem 3, there will be:

$$\begin{aligned} \mathfrak{D}[\prod_{i=1}^n VI_v(\Phi_i), x] &= \sum_{i=1}^n \mathfrak{D}[\Phi_i, x] \\ &+ \mathfrak{D}[v_1, \Phi_1, x] \mathfrak{D}[u_2, \Phi_2, x] \\ &+ \sum_{i=2}^{n-1} \mathfrak{D}[v_i, \prod_{h=1}^i VI_v(\Phi_h), x] \mathfrak{D}[u_{i+1}, \Phi_{i+1}, x], \end{aligned}$$

where $\prod_{h=1}^1 VI_v(\Phi_h) = \Phi_1$. Using Lemma 4, it is got the following:

$$\begin{aligned} \mathfrak{D}[\prod_{i=1}^n VI_v(\Phi_i), x] &= \sum_{i=1}^n \mathfrak{D}[\Phi_i, x] \\ &+ \mathfrak{D}[v_1, \Phi_1, x] \mathfrak{D}[u_2, \Phi_2, x] + \sum_{i=2}^{n-1} \{ \mathfrak{D}[v_i, \Phi_i, x] \\ &+ \sum_{h=1}^{i-1} \mathfrak{D}[v_h, \Phi_h, x] x^{\sum_{t=h+1}^i \mathfrak{D}(v_t, u_t)} \} \mathfrak{D}[u_{i+1}, \Phi_{i+1}, x]. \\ &= \sum_{i=1}^n \mathfrak{D}[\Phi_i, x] \\ &+ \sum_{i=1}^{n-1} \mathfrak{D}[v_i, \Phi_i, x] \mathfrak{D}[u_{i+1}, \Phi_{i+1}, x] \\ &+ \sum_{i=2}^{n-1} \sum_{h=1}^{i-1} \mathfrak{D}[v_h, \Phi_h, x] \mathfrak{D}[u_{i+1}, \Phi_{i+1}, x] x^{\sum_{t=h+1}^i \mathfrak{D}(v_t, u_t)}. \end{aligned}$$

Remark: Taking the derivative of $\mathfrak{D}[\prod_{i=1}^n VI_v(\Phi_i), x]$ given in Theorem 5 with respect to x and for $x = 1$, the detour index will be gotten of $\prod_{i=1}^n VI_v(\Phi_i)$.

If $G_i \equiv G$, for all $i = 1, 2, \dots, n$, $n \geq 2$, then $\prod_{i=1}^n VI_v(\Phi_i)$, is denoted by $VI_v^n(\Phi)$. It is obvious that $p(VI_v^n(\Phi)) = np(\Phi) - n + 1$, $q(VI_v^n(\Phi)) = nq(\Phi)$ and if $u_i = u$, $v_i = v$, $1 \leq i \leq n$, then the following result will be got:

Corollary 6: For any conn.graph. Φ and for all $n \in \mathbb{N}$, $n \geq 3$, there will be:

$$\begin{aligned} \mathfrak{D}[VI_v^n(\Phi), x] &= n\mathfrak{D}[\Phi, x] \\ &+ \mathfrak{D}[v, \Phi, x] \mathfrak{D}[u, \Phi, x] \sum_{i=0}^{n-2} (n - 1 - i) x^{i\mathfrak{D}(u, v)}. \end{aligned}$$

Proof: From Theorem 5, the following will be obtained:

$$\begin{aligned} \mathfrak{D}[VI_v^n(\Phi), x] &= \sum_{i=1}^n \mathfrak{D}[\Phi, x] \\ &+ \sum_{i=1}^{n-1} \mathfrak{D}[v, \Phi, x] \mathfrak{D}[u, \Phi, x] \\ &+ \sum_{i=2}^{n-1} \sum_{h=1}^{i-1} \mathfrak{D}[v, \Phi, x] \mathfrak{D}[u, \Phi, x] x^{\sum_{t=h+1}^i \mathfrak{D}(v, u)} \\ &= n\mathfrak{D}[\Phi, x] + (n - 1) \mathfrak{D}[v, \Phi, x] \mathfrak{D}[u, \Phi, x] \\ &+ \mathfrak{D}[v, \Phi, x] \mathfrak{D}[u, \Phi, x] \sum_{i=2}^{n-1} \sum_{h=1}^{i-1} x^{(i-h)\mathfrak{D}(u, v)} \\ &= n\mathfrak{D}[\Phi, x] + (n - 1) \mathfrak{D}[v, \Phi, x] \mathfrak{D}[u, \Phi, x] \\ &+ \mathfrak{D}(v, \Phi; x) \mathfrak{D}(u, \Phi; x) \sum_{i=1}^{n-2} (n - i - 1) x^{i\mathfrak{D}(u, v)} \\ &= n\mathfrak{D}[\Phi, x] \\ &+ \mathfrak{D}[v, \Phi, x] \mathfrak{D}[u, \Phi, x] \sum_{i=0}^{n-2} (n - 1 - i) x^{i\mathfrak{D}(u, v)}. \end{aligned}$$

Corollary 7: For a conn.graph. Φ and for all $n \in \mathbb{N}$, $n \geq 3$, there will be:

$$\begin{aligned} \mathfrak{D}(VI_v^n(\Phi)) &= n\mathfrak{D}(\Phi) + (p - 1)^2 \cdot \mathfrak{D}(u, v) \cdot \binom{n}{3} \\ &+ (p - 1) \cdot \{ \mathfrak{D}(v, \Phi) + \mathfrak{D}(u, \Phi) \} \cdot \binom{n}{2}. \end{aligned}$$

Proof: From Corollary 6, the following will be obtained:

$$\begin{aligned} \mathfrak{D}(VI_v^n(\Phi)) &= \mathfrak{D}'[VI_v^n(\Phi), x]|_{x=1} \\ &= n\mathfrak{D}(\Phi) \\ &+ \mathfrak{D}[v, \Phi, 1] \mathfrak{D}[u, \Phi, 1] \sum_{i=0}^{n-2} i(n - 1 - i) \mathfrak{D}(u, v) \\ &+ \{ \mathfrak{D}[v, \Phi, 1] \mathfrak{D}(u, \Phi) \\ &+ \mathfrak{D}(v, \Phi) \mathfrak{D}[u, \Phi, 1] \} \sum_{i=0}^{n-2} (n - 1 - i) \end{aligned}$$

$$= n\mathcal{D}(\Phi) + (p-1)^2\mathcal{D}(u, v) \binom{n}{3} + (p-1)\{\mathcal{D}(v, G) + \mathcal{D}(u, \Phi)\} \binom{n}{2},$$

where $[v, \Phi, 1] = p-1$, for any vertex v in Φ .

Examples:

In this section, detour polynomials and detour indices of generalized vertex identified of special graphs will be given a general formula such as: complete graph \mathcal{K}_p , wheel graph \mathcal{W}_p , cycle graph \mathcal{G}_p , quadruple circle with horizontal chord \mathcal{G}_4^+ and quadruple circle with vertical chord \mathcal{G}_4^\perp .

Example 1: If \mathcal{K}_p is a complete graph of order p , $p \geq 3$, then

$$\mathcal{D}(VI_v^n(\mathcal{K}_p); x) = n \binom{p}{2} x^{p-1} + (p-1)^2 x^{2p-2} \sum_{i=0}^{n-2} (n-1-i)x^{i(p-1)}.$$

Special case, if $p = 4$, see Fig. 2. Then

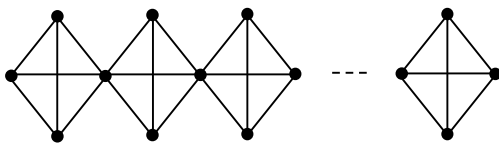


Figure 2. Graph $VI_v^n(\mathcal{K}_4)$

$$\mathcal{D}(I_v^n(\mathcal{K}_4); x) = 6nx^3 + 9x^6 \sum_{i=0}^{n-2} (n-1-i)x^{3i}.$$

Example 2: If $\Phi \equiv \mathcal{W}_p$, $p \geq 3$, then

$$\mathcal{D}(VI_v^n(\mathcal{W}_p); x) = n \binom{p}{2} x^{p-1} + (p-1)^2 x^{2p-2} \sum_{i=0}^{n-2} (n-1-i)x^{i(p-1)}.$$

Example 3: If $\Phi \equiv \mathcal{G}_p$, $p \geq 4$, and $\mathcal{D}(u, v) = \binom{p}{2}$, where u and v are identical vertices, then

(1) If p is odd,

$$\mathcal{D}(VI_v^n(\mathcal{G}_p); x) = np x^{\frac{p+1}{2}} \left[\frac{p-1}{2} \right] + 4x^{p+1} \left[\frac{p-1}{2} \right]^2 \sum_{i=0}^{n-2} (n-1-i)x^{i\left(\frac{p+1}{2}\right)}.$$

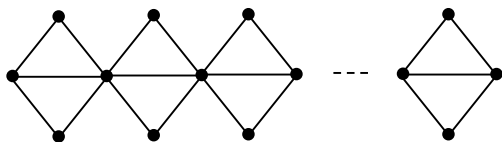


Figure 5.a. Graph $VI_v^n(\mathcal{G}_4^+)$

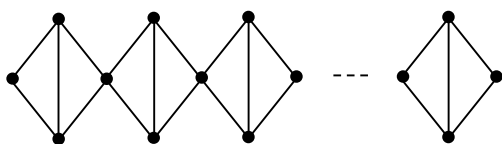


Figure 5.b. Graph $VI_v^n(\mathcal{G}_4^\perp)$

in which $\left[\frac{p-1}{2} \right] = 1 + x + x^2 + \dots + x^{\frac{p-1}{2}-1}$.

Special case, if $p = 5$, see Fig. 3, then

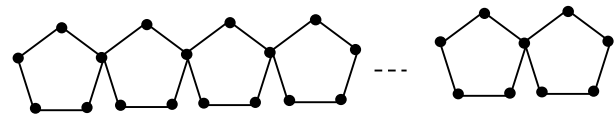


Figure 3. Graph $VI_v^n(\mathcal{G}_5)$

$$\mathcal{D}(I_v^n(\mathcal{G}_5); x) = 5nx^3(1+x) + 4x^6(1+x)^2 \sum_{i=0}^{n-2} (n-1-i)x^{3i}.$$

(2) If p is an even, then

$$\mathcal{D}(VI_v^n(\mathcal{G}_p); x) = np x^{\frac{p}{2}} \left(\left[\frac{p}{2} \right] - \frac{1}{2} \right) + 4x^p \left(\left[\frac{p}{2} \right] - \frac{1}{2} \right)^2 \sum_{i=0}^{n-2} (n-1-i)x^{2i},$$

in which $\left[\frac{p}{2} \right] = 1 + x + x^2 + \dots + x^{\frac{p}{2}-1}$.

Special case, if $p = 6$, see Fig. 4, then

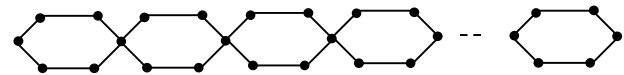


Figure 4. Graph $VI_v^n(\mathcal{G}_6)$

$$\mathcal{D}(I_v^n(\mathcal{G}_6); x) = 3nx^3(1+2x+2x^2) + x^6(1+2x+2x^2)^2 \sum_{i=0}^{n-2} (n-1-i)x^{3i}.$$

Example 4:

(1) If $\Phi \equiv \mathcal{G}_4^+$, see Fig. 5.a, then

$$\mathcal{D}(VI_v^n(\mathcal{G}_4^+); x) = nx^2(1+5x) + x^4(1+2x)^2 \sum_{i=0}^{n-2} (n-1-i)x^{2i}.$$

(2) If $\Phi \equiv \mathcal{G}_4^\perp$, see Fig. 5.b, then

$$\mathcal{D}(VI_v^n(\mathcal{G}_4^\perp); x) = nx^2(1+5x) + 9x^6 \sum_{i=0}^{n-2} (n-1-i)x^{3i}.$$

Corollary 8: For $n \geq 3$, there will be:

(1) $\mathcal{D}(VI_v^n(\mathcal{K}_p)) = n \cdot (p-1) \cdot \binom{p}{2} + (p-1)^3 \cdot \left(\binom{n}{3} + 2 \binom{n}{2} \right).$

(2) $\mathcal{D}(VI_v^n(\mathcal{G}_p)) = \frac{1}{8} np^2 \cdot (3p-4) + \binom{p}{2} \cdot \left((p-1) \cdot \binom{n}{3} + (3p-4) \cdot \binom{n}{2} \right),$

when p is an odd.

(3) $\mathcal{D}(VI_v^n(\mathcal{G}_p)) = \frac{1}{8} np \cdot (p-1) \cdot (3p-1) + \frac{1}{2} (p-1)^2 \cdot \left((p+1) \cdot \binom{n}{3} + (3p-1) \cdot \binom{n}{2} \right),$

where p is even

(4) $\mathcal{D}(VI_v^n(\mathcal{G}_4^+)) = n(3n^2 + 15n - 1).$

(5) $\mathcal{D}(VI_v^n(\mathcal{G}_4^\perp)) = \frac{1}{2} n(63n^2 - 81n + 52). \#$

The Graphs with Equal Detour Polynomials

The two polynomials will be equal if and only if the coefficients of x^i are equal for all i , that is, $\sum_{i=\alpha}^{\beta} a_i x^i = \sum_{i=\alpha}^{\beta} b_i x^i$ iff $a_i = b_i$, for all i . It is clear that if $\Phi_1 \cong \Phi_2$, then $\mathcal{D}[\Phi_1, x] = \mathcal{D}[\Phi_2, x]$, but the converse is not true, for example, the detour polynomials of complete graph \mathcal{K}_p and wheel graph \mathcal{W}_p , $p \geq 4$ are equal, but \mathcal{K}_p and \mathcal{W}_p , $p \geq 4$ are not isomorphic, $\mathcal{K}_p \not\cong \mathcal{W}_p$.

Corollary 9: Let Φ , Φ^* and Φ^{**} be conn.graphs. and the vertices identification $v^* \in V(\Phi)$, $v^{**} \in V(\Phi^*)$ and $v^{***} \in V(\Phi^{**})$. If they satisfy the conditions:

- $\mathcal{D}[\Phi^*, x] = \mathcal{D}[\Phi^{**}, x]$.
- $\mathcal{D}[v^{**}, \Phi^*, x] = \mathcal{D}[v^{***}, \Phi^{**}, x]$.

Then $\mathcal{D}[\Phi, \Phi^*, x] = \mathcal{D}[\Phi, \Phi^{**}, x]$.

Proof: Obviously. #

Corollary 10: Let, Φ_i^* and Φ_i^{**} be conn.graphs. and the vertices identification $v_i^*, u_i^* \in V(\Phi_i^*)$ and $v_i^{**}, u_i^{**} \in V(\Phi_i^{**})$, for all $i = 1, 2, \dots, n$, $n \geq 2$. If they satisfy the conditions for all $i = 1, 2, \dots, n$, $n \geq 2$:

- (1) $\mathcal{D}[\Phi_i^*, x] = \mathcal{D}[\Phi_i^{**}, x]$.
- (2) $\mathcal{D}[v_i^*, \Phi_i^*, x] = \mathcal{D}[v_i^{**}, \Phi_i^{**}, x]$ and $\mathcal{D}[u_i^*, \Phi_i^*, x] = \mathcal{D}[u_i^{**}, \Phi_i^{**}, x]$.
- (3) $\mathcal{D}(v_i^*, u_i^*) = \mathcal{D}(v_i^{**}, u_i^{**})$.

Then $\prod_{i=1}^n VI_v(\Phi_i^*) = \prod_{i=1}^n VI_v(\Phi_i^{**})$, see Examples 1 and 2.

Proof: Obviously. #

Conclusions:

The construct operation generalized by vertices identified of n – graphs are found from any two graphs from a sequence of n pairwise vertex disjoint connected graphs $\Phi_1, \Phi_2, \dots, \Phi_n$, $n \geq 3$. Moreover, the detour polynomial and detour index will be obtained for generalized vertex identified graphs when every Φ_i , $i = 1, 2, \dots, n$, $n \geq 3$ is an isomorphic to a special graph which has important applications in chemistry.

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Authors' declaration:

- Conflicts of Interest: None.

- We hereby confirm that all the Figures and Tables in the manuscript are mine ours. Besides, the Figures and images, which are not mine ours, have been given the permission for re-publication attached with the manuscript.
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Authors' contributions statement:

H. J. A., A. M. A. and G. A. M. S. uses their initials to explain his/her contribution (The definition: Generalized Vertex Identified of Graphs was presented by A. M. to PhD student H., with some problems which unsolved G. A. also made suggestions Something and gave other unresolved problems for some special graphs.

References:

1. Chartrand G, Lesniak L. Graphs and Digraphs. 6th ed. Wadsworth and Brooks / Cole. California, 2016.
2. Buckley F, Harary F. Distance in Graphs. 1st ed. Addison – Wesley, Longman, 1990.
3. Chartrand G, Escudero H, Zhang P. Detour distance in graphs. J Comb Math Comb Comput. 2005; 53: 75-9.
4. Chartrand G, Zhang P. Distance in graphs-taking the long view. AKCE Int J Graphs Comb. 2004; 1(1): 1-13.
5. Santhakumaran A P, Athisayanathan S. On the connected detour number of a graph. J Prime Res Math. 2009; 5: 149-170.
6. Ahmed M A, Ali A A. The Connected Detour Numbers of Special Classes of Connected Graphs. J Math. 2019; 2019: 1-9.
7. Ahmed M A, Ali A A. Characterizations of Graphs with Prescribed Connected Detour Numbers. Asian Eur J Math. 2021; 14: 1-18.
8. Manal N A, Rasha J M, Fatema A S. Variant Domination Types for a Complete h-ary Tree. Baghdad Sci J. 2021; 18 (1): 797-802.
9. Ahmed A O, Haneen H O. Hn-Domination in Graphs. Baghdad Sci J. 2019; 16(1): 242-247.
10. Manal N A, Parwen. A K B. Tadpole Domination in Graphs. Baghdad Sci J. 2018; 15(4): 266-471.
11. Raigorodskii A M , Trukhan T V. On the Chromatic Numbers of Distance Graphs. Dokl. Math., 2018, 98(2): 515–517.
12. Bobua A V, Kupriyanova A E, Raigorodskii A M, On the Number of Edges in a Uniform Hypergraph with a Range of Permitted Intersections. Dokl. Math., 2017, 96(1): 354–357.
13. Amić D, Trinajstić N. On the Detour Matrix. Croat Chem Acta. 1995; 68: 53-62.
14. Lukovits I. The detour index. Croat Chem Acta. 1996; 69: 873-882.
15. Shahkooni R J, Khormali O, Mahmiani A G. The Polynomial of Detour Index for a Graph. World Appl Sci J. 2011; 15(10): 1473-1483.

16. Wei F, Wei-Hua L, Jia-Bao L, Fu-Yuan C, Zhen-Mu H, Zheng-Jiang X. Maximum Detour–Harary Index for Some Graph Classes. *Symmetry*. 2018; 10:1-12.
17. Kavithaa S, Kaladevi V. Gutman Index and Detour Gutman Index of Pseudo-Regular Graphs. *J Appl Math*. Article ID 4180650. 2017; 2017: 1-8.
18. Kavithaa S, Kaladevi V. Wiener, Hyper Wiener and Detour Index of Pseudoregular Graphs. *J Infor Math Sci*. 2018; 5(1): 751–763.
19. Ahmed M A, Asmaa S A. A Relation between D -Index and Wiener Index for r -Regular Graphs. *Int J Math Math Sci*. 2020; 2020: 1-6.
20. Sakander H, Asad K, Suliman K, Jia-Bao L. Hamilton Connectivity of Convex Polytopes with Applications to Their Detour Index. *Compfs*. Article ID 6684784; 2021. 2021: 1- 23.
21. Abdulsattar M K, Ahmed M A, Shuaa' M A. Application of Width Distance on Semi–Star Link Satellite Constellation. *J Discret Math Sci. Cryptogr*. 2021; 24(3): 797–807.
22. Herish O A, Ivan D A. The restricted detour polynomials of a prism and some wheel related graphs. *JIOS*. 2021, 42: 773-784.
23. Gutman I. Some properties of the Wiener polynomial. *Graph theory notes N. Y. XXV. Ann N Y Acad Sci*. 1993: 13-17.
24. Haveen J A, Ahmed M A, Gashaw A M. Detour polynomials of vertex coalenscence and bridges coalenscence graphs. *Asian-European Journal of Mathematics* 2022; 15(2): 1-15.

متعددات حدود الالتفاف لتعميم تطابق الرأس للبيانات

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الخلاصة:

تعد مسافة الالتفاف من أهم أنواع المسافات التي لها تطبيقات حديثة في الكيمياء وشبكات الكمبيوتر، لذلك حصلنا في هذا البحث على متعددات حدود الالتفاف وأدلتها لـ n من البيانات المنفصلة عن بعضها البعض بالنسبة للرؤوس ، $n \geq 3$. أيضاً وجدنا متعددات حدود الالتفاف وأدلتها لبعض البيانات الخاصة والتي لها تطبيقات مهمة في الكيمياء.

الكلمات المفتاحية: مسافة الالتفاف، متعدد الحدود الالتفاف، دليل الالتفاف، بيانات خاصة، تطابق الرؤوس.